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A Cesàro-like operator from a class of analytic function spaces to analytic Besov spaces

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Abstract. Let μ be a finite positive Borel measure on $[0,1)$ and $f(z)=\sum_{n=0}^{\infty}a_nz^n\in H(\mathbb{D})$. For $0<\alpha<\infty$, the generalized Cesàro-like operator $\mathcal{C}_{\mu,\alpha}$ is defined by

$$\mathcal{C}_{\mu,\alpha}(f)(z)=\sum_{n=0}^{\infty}\left(\mu_n\sum_{k=0}^{n}\frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!}a_k\right)z^n,\ z\in\mathbb{D},$$

where, for $n\geq 0$, μ_n denotes the nth moment of the measure μ, that is, $\mu_n=\int_0^1 t^n d\mu(t)$.

For $s>1$, let X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}}\subset X\subset\mathcal{B}$. In this article, for $1\leq p<\infty$, we characterize the measure μ for which $\mathcal{C}_{\mu,\alpha}$ is bounded (resp. compact) from X into the analytic Besov space B_p.

1 Introduction

Let $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$. H^∞ denote the set of bounded analytical functions on $\mathbb{D}$.

The Bloch space $\mathcal{B}$ consists of those functions $f\in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}}=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)|<\infty.$$

For $1<p<\infty$, the analytic Besov space B_p consists of those functions $f\in H(\mathbb{D})$ such that

$$\|f\|_{B_p}=|f(0)|+\left(\int_{\mathbb{D}}|f'(z)|^p(1-|z|)^{p-2}\mathrm{d}A(z)\right)^{\frac{1}{p}}<\infty,$$

where $dA(z)=\frac{dxdy}{\pi}$ is the normalized area measure on $\mathbb{D}$. When $p=2$, then B_2 is just the classic Dirichlet space $\mathcal{D}$. If $1<p_1<p_2<\infty$, then $B_{p_1}\subsetneq B_{p_2}\subsetneq\mathcal{B}$. It is known that the analytic Besov spaces are Möbius invariant and the Bloch space $\mathcal{B}$ is the largest Möbius invariant space.

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The space B_1 consists of $f \in H(\mathbb{D})$ such that

$$\|f\|_{B_1} = |f(0)| + |f'(0)| + \int_{\mathbb{D}} |f''(z)| dA(z) < \infty.$$

The space B_1 is the smallest Möbius invariant Banach spaces of analytic function in $\mathbb{D}$ and $B_1 \subsetneq H^\infty$. See [32, Chapter 5] for the theory of these spaces.

Let $0 < p \leq \infty$, the classical Hardy space H^p consists of those functions $f \in H(\mathbb{D})$ for which

$$\|f\|_p = \sup_{0 \leq r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, \ 0 < p < \infty,$$
$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

Let $1 \leq p < \infty$ and $0 < \alpha \leq 1$, the mean Lipschitz space Λ_α^p consists of those functions $f \in H(\mathbb{D})$ having a non-tangential limit almost everywhere such that $\omega_p(t, f) = O(t^\alpha)$ as $t \to 0$. Here $\omega_p(\cdot, f)$ is the integral modulus of continuity of order p of the function $f(e^{i\theta})$. It is known (see [11]) that Λ_α^p is a subset of H^p and

$$\Lambda_\alpha^p = \left(f \in H(\mathbb{D}) : M_p(r, f') = O\left(\frac{1}{(1-r)^{1-\alpha}} \right), \text{ as } r \to 1 \right).$$

The space Λ_α^p is a Banach space with the norm $\|\cdot\|_{\Lambda_\alpha^p}$ given by

$$\|f\|_{\Lambda_\alpha^p} = |f(0)| + \sup_{0 \leq r < 1} (1-r)^{1-\alpha} M_p(r, f').$$

It is known (see, e.g., [8]) that

$$\Lambda_{\frac{1}{p}}^p \subsetneq \mathcal{B}. \ 1 < p < \infty.$$

For $f(z) = \sum_{n=0}^\infty a_n z^n \in H(\mathbb{D})$, the Cesàro operator $\mathcal{C}$ is defined by

$$\mathcal{C}(f)(z) = \sum_{n=0}^\infty \left(\frac{1}{n+1} \sum_{k=0}^n a_k \right) z^n, \ z \in \mathbb{D}.$$

The boundedness and compactness of the Cesàro operator $\mathcal{C}$ and its generalizations on various spaces of analytic functions such as Hardy spaces, Bergman spaces, Dirichlet spaces, the Bloch space, Q_p spaces, mixed norm spaces have been widely studied. See, e.g., [1, 2, 10, 19, 24–26, 30]. We refer the reader to the recent survey [22] for more on Cesàro operator.

Recently, Galanopoulos et al. [13] introduced a Cesàro-like operator $\mathcal{C}_\mu$ on $H(\mathbb{D})$, which is a natural generalization of the classical Cesàro operator $\mathcal{C}$. They considered the following generalization: For a positive Borel measure μ on the interval $[0, 1)$ they

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define the operator

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\mu_n \sum_{k=0}^{n} \widehat{f}(k)\right) z^n = \int_0^1 \frac{f(tz)}{1-tz} d\mu(t),\ z \in \mathbb{D}$$

where μ_n stands for the moment of order n of μ, that is, $\mu_n = \int_0^1 t^n d\mu(t)$. They studied the operators $\mathcal{C}_\mu$ acting on distinct spaces of analytic functions (e.g., Hardy spaces, Bergman spaces, the Bloch space, etc.).

The Cesàro-like operator $\mathcal{C}_\mu$ defined above has attracted the interest of many mathematicians. For instance, Jin and Tang [18] studied the boundedness (resp. compactness) of $\mathcal{C}_\mu$ from one Dirichlet-type space $\mathcal{D}_\alpha$ into another one $\mathcal{D}_\beta$. Bao et al. [4] studied the range of $\mathcal{C}_\mu$ acting on H^∞. Blasco [6] investigated the operators $\mathcal{C}_\mu$ induced by complex Borel measures on $[0,1)$, and extended the results of [13] to this more general case. Galanopoulos et al. [12] studied the behavior of the operators $\mathcal{C}_\mu$ on the Dirichlet space and on the analytic Besov spaces B_p. Recently, Bao et al. [3] have complectly characterized the measure μ such that $\mathcal{C}_\mu$ is bounded (resp. compact) on the Dirichlet space. The reader is referred to [5, 7, 14, 28, 29, 31] for more on Cesàro-like operators on spaces of analytic functions.

In [4], Bao et al. introduced a more general Cesàro-like operator. Suppose that $0 < \alpha < \infty$ and μ is a finite positive Borel measure on $[0,1)$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$, they defined

$$\mathcal{C}_{\mu,\alpha}(f)(z) = \sum_{n=0}^{\infty}\left(\mu_n \sum_{k=0}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!} a_k\right) z^n,\ z \in \mathbb{D}.$$

A simple calculation with power series gives the integral form of $\mathcal{C}_{\mu,\alpha}$ as follows.

$$\mathcal{C}_{\mu,\alpha}(f)(z) = \int_0^1 \frac{f(tz)}{(1-tz)^\alpha} d\mu(t).$$

For $1 < s < \infty$, let X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. There are many well known spaces located between the mean Lipschitz space $\Lambda^s_{\frac{1}{s}}$ and the Bloch space $\mathcal{B}$. In [4], the authors investigated the range of $\mathcal{C}_{\mu,\alpha}$ acting on H^∞. They proved that if $\max\{1, \frac{1}{\alpha}\} < s < \infty$, then $\mathcal{C}_{\mu,\alpha}(H^\infty) \subset X$ if and only if μ is an α-Carleson measure. Zhou [31] considered the same problem for the measure μ supported on $\mathbb{D}$. Guo et al. [17] investigated the boundedness (resp. compactness) of $\mathcal{C}_{\mu,\alpha}$ acting from X to weighted Bergman spaces A^p_β. Galanopoulos et al. [15] characterized the measure μ such that $\mathcal{C}_{\mu,\alpha}$ is bounded from weighted Bergman space A^p_β to A^q_β when $1 \le p \le q < \infty$ and $\beta > -1$. Sun et al. [27] studied the boundedness (resp. compactness) of the operator $\mathcal{C}_{\mu,1}$ acting from B_p to X. It remains open to characterize the boundedness and the compactness of $\mathcal{C}_{\mu,\alpha}$ from B_p to B_p when $p > 1$ and $p \neq 2$. The Besov spaces B_p and Bloch space $\mathcal{B}$ are Möbius invariant and the Bloch space $\mathcal{B}$ can be regarded as the limit case of B_p as $p \to +\infty$. The purpose of this article is describe the measure μ such that the operator $\mathcal{C}_{\mu,\alpha}$ is bounded (resp. compact) from X to B_p for $1 \le p < \infty$.

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Our main results are included in the following.

Theorem 1.1 *Suppose* $0 < \alpha < \infty, 1 < s < \infty$ *and* $\max\{1, \frac{1}{\alpha}\} \le p < \infty$. *Let* μ *be a finite positive Borel measure on* $[0,1)$ *and* X *be a Banach subspace of* $H(\mathbb{D})$ *with* $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. *Then the following statements are equivalent.*

(1) *The operator* $\mathcal{C}_{\mu,\alpha}$ *is bounded from* X *to* B_p.
(2) *The operator* $\mathcal{C}_{\mu,\alpha}$ *is compact from* X *to* B_p.
(3) *The measure* μ *satisfies*

$$\sum_{n=0}^{\infty} (n+1)^{p\alpha-1} \mu_n^p \log^p(n+2) < \infty.$$

For $p = 1$, we have the following corollary.

Corollary 1.2 *Suppose* $1 \le \alpha < \infty$ *and* $1 < s < \infty$. *Let* μ *be a finite positive Borel measure on* $[0,1)$ *and* X *be a Banach subspace of* $H(\mathbb{D})$ *with* $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. *Then the following statements are equivalent.*

(1) *The operator* $\mathcal{C}_{\mu,\alpha}$ *is bounded from* X *to* B_1.
(2) *The operator* $\mathcal{C}_{\mu,\alpha}$ *is compact from* X *to* B_1.
(3) *The measure* μ *satisfies*

$$\sum_{n=0}^{\infty} (n+1)^{\alpha-1} \mu_n \log(n+2) < \infty.$$

(4) *The measure* μ *satisfies*

$$\int_0^1 \frac{\log \frac{e}{1-t}}{(1-t)^{\alpha}} d\mu(t) < \infty.$$

When $p = 2$, the space B_2 is the classic Dirichlet space $\mathcal{D}$, so we have the following corollary.

Corollary 1.3 *Suppose* $1 \le \alpha < \infty$ *and* $1 < s < \infty$. *Let* μ *be a finite positive Borel measure on* $[0,1)$ *and* X *be a Banach subspace of* $H(\mathbb{D})$ *with* $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. *Then the following statements are equivalent.*

(1) *The operator* $\mathcal{C}_{\mu,\alpha}$ *is bounded from* X *to* $\mathcal{D}$.
(2) *The operator* $\mathcal{C}_{\mu,\alpha}$ *is compact from* X *to* $\mathcal{D}$.
(3) *The measure* μ *satisfies*

$$\sum_{n=0}^{\infty} (n+1)^{2\alpha-1} \mu_n^2 \log^2(n+2) < \infty.$$

Throughout the article, the letter C will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation "$P \lesssim Q$" if there exists a constant $C = C(\cdot)$ such that "$P \le CQ$," and "$P \gtrsim Q$" is understood in an analogous manner. In particular, if "$P \lesssim Q$" and "$P \gtrsim Q$," then we will write "$P \asymp Q$".

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2 Preliminaries

To prove our main results, we need some preliminary results which will be repeatedly used throughout the rest of the article. We begin with a characterization of the functions $f \in H(\mathbb{D})$ whose sequence of Taylor coefficients is decreasing which belong to B^p. For a proof, see, e.g., [16, Theorem 3.10] and [20].

Lemma 2.1 *Let $1 < p < \infty$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$. Suppose that the sequence $\{a_n\}_{n=0}^{\infty}$ is a decreasing sequence of nonnegative real numbers. Then $f \in B_p$ if and only if*

$$\sum_{n=1}^{\infty} n^{p-1} a_n^p < \infty.$$

The following lemma contains a characterization of L^p-integrability of power series with nonnegative coefficients. For a proof, see [21, Theorem 1].

Lemma 2.2 *Let $0 < \beta, p < \infty$, $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of nonnegative numbers. Then*

$$\int_0^1 (1-r)^{p\beta-1} \left(\sum_{n=0}^{\infty} \lambda_n r^n \right)^p dr \asymp \sum_{n=0}^{\infty} 2^{-np\beta} \left(\sum_{k \in I_n} \lambda_k \right)^p,$$

where $I_0 = \{0\}$, $I_n = [2^{n-1}, 2^n) \cap \mathbb{N}$ for $n \in \mathbb{N}$.

The following lemma is a consequence of Theorem 2.31 on page 192 of the classical monograph [33].

Lemma 2.3

(a) *The Taylor coefficients a_n of the function*

$$f(z) = \frac{1}{(1-z)^{\beta}} \log^{\gamma} \frac{2}{1-z}, \quad \beta > 0, \gamma \in \mathbb{R}, \ z \in \mathbb{D}$$

have the property $a_n \asymp n^{\beta-1}(\log(n+1))^{\gamma}$.

(b) *The Taylor coefficients a_n of the function*

$$f(z) = \log^{\gamma} \frac{2}{1-z}, \quad \gamma > 0, z \in \mathbb{D}$$

have the property $a_n \asymp n^{-1}(\log(n+1))^{\gamma-1}$.

We also need the following estimates (see, e.g., [23, Proposition 1.4.10]).

Lemma 2.4 *Let α be any real number and $z \in \mathbb{D}$. Then*

$$\int_0^{2\pi} \frac{d\theta}{|1 - ze^{-i\theta}|^{\alpha}} \asymp \begin{cases} 1, & \text{if } \alpha < 1, \\ \log \frac{2}{1-|z|^2}, & \text{if } \alpha = 1, \\ \frac{1}{(1-|z|^2)^{\alpha-1}}, & \text{if } \alpha > 1, \end{cases}$$

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The following lemma is useful in dealing with the compactness. The proof is similar to that of [9, Proposition 3.11]. The details are omitted.

Lemma 2.5 *Let $p \geq 1$, $s > 1$, X be a Banach subspace of $H(\mathbb{D})$ with $\Lambda^s_{\frac{1}{s}} \subset X \subset \mathcal{B}$. Suppose that T is a bounded operator from X to B_p. Then T is compact if and only if for any bounded sequence $\{f_k\}$ in X which converges to 0 uniformly on every compact subset of $\mathbb{D}$, we have $\lim_{k\to\infty} ||T(f_k)||_{B_p} = 0$.*

3 Proofs of the main results

We now present the proofs of Theorem 1.1.

Proof of the implication (1) ⇒ (3) Since the definition of the space B_1 is slightly different from B_p when $p > 1$, we split the proof into $p = 1$ and $p > 1$.

Case 1: $p = 1$.

Assume that $\mathcal{C}_{\mu,\alpha}$ is bounded from X to B_1. Let $g(z) = \log \frac{1}{1-z} = \sum_{k=1}^{\infty} \frac{z^k}{k}$. It is easy to check that $g \in \Lambda^s_{\frac{1}{s}} \subset X$. This implies that $\mathcal{C}_{\mu,\alpha}(g) \in B_1$. For $z \in \mathbb{D}$, by the definition of $\mathcal{C}_{\mu,\alpha}$ we get

$$\mathcal{C}_{\mu,\alpha}(g)''(z) = \sum_{n=0}^{\infty} \left((n+2)(n+1)\mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \right) z^n.$$

For $0 < r < 1$, by Hardy's inequality we have that

$$M_1(r, \mathcal{C}_{\mu,\alpha}(g)'') \gtrsim \sum_{n=0}^{\infty} \left((n+2)\mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \right) r^n.$$

It follows that

$$\begin{aligned} 1 &\gtrsim ||g||_X \gtrsim ||\mathcal{C}_{\mu,\alpha}(g)||_{B_1} = \int_{\mathbb{D}} |\mathcal{C}_{\mu,\alpha}(g)''(z)| dA(z) \\ &= 2\int_0^1 M_1(r, \mathcal{C}_{\mu,\alpha}(g)'') r dr \\ &\gtrsim \int_0^1 \sum_{n=0}^{\infty} \left((n+2)\mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \right) r^{n+1} dr \\ &\gtrsim \sum_{n=0}^{\infty} \mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k}. \end{aligned}$$

Using the Stirling's formula, we get

$$\sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \asymp \sum_{k=1}^{n+2} \frac{(n+3-k)^{\alpha-1}}{k}.$$

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For $n \geq 1$, simple estimations lead us to the following

$$\begin{aligned}\sum_{k=1}^{n+2} \frac{(n+3-k)^{\alpha-1}}{k} &= \left(\sum_{k=1}^{[\frac{n+2}{2}]} + \sum_{k=[\frac{n+2}{2}]+1}^{n+2}\right) \frac{(n+3-k)^{\alpha-1}}{k} \\ &\asymp (n+1)^{\alpha-1} \sum_{k=1}^{[\frac{n+2}{2}]} \frac{1}{k} + \frac{1}{n+1} \sum_{k=[\frac{n+2}{2}]+1}^{n+2} (n+3-k)^{\alpha-1} \\ &\asymp (n+1)^{\alpha-1} \log(n+2) + (n+1)^{\alpha-1} \\ &\asymp (n+1)^{\alpha-1} \log(n+2).\end{aligned}$$

Therefore,

$$\begin{aligned}1 &\gtrsim \sum_{n=0}^{\infty} \mu_{n+2} \sum_{k=1}^{n+2} \frac{\Gamma(n+2-k+\alpha)}{\Gamma(\alpha)(n+2-k)!k} \\ &\gtrsim \sum_{n=0}^{\infty} (n+1)^{\alpha-1} \mu_n \log(n+2).\end{aligned}$$

Case 2: $p > 1$.

Let q be the conjugate index of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. It is known that $(B_q)^* \cong B_p$ (see [32, Theorem 5.24]) under the paring

$$\langle F, G\rangle = \int_{\mathbb{D}} F'(z)\overline{G'(z)} dA(z), \quad F \in B_p, G \in B_q.$$

This means that $\mathcal{C}_{\mu,\alpha}$ is bounded from X to B_p if and only if

$$|\langle \mathcal{C}_{\mu,\alpha}(F), G\rangle| \lesssim \|F\|_X \|G\|_{B_q} \text{ for all } F \in X, G \in B_q.$$

Now, suppose that $\mathcal{C}_{\mu,\alpha}$ is bounded from X to B_p. Take $g(z) = \sum_{n=0}^{\infty} \widehat{g}(n) z^n \in B_q$ and the sequence of its Taylor coefficients is a decreasing sequence of the nonnegative real numbers. Let $f(z) = \log \frac{1}{1-z} = \sum_{n=1} \frac{z^n}{n} \in X$, we have that

$$|\langle \mathcal{C}_{\mu,\alpha}(f), g\rangle| \lesssim \|f\|_X \|g\|_{B_q}.$$

A simple calculation shows that

$$|\langle \mathcal{C}_{\mu,\alpha}(f), g\rangle| = \sum_{n=1}^{\infty} n\mu_n \left(\sum_{k=1}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!k}\right) \widehat{g}(n).$$

This implies that

$$|\langle \mathcal{C}_{\mu,\alpha}(f), g\rangle| = \sum_{n=1}^{\infty} n^{\frac{1}{q}} \mu_n \left(\sum_{k=1}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!k}\right) \widehat{g}(n) n^{\frac{q-1}{q}} < \infty.$$

By Lemma 2.1, the sequence $\{\widehat{g}(n) n^{\frac{q-1}{q}}\}_{n=1}^{\infty} \in l^q$. The well known duality $(l^q)^* = l^p$ yields that

$$\left\{ n^{\frac{1}{q}} \mu_n \left(\sum_{k=1}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!k}\right)\right\}_{n=1}^{\infty} \in l^p.$$

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Using the estimate $\sum_{k=1}^{n} \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)(n-k)!k} \asymp (n+1)^{\alpha-1}\log(n+2)$ we deduce that

$$\sum_{n=0}^{\infty}(n+1)^{p\alpha-1}\mu_n^p \log^p(n+2) < \infty.$$

■

Proof of the implication (3) ⇒ (2) Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in X which converges to 0 uniformly on every compact subset of $\mathbb{D}$. Without loss of generality, we may assume that $f_k(0) = 0$ and $\sup_{k\geq 1}\|f\|_X \leq 1$. It suffices to prove that $\lim_{k\to\infty}\|\mathcal{C}_{\mu,\alpha}(f_k)\|_{B_p} = 0$ by using Lemma 2.5. As before, we divide the proof into $p = 1$ and $p > 1$.

Case 1: $p > 1$.

Assume that $\sum_{n=1}^{\infty}(n+1)^{p\alpha-1}\mu_n^p\log^p(n+1) < \infty$, then

$$\begin{aligned}\sum_{n=1}^{\infty}(n+1)^{p\alpha-1}\mu_n^p\log^p(n+1) &= \sum_{n=1}^{\infty}\left(\sum_{k=2^{n-1}}^{2^n-1}(k+1)^{p\alpha-1}\mu_k^p\log^p(k+1)\right)\\ &\gtrsim \sum_{n=1}^{\infty}2^{np\alpha}\mu_{2^n}^p\log^p(2^n+1)\\ &\gtrsim \sum_{n=1}^{\infty}2^{-n(p-1)}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{\alpha-\frac{1}{p}}\mu_k\log(k+1)\right)^p.\end{aligned}$$

This shows that

$$\sum_{n=1}^{\infty}2^{-n(p-1)}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{\alpha-\frac{1}{p}}\mu_k\log(k+1)\right)^p < \infty.$$

By Lemma 2.2 we have that

$$\begin{aligned}&\int_0^1(1-r)^{p-2}\left(\sum_{n=0}^{\infty}(n+1)^{\alpha-\frac{1}{p}}\mu_n\log(n+1)r^n\right)^p dr\\ &\asymp \sum_{n=0}^{\infty}2^{-n(p-1)}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{\alpha-\frac{1}{p}}\mu_k\log(k+1)\right)^p < \infty.\end{aligned}$$

Therefore, for any $\varepsilon > 0$ there exists a $0 < r_0 < 1$ such that

$$\int_{r_0}^1(1-r)^{p-2}\left(\sum_{n=0}^{\infty}(n+1)^{\alpha-\frac{1}{p}}\mu_k\log(k+1)r^n\right)^p dr < \varepsilon. \tag{1}$$

It is clear that

$$\begin{aligned}\|\mathcal{C}_{\mu,\alpha}(f_k)\|_{B_p}^p &= \left(\int_{|z|\leq r_0} + \int_{r_0<|z|<1}\right)|\mathcal{C}_{\mu,\alpha}(f_k)'(z)|^p(1-|z|)^{p-2}dA(z)\\ &:= J_{1,k} + J_{2,k}.\end{aligned}$$

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By the integral representation of $\mathcal{C}_{\mu,\alpha}$ we get

$$\mathcal{C}_\mu(f_k)'(z) = \int_0^1 \frac{tf_k'(tz)}{(1-tz)^\alpha} d\mu(t) + \int_0^1 \frac{\alpha t f_k(tz)}{(1-tz)^{\alpha+1}} d\mu(t). \tag{2}$$

Cauchy integral theorem implies that the sequence $\{f_k'\}_{k=1}^\infty$ is also converge to 0 uniformly on every compact subset of $\mathbb{D}$. Thus, for $|z| \le r_0$ we have that

$$\begin{aligned}
|\mathcal{C}_{\mu,\alpha}(f_k)'(z)| &\lesssim \int_0^1 \frac{|f_k'(tz)|}{|1-tz|^\alpha} + \frac{|f_k(tz)|}{|1-tz|^{\alpha+1}} d\mu(t) \\
&\lesssim \sup_{|w|<r_0} (|f_k(w)| + |f_k'(w)|) \int_0^1 \frac{1}{(1-tr_0)^{\alpha+1}} d\mu(t) \\
&\lesssim \sup_{|w|<r_0} (|f_k(w)| + |f_k'(w)|).
\end{aligned}$$

It follows that

$$J_{1,k} \to 0, \ (k \to \infty).$$

Next, we estimate $J_{2,k}$.

Since $X \subset \mathcal{B}$, we have

$$|f_k(z)| \lesssim \log \frac{e}{1-|z|} \ \text{ and } \ |f_k'(z)| \lesssim \frac{1}{1-|z|} \ \text{ for all } \ k \ge 1, z \in \mathbb{D}. \tag{3}$$

By (2) and (3), Minkowski inequity, Lemma 2.4 we get

$$\begin{aligned}
M_p(r, \mathcal{C}_{\mu,\alpha}(f_k)') &= \left\{ \int_0^{2\pi} \left| \int_0^1 \frac{tf_k'(tre^{i\theta})}{(1-tre^{i\theta})^\alpha} + \frac{tf_k(tre^{i\theta})}{(1-tre^{i\theta})^{\alpha+1}} d\mu(t) \right|^p d\theta \right\}^{\frac{1}{p}} \\
&\lesssim \left\{ \int_0^{2\pi} \left(\int_0^1 \frac{1}{(1-tr)|1-tre^{i\theta}|^\alpha} d\mu(t) \right)^p d\theta \right\}^{\frac{1}{p}} \\
&\quad + \left\{ \int_0^{2\pi} \left(\int_0^1 \frac{\log \frac{e}{1-tr}}{|1-tre^{i\theta}|^{\alpha+1}} d\mu(t) \right)^p d\theta \right\}^{\frac{1}{p}} \\
&\lesssim \int_0^1 \frac{1}{1-tr} \left(\int_0^{2\pi} \frac{d\theta}{|1-tre^{i\theta}|^{p\alpha}} \right)^{\frac{1}{p}} d\mu(t) \\
&\quad + \int_0^1 \log \frac{e}{1-tr} \left(\int_0^{2\pi} \frac{d\theta}{|1-tre^{i\theta}|^{p(\alpha+1)}} \right)^{\frac{1}{p}} d\mu(t) \\
&\lesssim \int_0^1 H(t,r) d\mu(t),
\end{aligned}$$

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where

$$H(t,r)=\begin{cases}\dfrac{\log\frac{e}{1-tr}}{(1-tr)^{\alpha+1-\frac{1}{p}}}, & \text{if } p>\dfrac{1}{\alpha},\\[2ex] \dfrac{\log\frac{e}{1-tr}}{1-tr}, & \text{if } p=\dfrac{1}{\alpha}.\end{cases}$$

Lemma 2.3 yields that

$$M_p(r,\mathcal{C}_{\mu,\alpha}(f_k)')\lesssim\int_0^1 H(t,r)d\mu(t)\asymp\sum_{n=0}^{\infty}(n+1)^{\alpha-\frac{1}{p}}\mu_n\log(n+1)r^n.$$

This together with (1) implies

$$\begin{aligned}J_{2,k}&=\int_{r_0<|z|<1}|\mathcal{C}_{\mu,\alpha}(f_k)'(z)|^p(1-|z|)^{p-2}dA(z)\\&\lesssim\int_{r_0}^1(1-r)^{p-2}M_p^p(r,\mathcal{C}_{\mu,\alpha}(f_k)')dr\\&\lesssim\int_{r_0}^1(1-r)^{p-2}\left(\sum_{n=0}^{\infty}(n+1)^{\alpha-\frac{1}{p}}\mu_n\log(n+1)r^n\right)^p dr\\&\lesssim\varepsilon.\end{aligned}$$

Consequently,

$$\lim_{k\to\infty}\|\mathcal{C}_{\mu,\alpha}(f_k)\|_{B_p}=0.$$

Case 2: $p=1$.

When $p=1$, Lemma 2.3 shows that the condition $\sum_{n=0}^{\infty}(n+1)^{\alpha-1}\mu_n\log(n+2)<\infty$ is equivalent to $\int_0^1\frac{\log\frac{e}{1-t}}{(1-t)^\alpha}d\mu(t)<\infty$. Hence, for any $\varepsilon>0$ there exists a $0<t_0<1$ such that

$$\int_{t_0}^1\frac{\log\frac{e}{1-t}}{(1-t)^\alpha}d\mu(t)<\varepsilon. \tag{4}$$

By the integral representation of $\mathcal{C}_{\mu,\alpha}$ we have

$$\mathcal{C}_{\mu,\alpha}(f)''(z)=\int_0^1\left(\frac{t^2f''(tz)}{(1-tz)^\alpha}+\frac{2\alpha t^2f'(tz)}{(1-tz)^{\alpha+1}}+\frac{\alpha(\alpha+1)t^2f(tz)}{(1-tz)^{\alpha+2}}\right)d\mu(t). \tag{5}$$

For $0<r<1$, we have

$$\begin{aligned}M_1(r,\mathcal{C}_{\mu,\alpha}(f_k)'')\lesssim&\sup_{|w|\le t_0}(|f_k''(w)|+|f_k'(w)|+|f_k(w)|)\int_0^{t_0}\frac{d\mu(t)}{(1-t_0r)^{\alpha+2}}\\&+\int_0^{2\pi}\int_{t_0}^1\frac{|f_k''(tz)|}{|1-tre^{i\theta}|^\alpha}+\frac{|f_k'(tz)|}{|1-tre^{i\theta}|^{\alpha+1}}+\frac{|f_k(tz)|}{|1-tre^{i\theta}|^{\alpha+2}}d\mu(t)d\theta.\end{aligned}$$

Since $\{f_k\}\subset X\subset\mathcal{B}$, we see that

$$|f_k''(z)|\lesssim\frac{1}{(1-|z|)^2}\ \text{ for all }\ k\ge1. \tag{6}$$

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The assumption of p means that $\alpha \geq 1$. By Fubini's theorem, (3), (6), and Lemma 2.4, we have

$$\begin{aligned}
&\int_0^{2\pi}\int_{t_0}^1 \frac{|f_k''(tre^{i\theta})|}{|1-tre^{i\theta}|^{\alpha}} + \frac{|f_k'(tre^{i\theta})|}{|1-tre^{i\theta}|^{\alpha+1}} + \frac{|f_k(tre^{i\theta})|}{|1-tre^{i\theta}|^{\alpha+2}}\, d\mu(t)d\theta \\
&\lesssim \int_{t_0}^1\int_0^{2\pi}\left(\frac{1}{(1-tr)^2|1-tre^{i\theta}|^{\alpha}} + \frac{1}{(1-tr)|1-tre^{i\theta}|^{\alpha+1}} + \frac{\log\frac{e}{1-tr}}{|1-tre^{i\theta}|^{\alpha+2}}\right) d\theta d\mu(t) \\
&\lesssim \int_{t_0}^1 \frac{\log\frac{e}{1-tr}}{(1-tr)^{\alpha+1}} d\mu(t).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{t_0<|z|<1} |\mathcal{C}_{\mu,\alpha}(f_k)''(z)|dA(z) \\
&= 2\int_{t_0}^1 M_1(r, \mathcal{C}_{\mu,\alpha}(f_k)'')rdr \\
&\lesssim \sup_{|w|\leq t_0}(|f_k''(w)| + |f_k'(w)| + |f_k(w)|) + \int_{t_0}^1\int_{t_0}^1 \frac{\log\frac{e}{1-tr}}{(1-tr)^{\alpha+1}} d\mu(t)dr \\
&\lesssim \sup_{|w|\leq t_0}(|f_k''(w)| + |f_k'(w)| + |f_k(w)|) + \int_{t_0}^1 \log\frac{e}{1-t}\int_0^1 \frac{dr}{(1-tr)^{\alpha+1}} d\mu(t) \\
&\lesssim \sup_{|w|\leq t_0}(|f_k''(w)| + |f_k'(w)| + |f_k(w)|) + \int_{t_0}^1 \frac{\log\frac{e}{1-t}}{(1-t)^{\alpha}} d\mu(t) \\
&\lesssim \sup_{|w|\leq t_0}(|f_k''(w)| + |f_k'(w)| + |f_k(w)|) + \varepsilon.
\end{aligned}$$

The uniform convergence of $\{f_k\}$ on compact subsets of $\mathbb{D}$ implies that

$$\int_{|z|\leq t_0} |\mathcal{C}_{\mu,\alpha}(f_k)''(z)|dA(z) \lesssim \sup_{|w|\leq t_0}(|f_k''(w)| + |f_k'(w)| + |f_k(w)|) \to 0, \quad \text{as } k \to 0.$$

Since $f_k(0) = 0$, so we have that $|\mathcal{C}_{\mu,\alpha}(f_k)(0)| + |\mathcal{C}_{\mu,\alpha}(f_k)'(0)| \asymp |f_k'(0)| \to 0$ as $k \to \infty$.

Therefore, we deduce that

$$\lim_{k\to\infty} ||\mathcal{C}_{\mu,\alpha}(f_k)||_{B_1} = 0.$$

Thus, the operator $\mathcal{C}_{\mu,\alpha}$ is compact from X to B_1. ∎

Conflicts of interest The authors declare that there is no conflict of interest.

Availability of data and materials Data sharing not applicable to this article as no datasets were generated or analysed during the current study: the article describes entirely theoretical research.

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