SOLUTION OF A MOMENT PROBLEM FOR BOUNDED FUNCTIONS

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1. Given a sequence of variables $a_1, a_2, \ldots$, we define a sequence of polynomials $B_1, B_2, \ldots$ by

$$B_1 = -a_1, \tag{1}$$

$$(r+1)B_{r+1} = (-1)^{r+1}[a_{r+1} - B_1 a_r + B_2 a_{r-1} - \ldots + (-1)^r B_r a_1] \quad (r = 1, 2, \ldots). \tag{2}$$

For every positive integer n, B_n is a polynomial in $a_1, \ldots, a_n$, with rational coefficients. We write $B_n \equiv B_n(a_1, \ldots, a_n)$ to indicate the variables on which B_n depends. From the polynomials B_n, we form a new sequence of polynomials D_n in accordance with the following definitions. We write

$$D_0 = 1, \quad D_1 = B_1, \tag{3}$$

and, if $2m$ is an even integer, we define

$$D_{2m} = \begin{vmatrix} 1 & B_1 & B_2 & \ldots & B_m \\ B_1 & B_2 & \ldots & \ldots & B_{m+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ B_m & B_{m+1} & \ldots & \ldots & B_{2m} \end{vmatrix}, \tag{4}$$

while if $2m+1$ is an odd integer, we define

$$D_{2m+1} = \begin{vmatrix} B_1 & B_2 & \ldots & B_{m+1} \\ B_2 & B_3 & \ldots & B_{m+2} \\ \ldots & \ldots & \ldots & \ldots \\ B_{m+1} & \ldots & \ldots & B_{2m+1} \end{vmatrix}. \tag{5}$$

Thus, for every positive integer n, D_n is a polynomial in $a_1, \ldots, a_n$, with rational coefficients. We write $D_n \equiv D_n(a_1, \ldots, a_n)$ to indicate the variables on which D_n depends.

Given $k+1$ numbers $c_0, c_1, \ldots, c_k$, we form the two sets of numbers

$$\alpha_r = \tfrac{1}{2}(1 - rc_{r-1}), \tag{6}$$

$$\beta_r = \tfrac{1}{2}(1 + rc_{r-1}), \tag{7}$$

for $r = 1, 2, \ldots, k+1$. The object of this paper is to prove the following theorem:

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THEOREM A. *Given the $k+1$ numbers $c_0, \ldots, c_k$, it is necessary and sufficient in order that there shall exist a function $f(x)$ in $(0, 1)$ which satisfies*

$$\text{Max}\,|f(x)| < 1, \tag{8}$$

$$c_\nu = \int_0^1 f(x)\,x^\nu dx \quad (\nu = 0, 1, \ldots, k), \tag{9}$$

that, for $n = 1, 2, \ldots, k+1$,

$$D_n(\alpha_1, \ldots, \alpha_n) > 0, \quad D_n(\beta_1, \ldots, \beta_n) > 0 \quad n \not\equiv 1 \pmod 4, \tag{10}$$

$$D_n(\alpha_1, \ldots, \alpha_n) < 0, \quad D_n(\beta_1, \ldots, \beta_n) < 0 \quad n \equiv 1 \pmod 4. \tag{11}$$

2. Let ν be a positive integer. Let $y_1, \ldots, y_{2\nu}$ be 2ν numbers. We write

$$s_\mu = y_1^\mu + y_3^\mu + \ldots + y_{2\nu-1}^\mu \quad (\mu = 1, 2, \ldots), \tag{12}$$

and we denote the ν elementary symmetric functions of the y's with odd suffixes by $p_1, \ldots, p_\nu$; so that

$$p_1 = y_1 + y_3 + \ldots + y_{2\nu-1}; \quad p_2 = \Sigma y_1 y_3; \quad \ldots; \quad p_\nu = y_1 y_3 \ldots y_{2\nu-1}. \tag{13}$$

Similarly, we write

$$\sigma_\mu = y_2^\mu + y_4^\mu + \ldots + y_{2\nu}^\mu \quad (\mu = 1, 2, \ldots), \tag{14}$$

and we denote the ν elementary symmetric functions of the y's with even suffixes by $\pi_1, \ldots, \pi_\nu$; so that

$$\pi_1 = y_2 + y_4 + \ldots + y_{2\nu}; \quad \pi_2 = \Sigma y_2 y_4; \quad \ldots; \quad \pi_\nu = y_2 y_4 \ldots y_{2\nu}. \tag{15}$$

We have the following result:

LEMMA 1. *If*

$$s_r = a_r + \sigma_r \quad (r = 1, \ldots, \nu), \tag{16}$$

then

$$\pi_1 = p_1 + B_1, \tag{17}$$

$$\pi_r = p_r + \sum_{j=1}^{r-1} B_j p_{r-j} + B_r \quad (r = 2, \ldots, \nu), \tag{18}$$

where $B_j \equiv B_j(a_1, \ldots, a_j)$ is defined as in § 1.

The relation (17) is (16) with $r = 1$. The proof may be completed by induction, using Newton's formulae.

3. Suppose that we have n numbers $y_1, \ldots, y_n$, and that we define $n+1$ numbers $a_1, \ldots, a_{n+1}$ by

$$y_1^\mu - y_2^\mu + y_3^\mu - \ldots + (-1)^{n-1} y_n^\mu = a_\mu \quad (\mu = 1, \ldots, n+1). \tag{19}$$

LEMMA 2. *If* (19) *holds, then, with the notation of* § 1,

$$D_{n+1}(a_1, \ldots, a_{n+1}) = 0.$$

Let n be an even integer, say $n = 2m$. We can extend the set of n numbers $y_1, \ldots, y_n$ to a set of $2\nu = 2n+2$ numbers by the definition $y_i = 0$ for $i = n+1, \ldots, 2\nu$.

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Then the $\nu = n+1$ equations (19) are the same as (16). By Lemma 1, we have (17), (18). But

$$p_{m+1} = p_{m+2} = \ldots = p_{n+1} = 0,$$
$$\pi_{m+1} = \pi_{m+2} = \ldots = \pi_{n+1} = 0.$$

Hence, the last $m+1$ equations (18) become

$$0 = p_m B_r + p_{m-1} B_{r+1} + \ldots + p_1 B_{m+r-1} + B_{m+r} \quad (r = 1, \ldots, m+1).$$

On multiplying these equations by the co-factors of the elements in the last column of D_{2m+1} (cf. (5) § 1), and adding, we obtain $D_{2m+1} = 0$.

Let n be an odd integer, say $n = 2m-1$. We extend the set of y's to a set of $2\nu = 2n+2$ numbers as before. We again have (17), (18), but now

$$p_{m+1} = p_{m+2} = \ldots = p_{n+1} = 0,$$
$$\pi_m = \pi_{m+1} = \ \ldots \ = \ldots = \pi_{n+1} = 0.$$

Hence the last $m+1$ equations (17), (18), become

$$0 = p_m \quad + p_{m-1} B_1 + \ldots + p_1 B_{m-1} \quad + B_m,$$
$$0 = p_m B_r \qquad\qquad + \ldots + p_1 B_{m+r-1} + B_{m+r} \quad (r = 1, \ldots, m).$$

As before, we obtain $D_{2m} = 0$.

4. LEMMA 3. *Given $k+1$ numbers $c_0, \ldots, c_k$, it is necessary and sufficient in order that there shall exist a function $f(x)$ such that*

$$|f(x)| \leqslant M \quad (0 \leqslant x \leqslant 1), \tag{20}$$

$$c_\nu = \int_0^1 f(x)\, x^\nu dx \quad (\nu = 0, \ldots, k), \tag{21}$$

that, for every polynomial $T_k(x) = \lambda_0 + \lambda_1 x + \ldots + \lambda_k x^k$ (22)

of degree not greater than k,

$$\left| \sum_{j=0}^{k} c_j \lambda_j \right| \leqslant M \int_0^1 |T_k(x)|\, dx. \tag{23}$$

For the proof of a more general theorem, due to F. Riesz, see the author's paper, *Proc. London Math. Soc.* (2), 37 (1933), 33–62 (34).

LEMMA 4. *Let $c_0, \ldots, c_k$ be $k+1$ numbers of which the first k satisfy the following condition:*

(A) *For every polynomial $T_{k-1}(x)$ (cf. (22)) of degree not greater than $k-1$, we have*

$$\left| \sum_{j=0}^{k-1} c_j \lambda_j \right| < \int_0^1 |T_{k-1}(x)|\, dx.$$

In order that, for every polynomial $T_k(x)$ of degree not greater than k,

$$\left| \sum_{0}^{k} c_j \lambda_j \right| < \int_0^1 |T_k(x)|\, dx, \tag{24}$$

it is necessary and sufficient that (24) shall hold for all polynomials of the form

$$(x - x_1) \ldots (x - x_k) \quad (0 < x_1 < \ldots < x_k < 1). \tag{25}$$

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The condition is clearly necessary. To prove that it is sufficient, suppose that

$$\text{Max}\left|\sum_0^k c_j\lambda_j\right| \Big/ \int_0^1 |T_k(x)|\,dx = M \geqslant 1, \tag{26}$$

where the maximum is taken for all polynomials $T_k(x)$ of degree not greater than k. Then this maximum is attained for some polynomial $P(x)$; and, by condition (A), $P(x)$ is of degree k. We thus have

$$P(x) = \mu_0 + \mu_1 x + \ldots + \mu_\nu x^\nu \quad (\mu_\nu \neq 0),$$

and

$$\sum_0^k c_j\mu_j = \epsilon M \int_0^1 |P(x)|\,dx, \tag{27}$$

where $\epsilon = 1$ or -1 according as the first member (which cannot be zero) is positive or negative. By Lemma 3, it follows from (26) that there is a function f which satisfies (20), (21). Then (27) is

$$\int_0^1 fP\,dx = \int_0^1 \epsilon M \operatorname{sgn} P\,P\,dx,$$

i.e.

$$\int_0^1 [\epsilon M \operatorname{sgn} P - f]\,P\,dx = 0. \tag{28}$$

Now almost everywhere in the set at which $\epsilon M \operatorname{sgn} P - f$ is not zero, it is by (20) of the same sign as $\epsilon M \operatorname{sgn} P$. Hence, by (28),

$$\epsilon M \operatorname{sgn} P = f \tag{29}$$

almost everywhere, and we may suppose f so chosen that this equation holds everywhere.

Since $P(x)$ is of degree k, the number σ of points in the open interval (0, 1) at which it changes sign is not greater than k. Let these points be $x_1, \ldots, x_\sigma$ in increasing order of magnitude, so that $0 < x_1 < \ldots < x_\sigma < 1$. Consider the polynomial

$$\Pi = \prod_{i=1}^{\sigma} (x - x_i) = \lambda_0 + \lambda_1 x + \ldots + \lambda_\sigma x^\sigma.$$

This polynomial is, except for a finite set of points, either of the same sign as P everywhere in (0, 1), or of the opposite sign everywhere in (0, 1). Hence, by (29), $f = \epsilon' M \operatorname{sgn} \Pi$, where $\epsilon' = \operatorname{sgn} \epsilon$ in the first case and $\epsilon' = -\operatorname{sgn} \epsilon$ in the second. Thus

$$\sum_0^\sigma c_j\lambda_j = \int_0^1 f\Pi\,dx = \epsilon' M \int_0^1 |\Pi|\,dx,$$

and

$$\left|\sum_0^\sigma c_j\lambda_j\right| = M\int_0^1 |\Pi|\,dx \geqslant \int_0^1 |\Pi|\,dx, \tag{30}$$

since $M \geqslant 1$. By condition (A), Π is of degree k. Hence Π is of the form (25) of the enunciation. If the condition of the lemma is satisfied, then (30) cannot hold. But (30) is a consequence of (26). Hence $M < 1$. This proves the sufficiency.

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5. Let n be a positive integer. We denote the open set of points $X \equiv (x_1, \ldots, x_n)$, where $0 < x_1 < \ldots < x_n < 1$, by O; and the closed set of points X, where

$$0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant 1,$$

by G. Let

$$T(x) = \prod_{i=1}^{n} (x - x_i) = x^n - P_1 x^{n-1} + P_2 x^{n-2} - \ldots + (-1)^n P_n \quad (X \subset G).$$

Given a function $\phi(x)$, defined and integrable in $(0, 1)$, we write

$$F(\phi) = \int_0^1 |\phi| \, dx. \tag{31}$$

Suppose that we have $n+1$ numbers $c_0, \ldots, c_n$, of which the first n satisfy the following condition:

(B) *For every polynomial $T_{n-1}(x)$ of degree not greater than $n-1$,*

$$\left| \sum_0^{n-1} c_j \lambda_j \right| < F(T_{n-1}).$$

This is condition (A) of Lemma 4, with n instead of k. We write

$$L = c_n - P_1 c_{n-1} + P_2 c_{n-2} - \ldots + (-1)^n P_n c_0. \tag{32}$$

LEMMA 5. *Given $n+1$ numbers $c_0, \ldots, c_n$, of which the first n satisfy condition (B), it is necessary and sufficient in order that*

$$\left| \sum_0^n c_j \lambda_j \right| < \int_0^1 |T_n(x)| \, dx \tag{33}$$

for every polynomial of degree not greater than n that

$$\operatorname*{Max}_{X \subset O} \frac{|L|}{F(T)} < 1. \tag{34}$$

If this condition is not satisfied, then

$$\operatorname*{Max}_{X \subset O} \frac{|L|}{F(T)} \tag{35}$$

is attained in O.

Write $N = |L|/F(T)$. If (34) holds, then $N < 1$ for every $X \subset O$. By Lemma 4 with $k = n$, (33) holds for all polynomials of degree not greater than n. If (34) does not hold, then $N \geqslant 1$ for some $X \subset G$, so that (33) does not hold for all polynomials of degree not greater than n. This proves the first part.

If (34) does not hold, then by the remark just made, we have (26) with $k = n$. By the proof of Lemma 4, $N = M$ for some $X \subset O$. But by the definition of M, $N \leqslant M$ for all $X \subset G$. This proves the last part.

LEMMA 6. *Given n numbers $c_0, \ldots, c_{n-1}$ which satisfy the condition (B), it is possible to find c_n so that*

$$\operatorname*{Max}_{X \subset O} \frac{|L|}{F(T)} = \operatorname*{Max}_{X \subset O} \frac{L}{F(T)} = 1. \tag{36}$$

It is also possible to find c_n so that

$$\operatorname*{Max}_{X \subset O} \frac{|L|}{F(T)} = \operatorname*{Max}_{X \subset O} \frac{-L}{F(T)} = 1. \tag{37}$$

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We prove that (36) holds for some c_n. The second part is proved similarly. By Lemma 3, condition (B) implies the existence of a function ϕ, such that

$$|\phi| \leqslant \theta < 1 \quad (0 \leqslant x \leqslant 1),$$

$$c_\nu = \int_0^1 \phi x^\nu dx \quad (\nu = 0, \ldots, n-1).$$

If we give c_n the value

$$\int_0^1 \phi x^n dx, \tag{38}$$

we have, by Lemma 3, Max $L/F(T) \leqslant \theta$. On the other hand, for sufficiently large positive c_n, Max $L/F(T) > 1$. Further, the maximum in question is a continuous function of c_n. Hence there is a value $c_n(1)$ of c_n for which the maximum is equal to 1; i.e. writing $S = L/F(T)$, we have

$$\text{Max } S = 1 \quad (c_n = c_n(1)). \tag{39}$$

At least one of the equations

$$\text{Max } |S| = \text{Max } S, \tag{40}$$

$$\text{Max } |S| = \text{Max} - S, \tag{41}$$

must be satisfied. If (40) holds, then (39), (40) prove (36). If (40) does not hold, then, in the first place, the first member of (40) exceeds the second member, so that by (39)

$$\text{Max } |S| > 1 \quad (c_n = c_n(1)), \tag{42}$$

and, in the second place, (41) holds, so that by (42),

$$\text{Min } S < -1 \quad (c_n = c_n(1)). \tag{43}$$

For $c_n > c_n(1)$, (39) gives Max $S > 1$. For $c_n < c_n(1)$, (43) gives Min $S < -1$. Thus for $c_n \neq c_n(1)$,

$$\text{Max } |S| > 1, \tag{44}$$

and this holds for $c_n = c_n(1)$ by (42). Thus (44) holds for all c_n. This contradicts the fact that, if we give c_n the value (38), then, by Lemma 3, Max $|S| \leqslant \theta < 1$.

6. Let $c_0, \ldots, c_n$ be $n+1$ numbers of which the first n satisfy the condition (B), so that Lemmas 5, 6 are applicable. Write

$$H = F(T) - L, \quad K = F(T) + L. \tag{45}$$

We have, by continuity,

$$\underset{X \subset G}{\text{Min}}\, H = \underset{X \subset O}{\text{Min}}\, H = \mu, \tag{46}$$

say.

We assert that μ is attained in O. In the first place, since $-c_n$ occurs in H merely as an additive constant, the points of G at which the value μ is attained by H do not vary with c_n (though of course μ does). Suppose then that c_n is chosen so that (36) holds. Then (34) does not hold. By Lemma 5, (35) is attained in O. By (36), Max $L/F(T)$ is attained in O, say at X_0. By the second equation in (36), $H \geqslant 0$, $\mu = 0$, and this value is attained in O, namely, at X_0. This proves our assertion.

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It follows that the equations

$$\frac{\partial H}{\partial x_j}=0 \quad (j=1,\ldots,n) \tag{47}$$

have a solution in O. In order that μ, defined by (46), shall be positive, it is necessary and sufficient that H shall be positive at the (existent) solutions in O of (47). Similarly, in order that

$$\operatorname*{Min}_{X\subset G} K=\operatorname*{Min}_{X\subset O} K=\mu'$$

shall be positive, it is necessary and sufficient that K shall be positive at the (existent) solutions in O of

$$\frac{\partial K}{\partial x_j}=0 \quad (j=1,\ldots,n), \tag{48}$$

The two conditions, $\mu>0$, $\mu'>0$, are together equivalent to (34). We thus have by Lemma 5:

LEMMA 7. *Let $c_0,\ldots,c_n$ be $n+1$ numbers of which the first n satisfy condition (B). It is necessary and sufficient in order that* (33) *shall hold for all polynomials of degree not greater than n, that $H>0$ at the solutions in O of* (47), *and $K>0$ at the solutions in O of* (48).

7. Let $X\equiv(x_1,\ldots,x_n)$ denote a point of O. Given a function $\phi(x)$, defined and integrable in (0, 1), we write

$$A(\phi)=\int_{x_n}^{1}-\int_{x_{n-1}}^{x_n}+\ldots+(-1)^n\int_0^{x_1}\phi(x)\,dx.$$

In § 5 we defined the polynomial T. Manifestly, $F(T)=A(T)$, and it is easily seen that

$$\frac{\partial}{\partial x_j}F(T)=A\left(\frac{\partial T}{\partial x_j}\right).$$

Hence

$$\frac{\partial H}{\partial x_j}=A\left(\frac{\partial T}{\partial x_j}\right)-\frac{\partial L}{\partial x_j}=A\left(\frac{\partial T}{\partial x_j}\right)+c_{n-1}\frac{\partial P_1}{\partial x_j}-c_{n-2}\frac{\partial P_2}{\partial x_j}+\ldots-(-1)^n c_0\frac{\partial P_n}{\partial x_j}.$$

Since

$$\frac{\partial T}{\partial x_j}=-x^{n-1}\frac{\partial P_1}{\partial x_j}+x^{n-2}\frac{\partial P_2}{\partial x_j}-\ldots+(-1)^n\frac{\partial P_n}{\partial x_j},$$

the equations (47) may be written

$$\sum_{i=1}^{n}[c_{n-i}-A(x^{n-i})]\frac{\partial P_i}{\partial x_j}=0 \quad (j=1,\ldots,n). \tag{49}$$

We know that these equations have a solution with $0<x_1<\ldots<x_n<1$. But for such x_j, it may be shown that the determinant $\left|\frac{\partial P_i}{\partial x_j}\right|$ differs from zero. Hence, the solutions in O of (47) are the solutions in O of

$$c_\mu=A(x^\mu) \quad (\mu=0,\ldots,n-1). \tag{50}$$

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Now $$F(T) = A(T) = A(x^n) - P_1 A(x^{n-1}) + \ldots + (-1)^n P_n A(1)$$
$$= A(x^n) - P_1 c_{n-1} \quad + \ldots + (-1)^n P_n c_0$$
when (50) holds. Hence, $$H = F(T) - L = A(x^n) - c_n \tag{51}$$
when (50) holds. By the definition of the operation A,
$$A(x^\mu) = \left[1 - 2\sum_{i=0}^{n-1}(-1)^i x_{n-i}^{\mu+1}\right] \Big/ (\mu+1) \quad (\mu+1 \neq 0).$$
Thus the equations (50) take the form
$$x_n^\mu - x_{n-1}^\mu + x_{n-2}^\mu - \ldots = \tfrac{1}{2}(1 - \mu c_{\mu-1}) \quad (\mu = 1, \ldots, n), \tag{52}$$
and (51) becomes
$$H = \frac{1}{n+1}\left[\tfrac{1}{2}\{1 - (n+1)c_n\} - \{x_n^{n+1} - x_{n-1}^{n+1} + \ldots\}\right]. \tag{53}$$

8. We define α_r $(r = 1, \ldots, n+1)$ by (6), § 1. We write $y_j = x_{n+1-j}$ for $j = 1, \ldots, n$. The equations (52) are
$$y_1^\mu - y_2^\mu + y_3^\mu - \ldots = \alpha_\mu \quad (\mu = 1, \ldots, n).$$
We write $$a_s = \alpha_s \quad (s = 1, \ldots, n),$$
and $$a_{n+1} = x_n^{n+1} - x_{n-1}^{n+1} + \ldots = y_1^{n+1} - y_2^{n+1} + \ldots. \tag{54}$$
With these notations we have (19). By Lemma 2
$$D_{n+1}(\alpha_1, \ldots, \alpha_n, a_{n+1}) = 0. \tag{55}$$

Case 1. If $n = 2m$ is even, then (55) is
$$\begin{vmatrix} B_1 & \ldots & B_{m+1} \\ \ldots\ldots & & \ldots\ldots \\ B_{m+1} & \ldots & B_{2m+1} \end{vmatrix} = 0, \tag{56}$$
where $$B_i = B_i(\alpha_1, \ldots, \alpha_i) \quad (i = 1, \ldots, 2m),$$
$$B_{2m+1} = B_{2m+1}(\alpha_1, \ldots, \alpha_{2m}, a_{2m+1}).$$
Write $$B'_{2m+1} = B_{2m+1}(\alpha_1, \ldots, \alpha_{2m}, \alpha_{2m+1}).$$

The coefficient of B_{2m+1} in the determinant (56) is $D_{n-1}(\alpha_1, \ldots, \alpha_{n-1})$. Suppose that
$$(-1)^{n/2} D_{n-1}(\alpha_1, \ldots, \alpha_{n-1}) > 0. \tag{57}$$
Equation (56) is, on multiplying by $(-1)^{n/2}$,
$$(-1)^{n/2} D_{n-1} B_{2m+1} + (-1)^{n/2} C = 0, \tag{58}$$
where C denotes what the determinant in (56) becomes when B_{2m+1} is replaced by 0. Now by definition, ((2), §1),
$$B_{2m+1} = \frac{(-1)^{2m+1}}{2m+1}[a_{2m+1} - B_1 a_{2m} + \ldots]. \tag{59}$$
The condition $H > 0$ at the solutions in O of (47) is, by (53), (54),
$$\alpha_{n+1} > a_{n+1}. \tag{60}$$

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But B'_{2m+1} is obtained from B_{2m+1} by replacing a_{2m+1} by α_{2m+1}. Hence the condition (60) is, by (59), the same as

$$B'_{2m+1} < B_{2m+1}.$$

By (57), (58), this is the same as

$$(-1)^{n/2} D_{n-1} B'_{2m+1} + (-1)^{n/2} C < 0,$$

i.e. the same as $$(-1)^{(n+2)/2} D_{n+1}(\alpha_1, \ldots, \alpha_{n+1}) > 0.$$

Case 2. If $n = 2m-1$ is odd, then (55) is

$$\begin{vmatrix} 1 & B_1 & \ldots & B_m \\ \ldots & \ldots & \ldots & \ldots \\ B_m & \ldots & \ldots & B_{2m} \end{vmatrix} = 0, \tag{61}$$

where $$B_i = B_i(\alpha_1, \ldots, \alpha_i) \quad (i = 1, \ldots, 2m-1),$$

$$B_{2m} = B_{2m}(\alpha_1, \ldots, \alpha_{2m-1}, a_{2m}).$$

Write $$B'_{2m} = B_{2m}(\alpha_1, \ldots, \alpha_{2m-1}, \alpha_{2m}).$$

The coefficient of B_{2m} in the determinant in (61) is $D_{n-1}(\alpha_1, \ldots, \alpha_{n-1})$. Suppose that $$D_{n-1}(\alpha_1, \ldots, \alpha_{n-1}) > 0. \tag{62}$$

Equation (61) is $$D_{n-1} B_{2m} + E = 0, \tag{63}$$

where E denotes what the determinant in (61) becomes when B_{2m} is replaced by 0. Now

$$B_{2m} = \frac{1}{2m}[a_{2m} - B_1 a_{2m-1} + \ldots]. \tag{64}$$

The condition $H > 0$ at the solutions in O of (47) is, by (53), (54), the condition (60); and this, by (64), is the same as $B'_{2m} > B_{2m}$. By (62), (63), this is the same as

$$D_{n-1} B'_{2m} + E > 0,$$

i.e. the same as $$D_{n+1}(\alpha_1, \ldots, \alpha_{n+1}) > 0.$$

The treatment of K leads to similar results with β's (cf. (7), § 1) instead of α's. In virtue of Lemma 7, we have:

LEMMA 8. *Let $c_0, \ldots, c_n$ be $n+1$ numbers of which the first n satisfy the condition (B).*

(a) *If n is even, and*

$$(-1)^{n/2} D_{n-1}(\alpha_1, \ldots, \alpha_{n-1}) > 0, \quad (-1)^{n/2} D_{n-1}(\beta_1, \ldots, \beta_{n-1}) > 0,$$

then it is necessary and sufficient, in order that (33) *shall hold for all polynomials of degree not greater than n, that*

$$(-1)^{(n+2)/2} D_{n+1}(\alpha_1, \ldots, \alpha_{n+1}) > 0, \quad (-1)^{(n+2)/2} D_{n+1}(\beta_1, \ldots, \beta_{n+1}) > 0.$$

(b) *If n is odd, and*

$$D_{n-1}(\alpha_1, \ldots, \alpha_{n-1}) > 0, \quad D_{n-1}(\beta_1, \ldots, \beta_{n-1}) > 0,$$

a necessary and sufficient condition is

$$D_{n+1}(\alpha_1, \ldots, \alpha_{n+1}) > 0, \quad D_{n+1}(\beta_1, \ldots, \beta_{n+1}) > 0.$$

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9. We can now prove Theorem A. By Lemma 3 we must show that the condition of Theorem A is necessary and sufficient in order that (33) shall hold for all polynomials of degree not greater than k. The condition of Theorem A means that, in each set of $k+1$ expressions

$$D_1(\alpha_1), D_2(\alpha_1, \alpha_2), \ldots, D_{k+1}(\alpha_1, \ldots, \alpha_{k+1}), \tag{65}$$

$$D_1(\beta_1), D_2(\beta_1, \beta_2), \ldots, D_{k+1}(\beta_1, \ldots, \beta_{k+1}), \tag{66}$$

the D's with even suffixes are positive, while those with odd suffixes are alternately negative and positive.

For $k=0$, condition (33) is $|c_0|<1$; i.e. $1-c_0>0$, $1+c_0>0$; i.e. $\alpha_1>0$, $\beta_1>0$. Since $D_1(\alpha_1)=B_1(\alpha_1)=-\alpha_1$ by definition, the condition is $D_1(\alpha_1)<0$, $D_1(\beta_1)<0$. Thus Theorem A is true in this case, and we can establish it generally by induction.

Necessity. If (33) holds for all polynomials of degree not greater than k, then, *a fortiori*,

$$\left|\sum_0^{k-1} c_j\lambda_j\right| < \int_0^1 |T_{k-1}(x)|\,dx \tag{67}$$

for all polynomials of degree not greater than $k-1$. By the theorem for $k-1$, each of the expressions

$$D_1(\alpha_1), \ldots, D_k(\alpha_1, \ldots, \alpha_k), \tag{68}$$

$$D_1(\beta_1), \ldots, D_k(\beta_1, \ldots, \beta_k) \tag{69}$$

has the specified sign.

If k is even, take $k=n$ in Lemma 8 (a). The conditions of the lemma are satisfied; for condition (B) is now (67), and the two determinants D_{n-1} of the lemma are the two determinants D_{k-1}, which have the required signs. By Lemma 8 (a), the two determinants D_{k-1} have the required signs. If k is odd, take $k=n$ in Lemma 8 (b). (If $n=1$, we use the definition $D_0=1$, § 1, (3).)

Sufficiency. Suppose that (65), (66) have the signs specified by Theorem A; then (68), (69) have the signs specified by Theorem A for $k-1$. By Theorem A for $k-1$, $c_0, \ldots, c_{k-1}$ satisfy (67) for every polynomial of degree not greater than $k-1$. We now take $k=n$ in Lemma 8 (a) if k is even, in Lemma 8 (b) if k is odd. In both cases, the conditions of the lemma are satisfied, and we infer that (33) holds for all polynomials of degree not greater than k.

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