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*D*3-MODULES VERSUS *D*4-MODULES – APPLICATIONS TO QUIVERS

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Abstract. A module M is called a $D4$-module if, whenever A and B are submodules of M with $M = A \oplus B$ and $f : A \to B$ is a homomorphism with Imf a direct summand of B, then Ker$f$ is a direct summand of A. The class of $D4$-modules contains the class of $D3$-modules, and hence the class of semi-projective modules, and so the class of Rickart modules. In this paper we prove that, over a commutative Dedekind domain R, for an R-module M which is a direct sum of cyclic submodules, M is direct projective (equivalently, it is semi-projective) iff M is $D3$ iff M is $D4$. Also we prove that, over a prime PI-ring, for a divisible R-module X, X is direct projective (equivalently, it is Rickart) iff $X \oplus X$ is $D4$. We determine some $D3$-modules and $D4$-modules over a discrete valuation ring, as well. We give some relevant examples. We also provide several examples on $D3$-modules and $D4$-modules via quivers.

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1. Introduction. Throughout, rings R are associative with identity and modules are unitary right R-modules.

Let M be a module. M is called a $D4$-module if, $M = A \oplus B$ for submodules A and B of M, and $f : A \to B$ is a homomorphism with Imf a direct summand of B, then Ker$f$ is a direct summand of A (see [1] and [7]). From [7], we know that M is a $D4$-module if and only if for submodules A and B of M with $M = A \oplus B$ and any epimorphism $f : A \to B$, Ker$f$ is a direct summand of A. Let N be any module. We call M is *N-projective* if every homomorphism from M to a homomorphic image of N can be lifted to a homomorphism from M to N. M is called *quasi-projective* if M is M-projective. The module M is called *direct projective* [20] if, for every submodule A of M with M/A isomorphic to a direct summand of M, A is a direct summand of M. Direct projective modules are also called $D2$-modules [19]. Every quasi-projective module is direct projective. The module M is called $D3$ if, for every direct summands M_1 and M_2 of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is a direct summand of M. It was shown in [27] that every $D3$-module is a $D4$-module.

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The module M is called *semi-projective* provided for all endomorphisms α and β of M with $\beta(M) \subseteq \alpha(M)$ there exists an endomorphism γ of M such that $\beta = \alpha\gamma$ [**15**]. The module M is called *Rickart* if, the kernel of any endomorphism of M is a direct summand of M [**18**]. By [**15**, Lemma 2.1] and [**15**, Corollary 2.4], every semi-projective module is direct projective and every Rickart module is semi-projective. Therefore, we have the following implications:

$$\begin{array}{c} \text{quasi-projective} \Rightarrow \text{semi-projective} \Rightarrow \text{direct projective} \Rightarrow D3 \Rightarrow D4 \\ \Uparrow \\ \text{Rickart} \end{array}$$

Let R be any commutative integral domain with quotient field $F \neq R$. Then, $M_R = F \oplus R$ is semi-projective, but in general, it is not quasi-projective (see [**11**, p. 490]). The Prüfer p-group $\mathbb{Z}(p^\infty)$ is a $D3$-module which is not direct projective. Finally, there exist semi-projective modules which are not Rickart. Indeed, let S be a commutative integral domain and let I be a nonzero proper ideal of S. Consider the subring $R = \{(x, y) \in S \times S \mid x \equiv y(\text{mod}I)\}$ of $S \times S$. By [**16**, Example 2.20(i)], R_R is not Rickart, but it is clearly semi-projective. In the literature, there is no example of a direct projective module which is not semi-projective. In this paper, we provide some examples of direct projective modules which are not semi-projective. Note that examples of the $D4$-modules which are not $D3$ are not known.

The remainder of our work is organized as follows. In Section 2, we prove that, over a commutative Dedekind domain R, for an R-module M which is a direct sum of cyclic submodules the following are equivalent (Proposition 2.23):

(i) M is quasi-projective.
(ii) M is semi-projective.
(iii) M is direct projective.
(iv) M is $D3$.
(v) M is $D4$.

In [**7**, Proposition 2.11], it was proven that if $M \oplus M$ is a $D4$-module, then M is direct projective. In this paper, we prove that the converse is true for divisible modules defined over prime PI-rings. Namely we prove that, over a prime PI-ring R, for a divisible R-module X, the following are equivalent (Theorem 2.10):

(i) X is semi-projective.
(ii) X is direct projective.
(iii) X is semi-Hopfian.
(iv) X is nonsingular.
(v) $S = \text{End}(X)$ is von Neumann regular.
(vi) X is Rickart.
(vii) $X \oplus X$ is $D4$.

We also determine some $D3$-modules and $D4$-modules over a discrete valuation ring (Corollary 2.15, Theorem 2.18 and Proposition 2.20).

The focus of Section 3 is to construct several examples on $D3$-modules, $D4$-modules, quasi-projective modules, semi-projective modules, direct projective modules, semi-Hopfian modules, nonsingular modules, and Rickart modules via quivers. More precisely, we construct a module which is not $D4$ with exactly one submodule which is not a direct summand (Example 3.1). As we shall see, there is a quasi-projective and semi-Hopfian module which is not nonsingular (Example 3.3). We also show that there is a

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semi-projective module (of projective dimension one) which is neither Rickart nor quasi-projective (Example 3.20 and Remark 3.22). On the other hand, there are both injective and non-injective modules which are direct projective without being semi-projective (Examples 3.6 and 3.7). Moreover, only very special *D*4-modules may have a big free direct summand (Corollary 3.25).

We finally note that, as in [**24**], pictures of the form

$$\begin{matrix}1\\2\end{matrix}, \begin{matrix}2\\2\end{matrix}, \begin{matrix}1\\2\ 3\end{matrix}, \begin{matrix}1\ 2\\3\end{matrix}, \begin{matrix}1\ 3\ 5\ 7\\2\ 4\ 6\ 8\end{matrix}, \cdots$$

denote the composition series of indecomposable modules. Our convention for the composition of paths p, q is that pq stands for q *after* p whenever this composition is defined (because we are working with right modules). For more background on quivers, we refer to [**2**] and [**24**].

2. Modules over Dedekind domains. First, we exhibit in the following example a direct projective module which is not semi-projective. Note that some other observations in this direction (via quivers) will be given in Section 3.

EXAMPLE 2.1. Let $R = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ be the trivial extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Let S be a simple R-module. Consider an injective hull $M = E(S)$ of S. By [**4**, Section 3, p. 4765 Example 4], every submodule of M is fully invariant. It was shown in [**5**, Example 1.12] that the lattice of submodules of M is $\{0, S, K, L, N, M\}$, where K, L, and N are maximal submodules of M. Since R is a local ring, there exist isomorphisms $M/N \xrightarrow{\alpha_N} S$ and $M/L \xrightarrow{\alpha_L} S$. Suppose that M is a semi-projective module. Consider the following homomorphisms:

$$M \xrightarrow{\pi_N} M/N \xrightarrow{\alpha_N} S \xrightarrow{\mu} M \quad \text{and} \quad \mathrm{M} \xrightarrow{\pi_\mathrm{L}} \mathrm{M/L} \xrightarrow{\alpha_\mathrm{L}} \mathrm{S} \xrightarrow{\mu} \mathrm{M},$$

where π_N and π_L are the natural epimorphisms and μ is the inclusion map. Hence, $f = \mu\alpha_N\pi_N$ and $g = \mu\alpha_L\pi_L$ are two endomorphisms of M such that $f(M) = S = g(M)$. Since M is semi-projective, there exists an endomorphism γ of M with $f\gamma = g$. Note that $f\gamma(N) \subseteq f(N) = 0$ as N is a fully invariant submodule of M. This implies that $\mu\alpha_L\pi_L(N) = 0$ and hence $\pi_L(N) = 0$ since α_L is an isomorphism. This yields $N \subseteq L$, which is impossible. Thus, M is not semi-projective. On the other hand, note that $M/S = (K/S) \oplus (L/S)$ is not indecomposable. Therefore, M/S is not isomorphic to M. Moreover, it is clear that each of the modules M/K, M/L and M/N is not isomorphic to M. It follows that M is a direct projective module.

Our aim in this section is to investigate *D*3- and *D*4-modules over Dedekind domains. We begin with the following general results which will be useful to our work in this article.

LEMMA 2.2. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules $M_i (i \in I)$ such that* $\mathrm{Hom}_R(M_i, M_j) = 0$ *for all distinct $i, j \in I$. Then*

(i) *For every direct summand K of M, we have $K = \oplus_{i\in I}(K \cap M_i)$.*
(ii) *If A and B are two isomorphic direct summands of M, then $A \cap M_i \cong B \cap M_i$ for any $i \in I$.*

Proof. (i) Let K and N be two submodules of M such that $M = K \oplus N$. Let π be the endomorphism of M defined by $k + n \mapsto k$ for every $(k, n) \in K \times N$. By hypothesis,

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we have $\pi(M_i) \subseteq M_i$ for all $i \in I$. It is easily seen that $K \cap M_i = \pi(M_i)$ for all $i \in I$. Since $K = \pi(M) = \sum_{i\in I} \pi(M_i) = \oplus_{i\in I}\pi(M_i)$, it follows that $K = \oplus_{i\in I}(K \cap M_i)$.

(ii) Let A and B be two isomorphic direct summands of M. By (i), we have $A = \oplus_{i\in I}(A \cap M_i)$ and $B = \oplus_{i\in I}(B \cap M_i)$. Let $f : A \to B$ be an isomorphism. Then

$$f(A) = \oplus_{i\in I} f(A \cap M_i) = \oplus_{i\in I}(B \cap M_i).$$

Let A' be a submodule of M such that $M = A \oplus A'$. Let g be the endomorphism of M defined by $g(x + x') = f(x)$ for any $x \in A$ and $x' \in A'$. Note that $g(M_i) \subseteq M_i$ for all $i \in I$. Then $f(A \cap M_i) = g(A \cap M_i) \subseteq B \cap M_i$ for all $i \in I$. It follows that $f(A \cap M_i) = B \cap M_i$ for all $i \in I$. Thus, $A \cap M_i \cong B \cap M_i$ for all $i \in I$. □

The next lemma is taken from [7, Proposition 2.11(1)] and [19, Lemma 4.7].

LEMMA 2.3. *Any direct summand of a Di-module is again a Di-module* ($i \in \{3, 4\}$).

PROPOSITION 2.4. *Let a module $M = \oplus_{i\in I} M_i$ be a direct sum of submodules $M_i (i \in I)$ such that* $\mathrm{Hom}_R(M_i, M_j) = 0$ *for all distinct $i, j \in I$. Then, M is a $D3$-module (resp. $D4$-module) if and only if so is each $M_i (i \in I)$.*

Proof. The necessity follows from Lemma 2.3. Conversely, to show that M is a $D3$-module (resp. $D4$-module), let A and B be two direct summands of M such that $M = A + B$ (and $A \cong B$). Then, $A = \oplus_{i\in I}(A \cap M_i)$ and $B = \oplus_{i\in I}(B \cap M_i)$ (and $A \cap M_i \cong B \cap M_i$ for any $i \in I$) by Lemma 2.2. This yields $A \cap B = \oplus_{i\in I}((A \cap B) \cap M_i)$. On the other hand, we have $M = \oplus_{i\in I}((A \cap M_i) + (B \cap M_i))$. This implies that $M_i = (A \cap M_i) + (B \cap M_i)$ for all $i \in I$. Note that $A \cap M_i$ and $B \cap M_i$ are direct summands of M_i for all $i \in I$. Since each M_i is a $D3$-module (resp. $D4$-module), it follows that $(A \cap B) \cap M_i$ is a direct summand of M_i for any $i \in I$ (see [7, Theorem 2.2]). Hence, $\oplus_{i\in I}((A \cap B) \cap M_i)$ is a direct summand of M. That is, $A \cap B$ is a direct summand of M. Consequently, M is a $D3$-module (resp. $D4$-module) (see [7, Theorem 2.2]). □

REMARK 2.5. Proposition 2.4 is an answer to a question raised in [1, p. 4494].

The next lemma follows directly from the definition of $D4$-modules.

LEMMA 2.6. *Let M be a nonzero module and let N be a submodule of M which is not a direct summand of M. Then $M \oplus M/N$ cannot be a $D4$-module.*

PROPOSITION 2.7. *Let R be a commutative ring. Let I_1 and I_2 be two proper ideals of R such that $I_1 \subseteq I_2$ and R/I_1 is an indecomposable R-module. Then, the following are equivalent for the module $M = R/I_1 \oplus R/I_2$:*

(i) *M is a $D3$-module;*
(ii) *M is a $D4$-module;*
(iii) $I_1 = I_2$.

Proof. (i) ⇒ (ii) This is immediate.

(ii) ⇒ (iii) This follows from Lemma 2.6 and the fact that $R/I_2 \cong \frac{R/I_1}{I_2/I_1}$.

(iii) ⇒ (i) Since R is commutative, R/I_1 is quasi-projective. Hence, $M = R/I_1 \oplus R/I_1$ is quasi-projective. Therefore, M is a $D3$-module. □

REMARK 2.8. Let R be a commutative ring. Let M be a $D3$-module over R such that $Mr = M$ for some $0 \neq r \in R$ and $ann_M(r)$ is not a direct summand of M. For example, we can take any divisible torsion indecomposable module over a commutative domain (e.g.,

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M can be the $\mathbb{Z}$-module $M = \mathbb{Z}(p^\infty)$ for some prime integer p). Then $M \oplus M$ is not a $D4$-module. Indeed, let f be the endomorphism of M defined by $f(x) = xr$ for all $x \in M$. It is clear that $\mathrm{Im}f = M$ and $\mathrm{Ker}f = ann_M(r)$ is not a direct summand of M.

Let R be a commutative Dedekind domain with quotient field Q and let M be an R-module. An element $x \in M$ is said to be a *torsion* element if $xr = 0$ for some $0 \neq r \in R$. Let $T(M)$ denote the set of all torsion elements in M. It is well known that $T(M)$ is a submodule of M which is called the *torsion submodule* of M. The module M is said to be a *torsion module* if $T(M) = M$. If $T(M) = 0$, then M is called *torsion-free*. Let $\mathfrak{p}$ be a nonzero prime ideal of R. The set $T_\mathfrak{p}(M) = \{x \in M \mid x\mathfrak{p}^k = 0$ for some nonnegative integer $k\}$ is a submodule of M which is called the $\mathfrak{p}$-*primary component* of M. The $\mathfrak{p}$-primary component of the torsion R-module Q/R will be denoted by $R(\mathfrak{p}^\infty)$. It is well known that every torsion R-module is a direct sum of its $\mathfrak{p}$-primary components.

Now we are ready to give the following result which follows easily from Proposition 2.4.

COROLLARY 2.9. *Assume that R is a commutative Dedekind domain and let M be a torsion R-module. Then, M is a $D3$-module (resp. $D4$-module) if and only if so is every $\mathfrak{p}$-primary component of M (viewed as an R-module or as a module over the localization $R_\mathfrak{p}$).*

Recall that an element c of a ring R is called *regular* provided $cr \neq 0$ and $rc \neq 0$ for all nonzero $r \in R$. Following [**10**, p. 104], an R-module M is called *divisible* in case $M = Mc$ for every regular element c of R. A module M is called *semi-Hopfian* if the kernel of every epimorphism $\varphi : M \to M$ is a direct summand of M. An R-module M is said to be *nonsingular* if for every $0 \neq m \in M$, $mI \neq 0$ for any essential right ideal I of R ([**9**, p. 31]). A ring R is called a *prime* ring if whenever I and J are ideals of R with $IJ = 0$, then either $I = 0$ or $J = 0$. A *polynomial identity ring* (or *PI-ring* for short) is a ring which satisfies some monic polynomial identity.

Next, we investigate injective $D3$-modules (resp. $D4$-modules).

THEOREM 2.10. *Let R be a prime PI-ring. Then the following are equivalent for a divisible R-module M:*

(i) *M is a semi-projective module;*
(ii) *M is a direct projective module;*
(iii) *M is a semi-Hopfian module;*
(iv) *M is a nonsingular module;*
(v) *$S = \mathrm{End}(M)$ is a von Neumann regular ring;*
(vi) *M is a Rickart module;*
(vii) *$M \oplus M$ is a $D4$-module.*

Moreover, in this case, M is an injective module.

Proof. For the equivalences of (i)–(vi) see [**16**, Proposition 5.6].

(vii) $\Rightarrow$ (ii) follows by [**7**, Proposition 2.11(2)].

(iv) $\Rightarrow$ (vii) Note that $M \oplus M$ is a nonsingular divisible module since M is nonsingular and divisible. Using the equivalence (ii) $\Leftrightarrow$ (iv), it follows that $M \oplus M$ is a direct projective module. Therefore, $M \oplus M$ is a $D4$-module.

For the last part see [**15**, Proposition 2.19]. □

In the next result, we determine the structure of direct projective modules over Dedekind domains.

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COROLLARY 2.11. *Let R be a commutative Dedekind domain with quotient field Q and let M be an injective R-module. Then M is a direct projective module if and only if $M \cong Q^{(I)}$ for some index set I.*

Proof. Note that every commutative integral domain is a prime PI-ring. Moreover, an R-module M is nonsingular if and only if M is torsion-free. The result follows from Theorem 2.10 and [**14**, Theorem 7]. □

A module M is said to satisfy the *finite internal exchange property* if, for any direct summand N of M and any finite internal direct sum decomposition $M = \oplus_{i=1}^{n} M_i$, there exist submodules $M_i' \leq M_i$ $(i = 1, \ldots, n)$ such that $M = (\oplus_{i=1}^{n} M_i') \oplus N$.

A module M is said to have the *finite exchange property* if for any finite index set I, whenever $M \oplus N = \oplus_{i \in I} A_i$ for modules N and A_i, we have $M \oplus N = M \oplus (\oplus_{i \in I} B_i)$ for submodules $B_i \leq A_i$. It is well known that the finite exchange property implies the finite internal exchange property (see [**6**, 11.35]). Notice that quasi-injective modules have the finite internal exchange property (see [**19**, Theorem 1.21]). Also, every module which is a direct sum of a semisimple module and an injective module has the finite internal exchange property (see [**19**, Lemma 3.20]).

It was shown in [**7**, Proposition 2.23] that for a module M with the finite internal exchange property, M is a $D3$-module if and only if M is a $D4$-module.

REMARK 2.12. Let R be an artinian serial ring with Jacobson radical square zero. Then every R-module is a direct sum of an injective module and a semisimple module (see, for example, [**6**, 29.10]). Thus, every R-module has the finite internal exchange property. From the preceding note, it follows that every $D4$-module is a $D3$-module.

PROPOSITION 2.13. *Let R be a finite dimensional K-algebra over a field K and let M_R be a finite dimensional module. Then, M has the finite internal exchange property. Moreover, in this case, M is a $D3$-module if and only if M is a $D4$-module.*

Proof. By [**25**, Lemma 4.4 and Theorem 4.6], $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ with M_i indecomposable and $\mathrm{End}_R(M_i)$ a local ring. Hence, by [**6**, 11.9 and 12.2], M has the finite internal exchange property, as required. The last statement comes from [**7**, Proposition 2.23]. □

PROPOSITION 2.14. *Let R be any ring. Then the following statements are equivalent for a nonzero injective R-module M:*

(i) *M is a $D3$-module;*
(ii) *M is a $D4$-module;*

Moreover, if R is a commutative Dedekind domain with quotient field Q, then (i) and (ii) are equivalent to:

(iii) *$M \cong Q^{(I)}$ for some index set I, or M is a torsion module such that every nonzero $\mathfrak{p}$-primary component is isomorphic to $R(\mathfrak{p}^{\infty})$.*

Proof. (i) $\Leftrightarrow$ (ii) This follows from [**7**, Proposition 2.23] and the fact that injective modules have the finite internal exchange property.

(ii) $\Rightarrow$ (iii) It is well known that an R-module M is divisible if and only if M is an injective R-module if and only if M is a direct sum of copies of Q and $R(\mathfrak{p}^{\infty})$ for various nonzero prime ideals $\mathfrak{p}$. First, note that for every nonzero prime ideal $\mathfrak{p}$ of R, there exists a nonzero proper submodule L of Q such that $Q/L \cong R(\mathfrak{p}^{\infty})$. Moreover, Q is indecomposable. So, applying Lemma 2.6, we deduce that the module $Q \oplus R(\mathfrak{p}^{\infty})$ is not a $D4$-module. Now

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using Lemma 2.3, it follows that M is either torsion or torsion-free. If M is torsion-free, then $M \cong Q^{(I)}$ for some index set I by [**14**, Theorem 7]. Now assume that M is a torsion module. Hence, every $\mathfrak{p}$-primary component of M is a $D4$-module (see Corollary 2.4). By Lemma 2.3 and Remark 2.8, we conclude that for every nonzero prime ideal $\mathfrak{p}$ of R, we have $T_{\mathfrak{p}}(M) = 0$ or $T_{\mathfrak{p}}(M) \cong R(\mathfrak{p}^{\infty})$.

(iii) $\Rightarrow$ (i) This follows from Corollaries 2.9 and 2.11. □

The next result is a direct consequence of Proposition 2.14.

COROLLARY 2.15. *Let R be a discrete valuation ring with quotient filed Q. Then the following statements are equivalent for a nonzero injective R-module M:*

(i) *M is a $D3$-module;*
(ii) *M is a $D4$-module;*
(iii) *$M \cong Q/R$ or $M \cong Q^{(I)}$ for some index set I.*

Let R be a discrete valuation ring with maximal ideal $\mathfrak{m} = pR$ and let M be an R-module. Let x be a nonzero torsion element of M. Then $xp^k = 0$ for some positive integer k. If n is the smallest positive integer such that $xp^n = 0$, then p^n is called the *order* of the element x and it is denoted by $o(x)$. We also say that x is an *element of finite order*. If x is an element in M such that $xp^s \neq 0$ for every positive integer s, then x is called an *element of infinite order* and we write $o(x) = \infty$. We assume that the order of the zero element is equal to zero.

Let n be a nonnegative integer. An element x in M is said to have *height* n if x is in Mp^n but not in Mp^{n+1}. In this case, we write $h(x) = n$. If $x \in \cap_{k\geq 1} Mp^k$, then x is called an *element of infinite height* and we write $h(x) = \infty$.

Let N be a submodule of an R-module M. If x is an element in N, we may define two heights for x. When it is necessary to make a distinction, we will write $h_N(x)$ and $h_M(x)$ for the height of x in N and M, respectively. We always have $h_N(x) \leq h_M(x)$.

Recall that a module M over a commutative Dedekind domain is said to be *reduced* if 0 is the only divisible (injective) submodule of M. Next, we focus on the study of reduced $D3$- and $D4$-modules.

LEMMA 2.16. *Assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m} = pR$. Let M be a reduced R-module which is not torsion-free. If a is an element in M of order p and of finite height $n-1$ for some positive integer n, then bR is a direct summand of M for every $b \in M$ with $bp^{n-1} = a$.*

Proof. This follows from [**17**, Proof of Corollary 7.3]. □

The proof of the next lemma is taken from that of [**17**, Corollary 7.3] and [**13**, Lemma 8] but is given for completeness.

LEMMA 2.17. *Assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m} = pR$. Let M be a torsion R-module. Assume that all elements of order p in M are of infinite height. Then, M is a divisible module.*

Proof. By induction on the order, we prove that every element in M is of infinite height. Suppose that there exists $x \in M$ such that $h(x) = k < \infty$. By induction hypothesis, we have $h(xp) = \infty$. This implies that $xp = yp^{k+2}$ for some element $y \in M$. Put $z = yp^{k+1}$. Hence, $h(z) \geq k+1 > k$. It follows that $h(x-z) = k$ (see, e.g., [**17**, p. 59 Exercise 1]). This contradicts the fact that $(x-z)p = 0$. Therefore, all elements of M are of infinite height. Hence, M is a divisible module. □

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THEOREM 2.18. *Assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m} = pR$. Let M be a nonzero reduced R-module which is not torsion-free. Then the following are equivalent:*

(i) *M is a $D3$-module;*
(ii) *M is a $D4$-module;*
(iii) *$M \cong (R/\mathfrak{m}^n)^{(I)}$ for some positive integer n and an index set I.*

Proof. (i) $\Rightarrow$ (ii) This is clear.

(ii) $\Rightarrow$ (iii) Note that every direct summand of M is a $D4$-module by Lemma 2.3. From [17, Corollary 7.3], it follows that M has a cyclic direct summand v_0R such that $v_0R \cong R/\mathfrak{m}^n$ for some positive integer n. Hence, $M = v_0R \oplus N$ for some submodule N of M. Consider the submodule $K = Mp^n = Np^n$. Suppose that K is not torsion-free. Then, K contains an element of order p. Assume that K has an element of order p and of finite height. Therefore, there exists $u \in M$ such that $o(up^n) = p$ and $h_M(up^n) = k - 1 < \infty$, where k is a positive integer. So there exists $v \in M$ such that $up^n = vp^{k-1}$. By Lemma 2.16, vR is a direct summand of M. Moreover, $vR \cong R/\mathfrak{m}^k$ as $o(v) = p^k$. Note that $k - 1 \geq n$. Thus, $k \neq n$. By [12, Lemma 3.6], the sum $v_0R + vR$ is direct and it is a direct summand of M. Since M is a $D4$-module, $R/\mathfrak{m}^n \oplus R/\mathfrak{m}^k$ is a $D4$-module. Using Proposition 2.7, we obtain $k = n$; this is impossible. It follows that for every $x \in M$ with $o(xp^n) = p$, we have $h_M(xp^n) = \infty$. Now let $0 \neq y \in T(K)$ with $o(y) = p$. So $y = xp^n$ for some $x \in T(M)$. Let s be a nonnegative integer. So there exists $z \in M$ such that $xp^n = zp^{n+s}$. Thus $y = xp^n = (zp^n)p^s \in T(K)p^s$. This implies that $h_{T(K)}(y) = \infty$. It follows that for every $y \in T(K)$ with $o(y) = p$, we have $h_{T(K)}(y) = \infty$. By Lemma 2.17, $T(K)$ is a divisible module. But M is a reduced module. Then $T(K) = 0$, a contradiction. Consequently, K is torsion-free. This yields $T(M)p^n = 0$. Since $T(M)$ is a pure submodule of M, it follows that $T(M) \oplus L = M$ for some submodule L of M (see, e.g., [17, Theorem 7.2]). Moreover, $T(M)$ is a direct sum of cyclic submodules by [17, Theorem 7.1]. Hence $T(M) \cong (R/\mathfrak{m}^n)^{(I)}$ for some index set I by Proposition 2.7.

Suppose that $L/Lp^n \neq 0$. By [17, Theorem 7.1], $L/Lp^n = \oplus_{\lambda\in\Lambda}(L_\lambda/Lp^n)$ such that for each $\lambda \in \Lambda$, L_λ is a submodule of L which contains Lp^n and $L_\lambda/Lp^n \cong R/p^tR$ for some $t \in \{1, \ldots, n\}$. If $n = 1$, then clearly $L_\lambda/p^nL \cong R/pR$ for all $\lambda \in \Lambda$. Suppose that $n \geq 2$ and for each $\lambda \in \Lambda$, $L_\lambda/p^nL \cong R/p^tR$ for some $t \in \{1, \ldots, n-1\}$. Then, $(L/Lp^n)p^{n-1} = 0$. Hence, $Lp^{n-1} \subseteq Lp^n$. Therefore, $Lp^n = Lp^{n-1}$. That is, $(Lp^{n-1})p = Lp^{n-1}$. Thus, $\mathrm{Rad}(Lp^{n-1}) = Lp^{n-1}$. This implies that Lp^{n-1} is a divisible module. So $Lp^{n-1} = 0$. This contradicts the fact that L is torsion-free. Hence, there exists $\lambda_0 \in \Lambda$ such that $L_{\lambda_0}/Lp^n \cong R/p^nR$. Put $T_0 = v_0R$. Note that T_0 is a direct summand of $T(M)$ such that $T_0 \cong R/p^nR$. Let $f : L_{\lambda_0}/Lp^n \to T_0$ be an isomorphism. Then, $f\pi_2\pi_1 : L \to T_0$ is an epimorphism, where $\pi_1 : L \to L/Lp^n$ is the natural epimorphism and $\pi_2 : L/Lp^n \to L_{\lambda_0}/Lp^n$ is the projection map. Note that $L \oplus T_0$ is a $D4$-module. Hence, $\mathrm{Ker}(f\pi_2\pi_1) = \sum_{\lambda\neq\lambda_0} L_\lambda$ is a direct summand of L. Let X be a nonzero submodule of L such that $L = (\sum_{\lambda\neq\lambda_0} L_\lambda) \oplus X$. Then

$$X \cong L/\sum_{\lambda\neq\lambda_0} L_\lambda \cong \frac{L/Lp^n}{(\sum_{\lambda\neq\lambda_0} L_\lambda)/Lp^n} = \frac{\oplus_{\lambda\in\Lambda}(L_\lambda/Lp^n)}{\oplus_{\lambda\neq\lambda_0}(L_\lambda/Lp^n)} \cong L_{\lambda_0}/Lp^n.$$

This gives $Xp^n = 0$. This contradicts the fact that X is a nonzero torsion-free module. It follows that $L = Lp^n$. This yields $L = Lp$. That is, $\mathrm{Rad}(L) = L$. Hence, L is divisible. So $L = 0$ since M is reduced. We thus get $M = T(M)$.

(iii) $\Rightarrow$ (i) Assume that $M \cong (R/\mathfrak{m}^n)^{(I)}$ for some positive integer n and an index set I. Then M is quasi-projective by [21, Theorem 5.10]. So M is a $D3$-module. □

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Let M be an R-module over a commutative Dedekind domain R. It is well known that M contains a unique largest divisible submodule $d(M)$. Note that $d(M)$ is fully invariant in M. Moreover, there exists a reduced submodule E of M such that $M = d(M) \oplus E$. It is clear that if $M = A \oplus B$, then $d(M) = d(A) \oplus d(B)$ and $d(A) = A \cap d(M)$. Next, we will be concerned with torsion modules.

LEMMA 2.19. *Assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m} = pR$ and quotient field Q. Let M be a nonzero torsion R-module such that $M = d(M) \oplus N$ with $d(M) \cong Q/R$ and $N \cong (R/\mathfrak{m}^n)^{(I)}$, where n is a positive integer and I is an index set. Let K be a reduced direct summand of M. Then*

(i) *$K \cong (R/\mathfrak{m}^n)^{(J)}$ for some index set J.*
(ii) *The sum $d(M) + K$ is direct. Moreover, every submodule A of $d(M) \oplus K$ with $d(A) \neq 0$ can be written as $A = d(M) \oplus B$ such that B is a submodule of K.*
(iii) *M/K is K-projective.*

Proof. Let L be a submodule of M such that $M = L \oplus K$. Then $d(M) = d(L) \oplus d(K) = d(L)$. Let $L_1 \leq L$ such that $L = d(L) \oplus L_1$. Thus, $M = d(M) \oplus L_1 \oplus K$. This yields $L_1 \oplus K \cong N$.

(i) This follows from [**26**, Proposition 3].

(ii) The first part is clear. Now let A be a submodule of $d(M) \oplus K$ such that $d(A) \neq 0$. Let B be a submodule of A such that $A = d(A) \oplus B$. Since $d(A)$ is injective and $d(M)$ is indecomposable, we have $d(A) = d(M)$. Hence, $A = d(M) \oplus B$. Let $\pi : d(M) \oplus K \to K$ be the projection map. Set $A_1 = \pi(B)$. It is easy to check that $d(M) \oplus B = d(M) \oplus A_1$.

(iii) Applying [**21**, Theorem 5.10], we see that the module $L_1 \oplus K$ is quasi-projective. Therefore, L_1 is K-projective (see [**6**, 4.3]). Now let X be a submodule of K. Then, $(K/X)p^n = 0$. Hence, $\mathrm{Hom}_R(d(M), K/X) = 0$. This implies that $d(M)$ is K-projective. From [**6**, 4.3], it follows that $d(M) \oplus L_1$ is K-projective. □

In the next proposition, we determine all torsion $D3$-modules (resp. $D4$-modules) over a discrete valuation ring.

PROPOSITION 2.20. *Assume that R is a discrete valuation ring with maximal ideal $\mathfrak{m} = pR$ and quotient field Q. Then the following statements are equivalent for a nonzero torsion R-module M:*

(i) *M is a $D3$-module;*
(ii) *M is a $D4$-module;*
(iii) *$M \cong (Q/R)^a \oplus (R/\mathfrak{m}^n)^{(I)}$, where $a \in \{0, 1\}$, n is a positive integer and I is an index set.*

Proof. (i) ⇒ (ii) This is immediate.

(ii) ⇒ (iii) This follows from Lemma 2.3, Corollary 2.15, and Theorem 2.18.

(iii) ⇒ (i) Assume that $M = d(M) \oplus N$ such that $d(M) \cong Q/R$ and $N \cong (R/\mathfrak{m}^n)^{(I)}$, where n is a positive integer and I is an index set. Since direct summands of a $D3$-module are also $D3$-modules, the proof is completed by showing that M is a $D3$-module. To do this, we will use [**27**, Proposition 3]. Let A and B be direct summands of M such that $M = A + B$. Then, $A = d(A) \oplus A_1$ and $B = d(B) \oplus B_1$ for some submodules $A_1 \leq A$ and $B_1 \leq B$. So $M = d(A) + d(B) + A_1 + B_1$. By Lemma 2.19(i), we have $A_1 p^n = 0$ and $B_1 p^n = 0$. Hence, $d(M) = Mp^n = d(A)p^n + d(B)p^n + A_1 p^n + B_1 p^n = d(A) + d(B)$. Therefore, $M = d(M) + A_1 + B_1 = (d(M) \oplus A_1) + B_1$ (Lemma 2.19(ii)). Note that B_1 is a reduced direct summand of M. By Lemma 2.19(iii), M/B_1 is B_1-projective.

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By [6, 4.12], there exists a submodule $U \leq d(M) \oplus A_1$ such that $M = U \oplus B_1$. Clearly, $d(U) \neq 0$. Using Lemma 2.19(ii), we conclude that $U = d(M) \oplus U_1$ for some submodule $U_1 \leq A_1$. Therefore, $M = d(M) \oplus U_1 \oplus B_1$. If $d(B) = 0$, then $B_1 = B$ and $d(M) \oplus U_1 = d(A) \oplus U_1 \subseteq A$. If $d(B) \neq 0$, then $d(B) = d(M)$ and hence $M = U_1 \oplus B$ with $U_1 \subseteq A$. From [27, Proposition 3], it follows that M is a $D3$-module. □

Combining Corollary 2.9 and Proposition 2.20, we get the following two corollaries.

COROLLARY 2.21. *Assume that R is a commutative Dedekind domain. Then the following are equivalent for a nonzero torsion R-module M:*

(i) *M is a $D3$-module;*
(ii) *M is a $D4$-module;*
(iii) *For every nonzero prime ideal $\mathfrak{p}$ of R, there exist $a \in \{0, 1\}$, a positive integer n and an index set I such that $T_{\mathfrak{p}}(M) \cong (R(\mathfrak{p}^{\infty}))^a \oplus (R/\mathfrak{p}^n)^{(I)}$.*

COROLLARY 2.22. *The following are equivalent for a nonzero torsion $\mathbb{Z}$-module M:*

(i) *M is a $D3$-module;*
(ii) *M is a $D4$-module;*
(iii) *For every prime integer p, there exist $a \in \{0, 1\}$, a positive integer n and an index set I such that $T_{p\mathbb{Z}}(M) \cong (\mathbb{Z}(p^{\infty}))^a \oplus (\mathbb{Z}/p^n\mathbb{Z})^{(I)}$.*

In the following result, we characterize $D3$- and $D4$-modules which are direct sums of cyclic submodules.

PROPOSITION 2.23. *Let R be any commutative Dedekind domain. Then the following are equivalent for an R-module M which is a direct sum of cyclic submodules:*

(i) *M is a quasi-projective module;*
(ii) *M is a semi-projective module;*
(iii) *M is a direct projective module;*
(iv) *M is a $D3$-module;*
(v) *M is a $D4$-module.*

Proof. (i) ⇔ (ii) ⇔ (iii) follow by [15, Theorem 2.11].

(iii) ⇒ (iv) ⇒ (v) are clear.

(v) ⇒ (i) By Lemma 2.3 and Proposition 2.7, M is a free module or M is a torsion module. Clearly if M is free, then M is quasi-projective. Assume that M is torsion. Then for every nonzero prime ideal $\mathfrak{p}$ of R, there exist a positive integer n and an index set I such that $T_{\mathfrak{p}}(M) \cong (R/\mathfrak{p}^n)^{(I)}$ (see Corollary 2.21). By [21, Theorem 5.10], M is quasi-projective. □

Using Proposition 2.23 and its proof, we obtain the following corollary.

COROLLARY 2.24. *Let M be a nonzero $\mathbb{Z}$-module which is a direct sum of cyclic submodules (e.g., M is finitely generated). Then the following are equivalent:*

(i) *M is a $D3$-module;*
(ii) *M is a $D4$-module;*
(iii) *M is free or M is a torsion module such that every p-component is a direct sum of cyclic groups of the same order p^n.*

REMARK 2.25. From Theorem 2.18, it follows that a reduced $D4$-module M cannot be mixed; i.e., it is either a torsion module or torsion-free. But this is not true, in general, if the module M is not reduced as shown in the next example.

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EXAMPLE 2.26. Let R be a commutative Dedekind domain with quotient field Q. Let M_1 be an R-module such that $M_1 \cong Q^{(I)}$ for some nonempty index set I. Let M_2 be a nonzero torsion R-module and assume that for any nonzero prime ideal $\mathfrak{p}$ of R, there exists a positive integer n such that the $\mathfrak{p}$-primary component of M_2 is zero or isomorphic to a direct sum of copies of $R/\mathfrak{p}^n$. Consider the R-module $M = M_1 \oplus M_2$. By Proposition 2.14 and Corollary 2.21, M_1 and M_2 are $D3$-modules. Moreover, it is easily seen that $\mathrm{Hom}_R(M_1, M_2) = 0$ and $\mathrm{Hom}_R(M_2, M_1) = 0$. Applying Proposition 2.4, it follows that M is a $D3$-module.

3. Examples using quivers. Throughout this section K will be any field. We will use both hereditary and non-hereditary algebras given by quivers with at most three vertices and with paths of length at most two. Almost all these algebras will have finite representation type. Almost all modules over these algebras will be vector spaces of finite dimension over K. Hence, they will be direct sums of finitely many indecomposable summands.

EXAMPLE 3.1. There are a K-algebra R and a right R-module M with the following properties:

(i) M is injective and the projective dimension of M is one.
(ii) M has exactly one submodule which is not a direct summand.
(iii) M is not $D4$.
(iv) M is semi-Hopfian, but M is not nonsingular.

Construction: Let R be the hereditary K-algebra given by the quiver $\bullet_1 \xrightarrow{a} \bullet_2$. Then, R has a basis $\{e_1, e_2, a\}$, where e_1 and e_2 are primitive orthogonal idempotents, $a^2 = 0$ and $a = e_1 a = a e_2$. Hence, the K-linear map such that

$$e_1 \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right],\ e_2 \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right],\ a \mapsto \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$$

is an algebra isomorphism between R and the algebra of all 2×2 upper triangular matrices with entries in K. Next, let M denote the injective cogenerator $M = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 1$. Then, M has a basis $\{v_1, v_2, v_3\}$ such that the basis $\{e_1, e_2, a\}$ of R acts as follows on M:

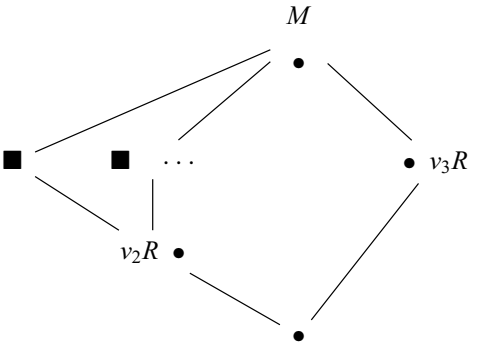
$$v_1e_1 = v_1 \qquad v_1e_2 = 0 \qquad v_1a = v_2$$
$$v_2e_1 = 0 \qquad v_2e_2 = v_2 \qquad v_2a = 0$$
$$v_3e_1 = v_3 \qquad v_3e_2 = 0 \qquad v_3a = 0$$

Let $A = v_1R$ and $B = v_3R$. Clearly, a picture describing M is as follows $\begin{smallmatrix} \bullet v_1 \\ \downarrow a \\ \bullet v_2 \end{smallmatrix}$ $\bullet v_3$.

Indeed, $v_1R = \begin{smallmatrix} \bullet v_1 \\ \downarrow a \\ \bullet v_2 \end{smallmatrix}$ corresponds to $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ and is a projective-injective module; $v_3R = \bullet v_3$ corresponds to 1 and is a simple injective module; $v_2R = \bullet v_2$ corresponds to 2 and is a simple projective module.

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Now, $M = A \oplus B$ and the K-linear map $f : A \to B$ satisfying $v_1 \mapsto v_3$, $v_2 \mapsto 0$ is an epimorphism of R-modules such that $\mathrm{Ker} f = v_2R$. Since the injective module v_3R is the unique simple direct summand of M, we conclude that M is not $D4$. We finally note that the indecomposable submodules of M of dimension two are the direct summands $(v_1 + tv_3)R$, where $t \in K$. Hence, the lattice $\mathcal{L}(M)$ of the submodules of M is of the following form



where the black squares denote the indecomposable direct summands of dimension two. Hence, M satisfies (i), (ii), and (iii). (The shape of this lattice gives a direct proof of the fact that M is not $D3$. Indeed, let $M_1 = v_1R$ and $M_2 = (v_1 + v_3)R$. Then, we clearly have $M = M_1 + M_2$ and $v_1a = v_2 = (v_1 + v_3)a$. Consequently, $M_1 \cap M_2 = v_2R$ is not a direct summand of M.) Since every epimorphism $M \to M$ is an automorphism, M is clearly semi-Hopfian. On the other hand, $ann_R(v_3) = e_2R \oplus aR = \mathrm{Soc}(R_R)$. Hence, M is not nonsingular. Thus, (iv) is satisfied.

The next lemma points out two obvious facts.

LEMMA 3.2. *Let R be a ring, let M_R be an R-module and let $\overline{R} = R/I$, where I is an ideal with $I \subseteq ann_R(M)$. The following facts hold:*

(i) *$M_{\overline{R}}$ is $D3$ (resp. $D4$) if and only if M_R is $D3$ (resp. $D4$).*
(ii) *$M_{\overline{R}}$ is quasi-projective if and only if M_R is quasi-projective.*

Proof. The assertions follow from the remark that M_R and $M_{\overline{R}}$ have the same submodules, direct summands and factor modules. □

As a first application of the above lemma, we obtain the following example.

EXAMPLE 3.3. There exists a module M_R with the following properties:

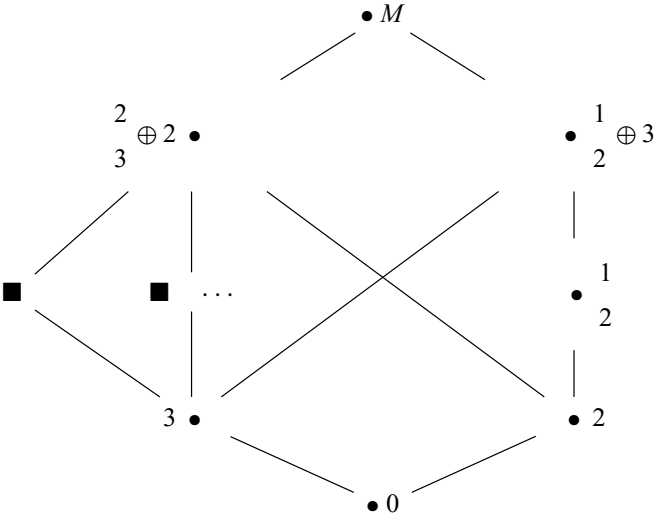
(i) The projective and injective dimensions of M are equal to one and $\mathrm{Ext}^1_R(M, M) \neq 0$.
(ii) M has exactly four submodules which are not direct summands.
(iii) M is a $D3$-module which is quasi-projective, but not Rickart.
(iv) M is semi-Hopfian, but not nonsingular.

Construction: Let R be the hereditary K-algebra given by the quiver $\overset{1}{\bullet} \xrightarrow{a} \overset{2}{\bullet} \xrightarrow{b} \overset{3}{\bullet}$. Then one easily sees that R is isomorphic to the algebra of all 3×3 upper triangular matrices with entries in K. Next, let M_R be the module $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$. Then, M_R is neither projective nor injective, and there is an Auslander–Reiten sequence of the form

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$$0 \longrightarrow \begin{matrix}2\\3\end{matrix} \longrightarrow \begin{matrix}1\\2\\3\end{matrix} \oplus 2 \longrightarrow \begin{matrix}1\\2\end{matrix} \longrightarrow 0.$$

Hence (i) holds. Moreover, 2, 3, $\begin{matrix}2\\3\end{matrix} \oplus 2$ and $3 \oplus \begin{matrix}1\\2\end{matrix}$ are the submodules of M which are not direct summands. Hence also (ii) holds. Since $ann_R(M)$ is the one-dimensional ideal generated by ab, it follows that $M_{\overline{R}}$ is a projective-injective module. Hence the conclusion that M is quasi-projective (and so $D3$) follows from Lemma 3.2. A direct proof of the fact that M is $D3$ follows from the remark that, if M_1 and M_2 are direct summands of M with $M = M_1 + M_2$ and $M_1 \neq M$, $M_2 \neq M$, then $dim M_1 = dim M_2 = 2$ and so $M_1 \cap M_2 = 0$. In this case, $\mathcal{L}(M)$ has the following form, where the black squares denote indecomposable projective direct summands of dimension two indexed by the elements of K, all isomorphic to $\begin{matrix}2\\3\end{matrix}$:



To end the proof of (iii), let f be an endomorphism of M such that $f\begin{pmatrix}1\\2\end{pmatrix} = 0$ and $f\begin{pmatrix}2\\3\end{pmatrix} = 2$. Then, $\mathrm{Ker} f = \begin{matrix}1\\2\end{matrix} \oplus 3$ is not a direct summand of M, and so M is not Rickart. Since any epimorphism $M \to M$ is an isomorphism, M is obviously semi-Hopfian. For every i, let e_i be the path of length zero around the vertex i. Then, $\begin{matrix}1\\2\end{matrix}$ is generated by a vector v such that $v = ve_1$ and $\mathrm{Soc}(R_R)$ is the vector space generated by e_3, b and ab. Consequently, $\mathrm{Soc}(R_R) \subseteq ann_R(v)$, and so M is not nonsingular. Hence, also (iv) holds.

It is easy to find an example of a faithful decomposable $D3$-module which is not projective.

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EXAMPLE 3.4. There is a module M_R with the following properties:

(i) M is injective, faithful, and the projective dimension of M is one.
(ii) M has no direct summand of dimension one and exactly two submodules of dimension one.
(iii) M has exactly two indecomposable submodules of dimension two, both injective.
(iv) M has no direct summand of dimension three and exactly $|K\backslash\{0\}|$ submodules of dimension three, all isomorphic and projective.
(v) M is a $D3$-module which is Rickart, but not quasi-projective.

Construction: Let R be the hereditary K-algebra given by the quiver $\overset{2}{\bullet} \xleftarrow{a} \overset{1}{\bullet} \xrightarrow{b} \overset{3}{\bullet}$ and let M be the faithful injective module $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$. Then M is not projective and $2, 3, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}1\\3\end{smallmatrix}$ are the unique indecomposable submodules of M of dimension ≤ 2. Hence, (i), (ii), and (iii) hold. On the other hand, the indecomposable submodules of M of dimension three are all isomorphic to $\begin{smallmatrix}1\\2\ 3\end{smallmatrix}$ and generated by a vector $(1, a) \in K^2$ with $a \neq 0$. Thus, also (iv) holds. Assume now that $M = M_1 + M_2$, where M_1 and M_2 are direct summands of M of dimension < 4. Then we deduce from (ii), (iii), and (iv) that $dim M_1 = dim M_2 = 2$. Hence, $M_1 \cap M_2 = 0$, and so M is a $D3$-module. Since $\begin{smallmatrix}1\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\\3\end{smallmatrix}$ are fully invariant submodules of M with endomorphism ring K, it follows that $\mathrm{Ker} f$ is a direct summand of M for every endomorphism f of M. To end the proof of (v), let N be an injective module of the form $N = N_1 \oplus N_2$ with $N_1 \cong N_2 \cong 1$. Next, let $\alpha : M \to N$, $\beta : M \to N$ be two morphisms such that $\alpha\begin{pmatrix}1\\2\end{pmatrix} = N_1 = \beta\begin{pmatrix}1\\3\end{pmatrix}$ and $\alpha\begin{pmatrix}1\\3\end{pmatrix} = N_2 = \beta\begin{pmatrix}1\\2\end{pmatrix}$. Then α is surjective and for any endomorphism γ of M, we have $(\alpha \circ \gamma)\begin{pmatrix}1\\2\end{pmatrix} \subseteq \alpha\begin{pmatrix}1\\2\end{pmatrix} = N_1$. Hence, $\alpha \circ \gamma \neq \beta$, and so M is not quasi-projective.

COROLLARY 3.5. *Let R be a K-algebra of finite representation type. Let M be a decomposable finite dimensional $D3$-module. The following cases may occur:*

(a) *M has only finitely many indecomposable direct summands.*
(b) *The indecomposable direct summands of M are parameterized by the projective line over K.*

Proof. (a) The module $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$ constructed in Example 3.4 has exactly two indecomposable direct summands.

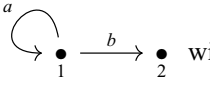
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(b) The module $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$ constructed in Example 3.3 has exactly one indecomposable direct summand with top 1 and exactly $|K|$ indecomposable direct summands with socle 3. □

The minimal injective cogenerators over the algebras considered in Examples 3.3 and 3.4 (that is the modules $\begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$ and $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus 1$) are not *D*4-modules. However, it suffices to consider an algebra with exactly two simple modules to construct an injective cogenerator which is *D*3, but not projective.

EXAMPLE 3.6. There is a module M_R with the following properties:

(i) M is an injective cogenerator of infinite projective dimension.
(ii) M has no direct summand of dimension one and exactly two submodules of dimension one.
(iii) Any indecomposable submodule of M of dimension two is injective, and the indecomposable direct summands of M of dimension two are parameterized by the projective line over K.
(iv) M has $|K\backslash\{0\}|$ indecomposable submodules of dimension three. Moreover, all these modules are isomorphic, projective and they are not direct summands of M.
(v) M is a direct projective module, but not a semi-projective module.

Construction: Let R be the K-algebra given by the quiver 

with relations $a^2 = 0$ and $ab = 0$. Then the indecomposable R-modules are $1, 2, \begin{smallmatrix}1\\1\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix}$, and $\begin{smallmatrix}1\\1\ 2\end{smallmatrix}$. Let M be the minimal injective cogenerator $\begin{smallmatrix}1\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$. Then 1 and 2 are the unique simple submodules of M. On the other hand, M has a unique direct summand isomorphic to $\begin{smallmatrix}1\\1\end{smallmatrix}$ and $|K|$ direct summands isomorphic to $\begin{smallmatrix}1\\2\end{smallmatrix}$. Moreover, M has $|K\backslash\{0\}|$ submodules isomorphic to $\begin{smallmatrix}1\\1\ 2\end{smallmatrix}$. Hence, (i), (ii), (iii), and (iv) hold. Assume now that A is a submodule of M such that M/A is isomorphic to an indecomposable direct summand of M. Then, $dim M/A = 2 = dim A$ and A is indecomposable. Consequently, we deduce from (iii) that A is a direct summand of M. Hence, M is direct projective. Let now α and β be two endomorphisms of M such that $\alpha\begin{pmatrix}1\\1\end{pmatrix} = 0$, $\alpha\begin{pmatrix}1\\2\end{pmatrix} = 1$, $\beta\begin{pmatrix}1\\1\end{pmatrix} = 1$, and $\beta\begin{pmatrix}1\\2\end{pmatrix} = 0$. Then we have $\mathrm{Im}\alpha = \mathrm{Im}\beta$. Since $\begin{smallmatrix}1\\1\end{smallmatrix}$ is a fully invariant submodule of M, we

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obtain $(\alpha \circ \gamma)\begin{pmatrix}1\\1\end{pmatrix} \subseteq \alpha\begin{pmatrix}1\\1\end{pmatrix} = 0 \neq \beta\begin{pmatrix}1\\1\end{pmatrix}$ for any endomorphism γ of M. Hence, M is not semi-projective, as claimed in (v).

The modules $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\end{smallmatrix}$, $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$, and $\begin{smallmatrix}1\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}$, constructed in the previous examples, are direct sums of uniserial modules with at least one indecomposable fully invariant direct summand. In the next example, we construct a $D3$-module which does not have these properties.

EXAMPLE 3.7. There is a module M_R with the following properties:

(i) M is of the form $M = P \oplus I$, where P is an indecomposable projective module and I is an indecomposable injective module (of infinite projective dimension) such that $I/\mathrm{Soc}I$ is decomposable.
(ii) P and I are not fully invariant.
(iii) M is a direct projective module but not a semi-projective module.

Construction: Let R be the K-algebra given by the quiver $\bullet_1 \xrightarrow{a} \bullet_2 \circlearrowleft b$ with relations $ab = 0$ and $b^2 = 0$. Then, the indecomposable R-modules are $1, 2, \begin{smallmatrix}1\\2\end{smallmatrix}, \begin{smallmatrix}2\\2\end{smallmatrix}$ and $\begin{smallmatrix}1&2\\&2\end{smallmatrix}$. Let M be a module of the form $M = P \oplus I$ where $P \cong \begin{smallmatrix}2\\2\end{smallmatrix}$ and $I \cong \begin{smallmatrix}1&2\\&2\end{smallmatrix}$. Then (i) and (ii) clearly hold. Let X be a submodule of M such that $0 \neq X \neq M$ and M/X is isomorphic to a direct summand of M. Since $M/\mathrm{Soc}M$ is semisimple and M/X is not semisimple, it follows that $\mathrm{Soc}M \not\subseteq X$. Consequently, X is an indecomposable non-simple module. We claim that X is a direct summand of M. If $X \cong \begin{smallmatrix}1&2\\&2\end{smallmatrix}$, then there is nothing to prove. Assume now that $X \cong \begin{smallmatrix}2\\2\end{smallmatrix}$ and that $\mathrm{Soc}X \neq \mathrm{Soc}I$. Then we have $X \cap I = 0$ and so $X + I = X \oplus I = M$, as desired. Suppose, on the contrary, that $X \cong \begin{smallmatrix}2\\2\end{smallmatrix}$ and that $\mathrm{Soc}X = \mathrm{Soc}I$. Then, $X \cap P = 0$ and we can find a direct summand $Y \cong \begin{smallmatrix}1&2\\&2\end{smallmatrix}$ of M such that $X \subseteq Y$. Consequently, $P + Y = P \oplus Y = M$, and we have $M/X \cong P \oplus 1$. This is the desired contradiction. Assume, again on the contrary, that X is the unique submodule of M isomorphic to $\begin{smallmatrix}1\\2\end{smallmatrix}$. Then we have $X \subseteq I$ and so $M/X \cong P \oplus 2$. This contradiction completes the proof that M is direct projective. To end the proof of (iii), let α and β be two endomorphisms of M such that $\alpha(P) = \mathrm{Soc}I = \beta(I)$ and $\alpha(I) = 0 = \beta(P)$. Then, we have

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$\mathrm{Im}\alpha = \mathrm{Im}\beta$ and $\mathrm{Ker}\alpha = \mathrm{Soc}P \oplus I$. Take any endomorphism γ of M. Then, we clearly have $\gamma(I) \subseteq \mathrm{Soc}P \oplus I$ and so $\alpha \circ \gamma = 0 \neq \beta$. Hence, M is not semi-projective. Thus, (iii) holds.

We shall say in the sequel that a finite dimensional module M is *multiplicity free* if M is the direct sum of pairwise non-isomorphic indecomposable modules. The next lemma describes a strategy to construct many $D4$-modules.

LEMMA 3.8. *Let R be a finite dimensional K-algebra and let M_R be a finite dimensional multiplicity free module. Then the following are equivalent:*

(a) *For any decomposition of M of the form $M = X \oplus Y$ with Y indecomposable there is no surjective morphism $X \to Y$;*
(b) *M is $D4$;*
(c) *M is $D3$.*

Proof. (a) $\Rightarrow$ (b) Let $M = A \oplus B$ and let $f : A \to B$ be an epimorphism. We claim that $B = 0$. Assume that B is nonzero. Since M is finite dimensional, it follows that $B = L \oplus Y$ with Y indecomposable. If $X = A \oplus L$, then we have an epimorphism from X to Y and $M = X \oplus Y$. This is a contradiction to (a). Hence, $B = 0$ and $\mathrm{Ker}f = A$ is a direct summand of A, as desired.

(b) $\Rightarrow$ (a) Assume that M is a $D4$-module such that (a) does not hold. Then, M has a decomposition of the form $M = X \oplus Y$ with Y indecomposable, and there is an epimorphism $f : X \to Y$. Since M is $D4$, we have $X = \mathrm{Ker}f \oplus C$ for some $C \cong Y$. Hence, we have $M = X \oplus Y = \mathrm{Ker}f \oplus C \oplus Y$. Since M is multiplicity free, this contradiction proves that (a) holds.

(b) $\Leftrightarrow$ (c) This follows from Proposition 2.13. □

Note that in the above proof we do not need the multiplicity free condition of M_R to prove (a) $\Rightarrow$ (b).

The multiplicity free $D4$-modules constructed up to now are direct sums of at most n indecomposable direct summands, where n is the number of simple modules (up to isomorphism). Hence, the following question naturally arises:

Is the number of the isomorphism classes of the indecomposable direct summands of a multiplicity free $D4$-module at most equal to the rank of the Grothendieck group? (We recall that the *rank* of the Grothendieck group is the number of the isomorphism classes of a multiplicity free tilting module over a finite dimensional algebra [**22**, p. 167]).

EXAMPLE 3.9. There are a K-algebra R of finite representation type and a multiplicity free module M_R with the following properties:

(i) R admits two simple modules (up to isomorphism).
(ii) M is the direct sum of three indecomposable modules and two of them are injective.
(iii) M is a $D3$-module.
(iv) Any multiplicity free module of the form $M \oplus X$ with X indecomposable is not a $D4$-module.

Construction: Let R be the K-algebra used in Example 3.6 and let M be the module $\begin{smallmatrix}1\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2$. Then (i) and (ii) clearly hold and (iii) follows from Lemma 3.8. Assume first

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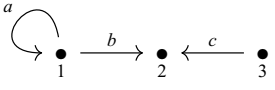
that $X = 1$. Then there is an epimorphism $f : M \to X$ such that $\mathrm{Ker}f = 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2$. Hence, $\mathrm{Ker}f$ is not a direct summand of M. Assume now that $X = \begin{smallmatrix}1\\1\ 2\end{smallmatrix}$, $A = \begin{smallmatrix}1\\1\ 2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2$ and $B = \begin{smallmatrix}1\\1\end{smallmatrix}$. Then, $M \oplus X = A \oplus B$ and there is an epimorphism $g : A \to B$ such that $\mathrm{Ker}g = 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2$. Hence, $\mathrm{Ker}g$ is not a direct summand of A. Consequently, in both cases, $M \oplus X$ is not a $D4$-module, and the proof of (iv) is complete.

REMARK 3.10. The module M in the previous example is direct projective, but not semi-projective. Indeed, let $f : M \to N$ be an epimorphism with N isomorphic to a direct summand of M. Then, there exist $a, b, c, d \in \{0, 1\}$ such that $1 \leq a + b + c \leq 2$ and $N \cong \begin{smallmatrix}1\\1\end{smallmatrix}^{a} \oplus \begin{smallmatrix}1\\2\end{smallmatrix}^{b} \oplus 2^c$, and M is of the form $M_1 \oplus M_2$ with $M_1 \cong N$. A case-by-case verification shows that $f(M_1) = N$. Hence, there exist two morphisms $s : M_1 \to N$ and $t : M_2 \to N$ such that s is an isomorphism and $f = [s\ t]$. Let $g : N \to M$ be the morphism $\left[\begin{smallmatrix}s^{-1}\\0\end{smallmatrix}\right]$. Then, $f \circ g$ is the identity on N. Hence, $\mathrm{Ker}f$ is a direct summand of M, and so M is direct projective. Finally, let α and β be endomorphisms of M such that $\alpha\begin{pmatrix}1\\2\end{pmatrix} = \beta\begin{pmatrix}1\\1\end{pmatrix} = 1$ and $\alpha\left(\begin{smallmatrix}1\\1\end{smallmatrix} \oplus 2\right) = \beta\left(\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2\right) = 0$. Then, we have $\mathrm{Im}\alpha = \mathrm{Im}\beta$. Since $\begin{smallmatrix}1\\1\end{smallmatrix}$ is fully invariant in M, we have $(\alpha \circ \gamma)\begin{pmatrix}1\\1\end{pmatrix} \subseteq \alpha\begin{pmatrix}1\\1\end{pmatrix} = 0 \neq \beta\begin{pmatrix}1\\1\end{pmatrix}$ for every endomorphism γ of M. Consequently, M is not semi-projective.

The next example shows that a non-projective maximal $D4$-module does not necessarily admit an indecomposable injective direct summand.

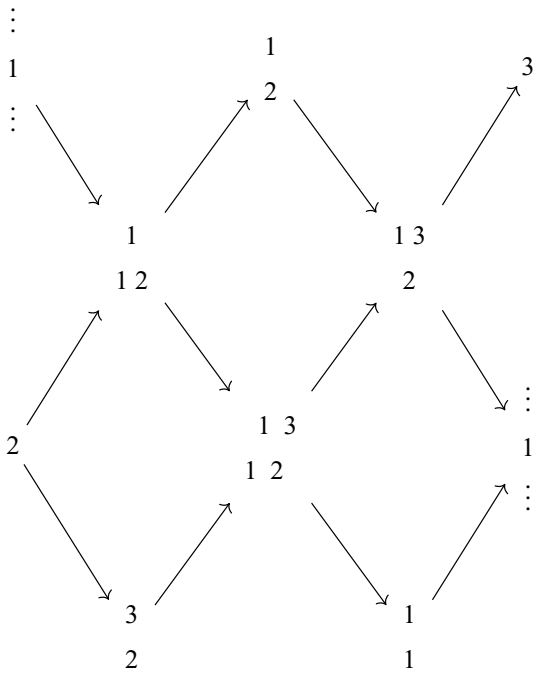
EXAMPLE 3.11. There are a K-algebra R of finite representation type and a multiplicity free module M_R with the following properties:

(i) R admits three simple modules (up to isomorphism).
(ii) M is the direct sum of four indecomposable non-injective modules.
(iii) M is a faithful $D3$-module.
(iv) Any multiplicity free module of the form $M \oplus X$ with X indecomposable is not a $D4$-module.

Construction: Let R be the K-algebra given by the quiver 

with relations $a^2 = 0$ and $ab = 0$. Then, the Auslander–Reiten quiver of R is of the following form:

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Let $M = 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus \begin{smallmatrix}1&3\\1&2\end{smallmatrix}$. Then (i) and (ii) clearly hold and (iii) follows from Lemma 3.8. Let X denote one of the indecomposable modules $\begin{smallmatrix}1&3\\2\end{smallmatrix}$, $\begin{smallmatrix}1\\1\end{smallmatrix}$, 1 and 3. Then there is an epimorphism $g : \begin{smallmatrix}1&3\\1&2\end{smallmatrix} \to X$ such that Ker$g$ is one of the modules 1, $\begin{smallmatrix}3\\2\end{smallmatrix}$, $1 \oplus \begin{smallmatrix}3\\2\end{smallmatrix}$ and $\begin{smallmatrix}1\\1&2\end{smallmatrix}$. Consequently, there is an epimorphism $f : M \to X$ such that Ker$f = 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \oplus$ Ker$g$. Since M is multiplicity free and the modules 1 and $\begin{smallmatrix}1\\1&2\end{smallmatrix}$ are not direct summands of M, it follows that Ker$f$ is not a direct summand of M. Finally, let $X = \begin{smallmatrix}1\\1&2\end{smallmatrix}$, $A = 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\1&2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix}$ and $B = \begin{smallmatrix}1&3\\1&2\end{smallmatrix}$. Then, $M \oplus X = A \oplus B$ and there is an epimorphism $g : \begin{smallmatrix}1\\1&2\end{smallmatrix} \oplus \begin{smallmatrix}3\\2\end{smallmatrix} \to B$ such that Ker$g \cong 2$. Hence there is an epimorphism $f : A \to B$ such that Ker$f \cong 2 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus$ Ker$g$. Since A is multiplicity free, it follows that Ker$f$ is not a direct summand of A. Hence, in both cases, $M \oplus X$ is not a $D4$-module and the proof of (iv) is complete.

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REMARK 3.12. It is easy to check that the module M in the previous example is direct projective, but not semi-projective. Indeed, a word by word repetition of the first part of Remark 3.10 shows that the module M in the previous example is direct projective. To see that it is not semi-projective, let α and β be two endomorphisms of M such that $\alpha\begin{pmatrix}1\\2\end{pmatrix}=\beta\begin{pmatrix}1&3\\1&2\end{pmatrix}=1$ and $\alpha\left(2\oplus\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}1&3\\1&2\end{smallmatrix}\right)=\beta\left(2\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}3\\2\end{smallmatrix}\right)=0$. Then, we have $\mathrm{Im}\alpha=\mathrm{Im}\beta$. Since $\begin{smallmatrix}1&3\\1&2\end{smallmatrix}$ is fully invariant in M, we have $(\alpha\circ\gamma)\begin{pmatrix}1&3\\1&2\end{pmatrix}\subseteq\alpha\begin{pmatrix}1&3\\1&2\end{pmatrix}=0\neq\beta\begin{pmatrix}1&3\\1&2\end{pmatrix}$ for any endomorphism γ of M. Hence, M is not semi-projective.

The previous examples also show that a non-projective $D4$-module may have many projective direct summands.

COROLLARY 3.13. *There exist a natural number n, a K-algebra R of finite representation type with n simple modules (up to isomorphism) and a module M_R with the following properties:*

(a) *M is a faithful and multiplicity free non-projective $D3$-module.*
(b) *M is the direct sum of $n+1$ indecomposable modules and exactly $n-1$ of them are projective.*

Proof. The modules $\begin{smallmatrix}1\\1\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2$ and $2\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}1&3\\1&2\end{smallmatrix}$ constructed in Examples 3.9 and 3.11 satisfy (a) and (b) with $n=2$ and $n=3$, respectively. □

It is easy to see that "maximal" $D3$-modules may have quite different decompositions.

COROLLARY 3.14. *There are a K-algebra R and multiplicity free modules L_R and M_R with the following properties:*

(a) *L and M are faithful $D3$-modules.*
(b) *The multiplicity free modules of the form $L\oplus X$ and $M\oplus X$ with X indecomposable are not $D4$-modules.*
(c) *$L=\oplus_{i=1}^{s}L_i$ and $M=\oplus_{j=1}^{t}M_j$ where $s\neq t$ and the summands L_i and M_j are indecomposable for every i and j.*
(d) *L and M generate all but one indecomposable modules.*

Proof. Let R be the K-algebra considered in Example 3.11. Then, the module $M=2\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}1&3\\1&2\end{smallmatrix}$ satisfies (a), (b), and (c) with $t=4$. Finally, let L denote the module $2\oplus\begin{smallmatrix}1\\1&2\end{smallmatrix}\oplus\begin{smallmatrix}1&3\\1&2\end{smallmatrix}$. Then, Lemma 3.8 implies that L is a faithful $D3$-module. Hence, (a) holds and (c) holds with $s=3$. We also note that there exist epimorphisms $\begin{smallmatrix}1\\1&2\end{smallmatrix}\to\begin{smallmatrix}1\\2\end{smallmatrix},\ \begin{smallmatrix}1\\1&2\end{smallmatrix}\oplus$

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$\begin{matrix}3\\2\end{matrix} \to \begin{matrix}&1&3\\1&&2\end{matrix}$ and $\begin{matrix}&1&3\\1&&2\end{matrix} \to Y$ with $Y \in \left\{ \begin{matrix}1\\1\end{matrix}, 1, \begin{matrix}&1&3\\2&&\end{matrix}, 3 \right\}$. Hence, proceeding as in Example 3.11, we conclude that L satisfies (b). On the other hand, $\begin{matrix}3\\2\end{matrix}$ $\left(\text{resp. } \begin{matrix}&1\\1&2\end{matrix}\right)$ is the unique indecomposable module which is not generated by L (resp. M). Hence also (d) holds. □

In order to compare *D*3-modules and *D*4-modules, we compare two special decompositions of a module M.

PROPOSITION 3.15. *Let M be a module and let M_i, X_i, for $i = 1, 2$, be direct summands of M such that $M = M_1 + M_2 = M_1 \oplus X_1 = M_2 \oplus X_2$. Then the following facts hold:*

(i) *Let $\pi_{1|_{M_2}} : M_2 \to X_1$ be the restriction of the projection map $\pi_1 : M \to X_1$, $\pi_{2|_{M_1}} : M_1 \to X_2$ the restriction of the projection map $\pi_2 : M \to X_2$, and $\pi : M = M_1 \oplus X_1 \to X_2 \oplus X_1$ the map defined by $\pi(m_1 + x_1) = \pi_2(m_1) + x_1$ for all $m_1 \in M_1$ and $x_1 \in X_1$. Then, the maps $\pi_{1|_{M_2}}, \pi_{2|_{M_1}}$ and π are surjective and their kernels coincide with $M_1 \cap M_2$.*
(ii) *If one of the modules M_1 and M_2 is semisimple, then $M_1 \cap M_2$ is a direct summand of M.*
(iii) *If M is a D4-module and $X_1 \cong X_2$, then $M_1 \cap M_2$ is a direct summand of M.*
(iv) *If one of the modules X_i is projective, then $M_1 \cap M_2$ is a direct summand of M.*

Proof. (i) Let $\pi_1 : M \to X_1$ and $\pi_2 : M \to X_2$ be the projection maps. Then, we clearly have $\pi_1(M_2) = \pi_1(M_2 + \mathrm{Ker}\pi_1) = \pi_1(M) = X_1$, $\pi_2(M_1) = \pi_2(M_1 + \mathrm{Ker}\pi_2) = \pi_2(M) = X_2$ and $M_1 \cap M_2 = M_2 \cap \mathrm{Ker}\pi_1 = M_1 \cap \mathrm{Ker}\pi_2$. Finally, let $\pi : M = M_1 \oplus X_1 \to X_2 \oplus X_1$ be the map sending $m_1 + x_1$ to $\pi_2(m_1) + x_1$ for all $m_1 \in M_1$ and $x_1 \in X_1$. Then, π is surjective and $\mathrm{Ker}\pi = M_1 \cap M_2$.

(ii) This is clear.

(iii) This follows from [7, Theorem 2.2(5)].

(iv) Assume that X_1 is projective. Then this hypothesis and (i) imply that the epimorphism $\pi_{1|_{M_2}} : M_2 \to X_1$ splits. Hence, its kernel $M_1 \cap M_2$ is a direct summand of both M_2 and M. □

REMARK 3.16. The hypothesis that M is a *D*4-module is essential in condition (iii) of Proposition 3.15. Indeed, let M be the module constructed in Example 3.1. Next, let M_1 and M_2 be the injective modules v_1R and $(v_1 + v_3)R$ considered at the end of the construction. Finally, let X be the injective module v_3R. Then, we have $M = M_1 + M_2 = M_1 \oplus X = M_2 \oplus X$. However, $M_1 \cap M_2 = v_2R$ is not a direct summand of M. We also note that M_1 and M_2 have Loewy length two. Hence, in condition (ii) of Proposition 3.15, we cannot replace the hypothesis that one of the modules M_1 and M_2 is semisimple with the hypothesis that these modules have Loewy length ≤ 2, or with the hypothesis that $M/\mathrm{Soc}(M)$ is either zero or simple.

THEOREM 3.17. *Let M be a D4-module which is not D3. Then there exist direct summands M_1, M_2, X_1, X_2 of M with the following properties:*

(a) *$M = M_1 + M_2 = M_1 \oplus X_1 = M_2 \oplus X_2$.*
(b) *$M_1 \oplus X_2$ and $M_2 \oplus X_1$ are not D4-modules.*
(c) *If $M = X_1 + X_2$, then M_1, M_2, X_1, X_2 are not of finite length.*

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(d) *M_1 and M_2 are not isomorphic to direct summands of M_2 and M_1, respectively.*
(e) *X_1 and X_2 are not isomorphic to direct summands of X_2 and X_1, respectively.*
(f) *$M_1 \not\subseteq X_2$ and $M_2 \not\subseteq X_1$.*
(g) *$X_1 \not\subseteq M_2$ and $X_2 \not\subseteq M_1$.*

Proof. Since M is not $D3$, there exist M_i, X_i such that (a) holds and

(1) $M_1 \cap M_2$ is not a direct summand of M.
On the other hand, by Proposition 3.15(i), there exist morphisms $f : M_1 \to X_2$ and $g : M_2 \to X_1$ such that
(2) f and g are surjective and $\mathrm{Ker}f = \mathrm{Ker}g = M_1 \cap M_2$.
Putting (1) and (2) together, we conclude that (b) holds. Assume that $M = X_1 + X_2$. Then, Proposition 3.15(i) guarantees the existence of morphisms $h : X_1 \to M_2$ and $l : X_2 \to M_1$ such that
(3) h and l are surjective and $\mathrm{Ker}h = \mathrm{Ker}l = X_1 \cap X_2$.
Hence, we deduce from (1), (2) and (3) that $l \circ f : M_1 \to M_1$, $h \circ g : M_2 \to M_2$, $g \circ h : X_1 \to X_1$ and $f \circ l : X_2 \to X_2$ are surjective morphisms which are not injective. Consequently, also (c) holds. To prove (d), assume by contradiction that $M_2 = N_1 \oplus Y$ with $N_1 \cong M_1$. Then, we have
(4) $M = M_2 \oplus X_2 = N_1 \oplus Y \oplus X_2 \cong (M_1 \oplus X_2) \oplus Y$.
Putting together (4), (b) and Lemma 2.3, we conclude that M is not a $D4$-module. This contradiction proves that $M_2 \neq N_1 \oplus Y$ with $N_1 \cong M_1$. This implies that (d) holds. To prove (e), assume by contradiction that $X_2 = N_1 \oplus Y$ with $N_1 \cong X_1$. Then, we have
(5) $M = M_2 \oplus X_2 = M_2 \oplus N_1 \oplus Y \cong (M_2 \oplus X_1) \oplus Y$.

Combining (5), (b) and Lemma 2.3, we conclude that M is not $D4$. This contradiction shows that $X_2 \neq N_1 \oplus Y$ with $N_1 \cong X_1$. This implies that (e) holds. To prove (f), assume by contradiction that $M_1 \subseteq X_2$. Then, we have $M_1 \cap M_2 = 0$. Since this is a contradiction to (1), it follows that $M_1 \not\subseteq X_2$. This implies that also (f) holds. Finally, assume by contradiction that $X_1 \subseteq M_2$. Since $M = M_1 \oplus X_1$, we clearly have $M_2 = X_1 \oplus (M_1 \cap M_2)$. This implies that $M_1 \cap M_2$ is a direct summand of M, another contradiction to (1). Consequently, $X_1 \not\subseteq M_2$. This implies that also (g) holds. □

Note that only the proof of (d) and (e) of the above theorem uses the hypothesis that M is $D4$.

The next statement describes a sufficient condition for the existence of many "big" direct summands.

PROPOSITION 3.18. *Let R be a K-algebra. Let M, A, B be nonzero R-modules such that there is an epimorphism $f : A \to B$ and $M = A \oplus B$. Then, M admits $|K|$ direct summands A_s with $s \in K$, such that $A_s \cong A$ and $A_s \oplus B = M$ for any s. Moreover, we have $A_s + A_t = M$ and $A_s \cap A_t = \mathrm{Ker}f$ for any $s, t \in K$ with $s \neq t$.*

Proof. For any $s \in K$ let A_s be the following submodule of M: $A_s = \{a + sf(a) \mid a \in A\}$. Then, the map $a \mapsto a + sf(a)$ is an isomorphism from A to A_s. On the other hand, $A_s \cap B = 0$ and $A_s \oplus B = M$. Assume $s \neq t$. Then, we have $A_s \cap A_t = \mathrm{Ker}f$. We also note that $(s - t)f(a) = (a + sf(a)) - (a + tf(a))$ for any $a \in A$. Consequently, $B = f(A) \subseteq A_s + A_t$. Since $A \subseteq A_s + B$, it follows that $M = A \oplus B \subseteq A_s + A_t$ and so $A_s + A_t = M$. □

REMARK 3.19. It is easy to find a module M, which is not $D4$, such that M admits exactly one isoclass, say $[A]$, of submodules with more than one element. Indeed, let R be

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the algebra considered in Example 3.1, and let M be the injective cogenerator $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1$. Then, as already observed, $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ contains $|K|$ submodules, all of the form considered in the proof of Proposition 3.18. On the other hand, [1] and [2] contain only one submodule.

The next example shows that a projective generator may be the direct summand of a non-projective *D*3-module.

EXAMPLE 3.20. There are a finite dimensional K-algebra R and a finite dimensional module M_R of the form $R \oplus N$ such that N is not projective and M is a *D*3-module.

Construction: Let R be the Kronecker algebra given by the quiver $\bullet_1 \rightrightarrows \bullet_2$, and let M denote the module $\begin{smallmatrix}1\\2\ 2\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}1\ 1\\2\end{smallmatrix}$. Then, we clearly have $\mathrm{Hom}_R\left(\begin{smallmatrix}1\\2\ 2\end{smallmatrix}, 2\right) = 0$, $\mathrm{Hom}_R\left(\begin{smallmatrix}1\ 1\\2\end{smallmatrix}, \begin{smallmatrix}1\\2\ 2\end{smallmatrix} \oplus 2\right) = 0$ and there is no surjective homomorphism $\begin{smallmatrix}1\\2\ 2\end{smallmatrix} \oplus 2 \to \begin{smallmatrix}1\ 1\\2\end{smallmatrix}$. Since the theorem of Krull–Remak–Schmidt holds in the category of finitely generated R-modules [**23**, p. 3], we conclude that there is no surjective homomorphism $f : A \to B$ with the property that B is indecomposable and $M = A \oplus B$. By Lemma 3.8, this implies that M is a *D*3-module of projective dimension one. Note also that M_R is neither Rickart nor quasi-projective. Indeed, take a morphism $f : M \to M$ such that $f\left(2 \oplus \begin{smallmatrix}1\ 1\\2\end{smallmatrix}\right) = 0$ and $0 \neq f\left(\begin{smallmatrix}1\\2\ 2\end{smallmatrix}\right) \subseteq \begin{smallmatrix}1\ 1\\2\end{smallmatrix}$. Then, $\mathrm{Ker} f = 2 \oplus 2 \oplus \begin{smallmatrix}1\ 1\\2\end{smallmatrix}$ is not a direct summand of M. Hence, M is not Rickart. Finally, let S denote the simple module 1 and let $f : M \to S$ and $g : M \to S$ be two epimorphisms such that $f\left(2 \oplus \begin{smallmatrix}1\ 1\\2\end{smallmatrix}\right) = 0$ and $g\left(\begin{smallmatrix}1\\2\ 2\end{smallmatrix} \oplus 2\right) = 0$. Using again the fact that $\begin{smallmatrix}1\ 1\\2\end{smallmatrix}$ is fully invariant in M, we have $(f \circ h)\left(\begin{smallmatrix}1\ 1\\2\end{smallmatrix}\right) \subseteq f\left(\begin{smallmatrix}1\ 1\\2\end{smallmatrix}\right) = 0 \neq g\left(\begin{smallmatrix}1\ 1\\2\end{smallmatrix}\right)$ for any endomorphism h of M. Hence, M is not quasi-projective.

LEMMA 3.21. *Let R be a K-algebra and let M_R be a module with the following properties:*

(i) *$M = P \oplus X$ with P projective, X fully invariant and* $\mathrm{End}_R(X) \cong K$.
(ii) *If $\sigma \in \mathrm{End}_R(M)$ and $X \subseteq \mathrm{Im}\sigma$, then we have $\sigma(X) = X$.*

Then, M is semi-projective.

Proof. Assume $\alpha, \beta \in \mathrm{End}_R(M)$ and $\mathrm{Im}\beta \subseteq \mathrm{Im}\alpha$. We first deduce from (i) that

(1) There is a morphism $\delta : P \to M$ such that $\overline{\alpha} \circ \delta = \overline{\beta}$, where $\overline{\alpha} : M \to \mathrm{Im}\alpha$ is the epimorphism induced by α and $\overline{\beta} : P \to \mathrm{Im}\alpha$ is the morphism induced by β.
We next deduce from (i) that

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(2) There exist $a, b \in K$ such that $\alpha(x) = ax$ and $\beta(x) = bx$ for any $x \in X$.
Since $bX = \beta(X) \subseteq \mathrm{Im}\beta \subseteq \mathrm{Im}\alpha$, we deduce from (ii) that
(3) $b \neq 0$ implies $a \neq 0$.
Finally, let $\gamma : M \to M$ be the endomorphism defined by the formula
(4) $\gamma(m) = \delta(m)$ if $m \in P$
$= 0$ if $m \in X$ and $b = 0$
$= a^{-1}bm$ if $m \in X$ and $b \neq 0$.

Then, we deduce from (1) and (2) that $\alpha \circ \gamma = \beta$, as desired. □

REMARK 3.22. (1) The module M_R in Example 3.20 satisfies conditions (i) and (ii) of the above lemma. Therefore, it is semi-projective, as well. However, as already observed in Example 3.20, M_R is neither Rickart nor quasi-projective.

(2) Let N be an indecomposable module over the Kronecker algebra R satisfying the hypotheses of Example 3.20. Since N is not cyclic (as we shall see in Corollary 3.25(a)), the classification of Kronecker modules ([**2**, Chapter VIII] or [**22**, p. 122]) tells us that one of the following cases occurs:

- N is preprojective and $dim N \geq 5$;
- N is regular and $dim N \geq 4$;
- N is preinjective and $dim N \geq 3$.

Hence, the module $\begin{smallmatrix} & 1 & 1 \\ 2 & & \end{smallmatrix}$ has the smallest possible dimension. Moreover, it is easy to see that the modules $D = A \oplus B$ with A preprojective and B preinjective and containing the module $\begin{smallmatrix} & 1 & \\ 2 & & 2 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} & 1 & 1 \\ 2 & & \end{smallmatrix}$ of Example 3.20 as a proper direct summand are not $D4$. Indeed, let X be an indecomposable preprojective or preinjective module of dimension ≥ 5. Then, X is of the form $\begin{smallmatrix} 1 & \cdots & 1 \\ 2 & \cdots & 2 \end{smallmatrix}$ or $\begin{smallmatrix} 1 & \cdots & 1 \\ & 2 \cdots 2 & \end{smallmatrix}$. Consequently, there is an obvious non-split epimorphism $X \to \begin{smallmatrix} & 1 & 1 \\ 2 & & \end{smallmatrix}$. Hence, $\begin{smallmatrix} & 1 & 1 \\ 2 & & \end{smallmatrix} \oplus X$ is not a $D4$-module. Applying again Lemma 2.3, we conclude that $\begin{smallmatrix} & 1 & \\ 2 & & 2 \end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix} & 1 & 1 \\ 2 & & \end{smallmatrix} \oplus X$ and $D = A \oplus B$ are not $D4$-modules. Note that we have $A = \begin{smallmatrix} & 1 & \\ 2 & & 2 \end{smallmatrix} \oplus 2 \oplus A_1$ and $B = \begin{smallmatrix} & 1 & 1 \\ 2 & & \end{smallmatrix} \oplus B_1$, where A_1 (resp. B_1) is preprojective (resp. preinjective) and at least one of the direct summands A_1 and B_1 is different from zero.

As the following corollary shows, projective and injective modules play a symmetric role as direct summands of $D4$-modules.

COROLLARY 3.23. *There exist K-algebras R and $D3$-modules of the form $P \oplus I$ such that, among others, the following cases are possible:*

(a) *P is the regular module R and I is an indecomposable injective module which is neither simple, nor projective, nor a cogenerator.*

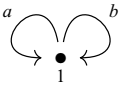
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(b) *I is the minimal injective cogenerator E and P is a simple projective module which is neither injective nor a generator.*
(c) *P is the regular module R, I is the minimal injective cogenerator E, and both P and I are indecomposable.*

Proof. (a) This follows from Example 3.20 with $P = \begin{smallmatrix} & 1 & \\ 2 & & 2 \end{smallmatrix} \oplus 2$ and $I = \begin{smallmatrix} 1 & & 1 \\ & 2 & \end{smallmatrix}$.

(b) This follows from Example 3.9 with $I = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ and $P = 2$.

(c) Let R be the K-algebra given by the quiver



with relations $a^2 = b^2 = ab = ba = 0$. Then, $R = \begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix}$, $E = \begin{smallmatrix} 1 & & 1 \\ & 1 & \end{smallmatrix}$ and there are no epimorphisms of the form $\begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix} \to \begin{smallmatrix} 1 & & 1 \\ & 1 & \end{smallmatrix}$ and $\begin{smallmatrix} 1 & & 1 \\ & 1 & \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix}$. Hence, the same arguments used in Example 3.20 prove that $\begin{smallmatrix} & 1 & \\ 1 & & 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & 1 \\ & 1 & \end{smallmatrix}$ is a $D3$-module satisfying (c). □

REMARK 3.24. Since $\begin{smallmatrix} 1 & & 1 \\ & 2 & \end{smallmatrix} \oplus 1$ and $\begin{smallmatrix} & 1 & \\ 1 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$ are not $D4$-modules, we deduce from Lemma 2.3 that the algebras used to prove conditions (a) and (b) of above corollary do not satisfy condition (c) of the same corollary.

COROLLARY 3.25. *Let M_R be a $D4$-module of the form $F \oplus X$, where F is a nonzero free module of rank d. The following facts hold:*

(a) *If Y is a direct summand of X generated by d elements, then Y is projective.*
(b) *If $d = 1$, then X may admit an indecomposable non-projective direct summand generated by two elements and $M \oplus M$ is not necessarily a $D4$-module.*
(c) *$M^{(I)}$ is a $D4$-module for any set I if and only if X is projective.*

Proof. (a) The hypotheses that F is free of rank d and that Y can be generated by d elements guarantee the existence of an epimorphism $f : F \to Y$. Since $F \oplus Y$ is a $D4$-module (Lemma 2.3), it follows that Y is isomorphic to a direct summand of F. Hence, Y is projective.

(b) This follows from Example 3.20, where $F = \begin{smallmatrix} & 1 & \\ 2 & & 2 \end{smallmatrix} \oplus 2$ and $X = \begin{smallmatrix} 1 & & 1 \\ & 2 & \end{smallmatrix}$. Indeed, $\begin{smallmatrix} 1 & & 1 \\ & 2 & \end{smallmatrix}$ is generated by two elements. Moreover, there is a non-split epimorphism $\begin{smallmatrix} 1 & & 1 \\ 2 & & 2 \end{smallmatrix} \oplus \begin{smallmatrix} 1 & & 1 \\ 2 & & 2 \end{smallmatrix} \to$

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$\begin{smallmatrix}1&1\\&2\end{smallmatrix}$. Hence, the summand $\begin{smallmatrix}1\\2&2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2&2\end{smallmatrix} \oplus \begin{smallmatrix}1&1\\&2\end{smallmatrix}$ of $M \oplus M$ is not a $D4$-module. This remark and Lemma 2.3 prove that $M \oplus M$ is not a $D4$-module.

(c) Suppose first X is projective. Then, $M^{(I)}$ is a projective module, hence a $D4$-module for any set I. Assume now that X is not projective. Then, there is a non-split epimorphism $F^{(I)} \to X$ for some set I. Consequently, the summand $F^{(I)} \oplus X$ of $M^{(I)}$ is not a $D4$-module. This observation and Lemma 2.3 prove that $M^{(I)}$ is not a $D4$-module. □

REMARK 3.26. The previous result suggests the following question: Are there non-projective modules M_R such that $M^{(I)}$ is a $D4$-module for every set I? The positive answer to this question follows from Example 3.4, where R is a finite dimensional algebra. Indeed, let M be the module $\begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$ and let $f : M^{(I)} \to M^{(I)}$ be a non-injective endomorphism. Then, $\mathrm{Ker}f$ is a direct sum of indecomposable injective modules isomorphic to $\begin{smallmatrix}1\\2\end{smallmatrix}$ or $\begin{smallmatrix}1\\3\end{smallmatrix}$. Since R is noetherian, it follows that $\mathrm{Ker}f$ is injective. Hence, $\mathrm{Ker}f$ is a direct summand of $M^{(I)}$, and so $M^{(I)}$ is a Rickart module. Consequently, $M^{(I)}$ is a $D4$-module.

We describe in the sequel another consequence of Example 3.20.

PROPOSITION 3.27. *Let M, A, B be nonzero modules such that $M = A \oplus B$ and $\mathrm{Hom}_R(B, A) = 0$. The following facts hold:*

(i) *If there is an epimorphism $f : A \to B$, then M is not a $D4$-module.*
(ii) *If there exist finitely many morphisms $f_i : A \to B$ such that $f_1(A) + \cdots + f_n(A) = B$, then M may be a $D3$-module.*

Proof. (i) This follows from the hypothesis that $\mathrm{Hom}_R(B, A) = 0$.

(ii) Let M be the $D3$-module $\begin{smallmatrix}1\\2&2\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}1&1\\&2\end{smallmatrix}$ considered in Example 3.20. Next let $A = \begin{smallmatrix}1\\2&2\end{smallmatrix} \oplus 2$ and let $B = \begin{smallmatrix}1&1\\&2\end{smallmatrix}$. Since A is projective and B is injective, we have $\mathrm{Hom}_R(B, A) = 0$. On the other hand, there is an obvious epimorphism $\begin{smallmatrix}1\\2&2\end{smallmatrix} \oplus \begin{smallmatrix}1\\2&2\end{smallmatrix} \to \begin{smallmatrix}1&1\\&2\end{smallmatrix}$. Hence, there are two morphisms $f_1, f_2 : A \to B$ such that $f_1(A) + f_2(A) = B$. Thus, (ii) holds. □

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