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Dagger extension theorem[†]

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Partial iterative theories are algebraic theories such that for certain morphisms f the equation $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$ has a unique solution. Iteration theories are algebraic theories satisfying a certain set of identities. We investigate some similarities between partial iterative theories and iteration theories.

In our main result, we give a sufficient condition ensuring that the partially defined dagger operation of a partial iterative theory can be extended to a totally defined operation so that the resulting theory becomes an iteration theory. We show that this general extension theorem can be instantiated to prove that every Elgot iterative theory with at least one constant morphism $1 \to 0$ can be extended to an iteration theory. We also apply our main result to theories equipped with an additive structure.

1. Introduction

Fixed-point operations occur in virtually all areas of theoretical computer science including automata and languages, semantics of programming languages, process algebra, logical theories of computational systems, programming logics, recursive types and proof theory, computational complexity, and so on. The equational properties of the fixed-point or dagger operation can be best described in the context of Lawvere theories of functions over a set equipped with structure, or, more generally, in the context of abstract Lawvere theories (or just theories), or cartesian or co-cartesian categories, *cf.* Lawvere (1963), Bloom and Ésik (1993), Elgot (1973) and Simpson and Plotkin (2000).

Iteration theories were introduced in Bloom *et al.* (1980a), and, independently, in Ésik (1980) in order to describe the equational properties of the dagger operation in iterative and rational algebraic theories, *cf.* Elgot (1973) and Wright *et al.* (1976). In an iterative theory, dagger is defined by unique fixed points, and in rational theories, by least fixed points. In both types of theories, the dagger operation satisfies the same set of identities. These identities define iteration theories. In Simpson and Plotkin (2000), it is argued that any non-trivial fixed-point model satisfies the iteration theory identities.

The dagger operation is necessarily a partial operation in (non-trivial) iterative theories. It was shown in Bloom *et al.* (1980b) and Ésik (1982) (see also Bloom and Ésik (1993,

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Theorem 6.4.5)), that any iterative theory with at least one 'constant' (that is, morphism $1 \to 0$) can be turned into an iteration theory with a total dagger operation. Moreover, the extension of the dagger operation to a total operation only depends on the choice of the constant that serves as the canonical solution of the fixed-point equation associated with the identity morphism $1 \to 1$.

In this paper, we provide a generalisation of this construction that is applicable to partial iterative theories. In the main result, we give a sufficient condition ensuring that a partially defined dagger operation of a partial iterative theory can be extended to a total operation so that the resulting theory becomes an iteration theory. We show that this general result can be instantiated to prove that every iterative theory with at least one constant can be extended to an iteration theory. We also apply our main result to theories equipped with an additive structure. We show that our result implies the Matrix Extension Theorem of Bloom and Ésik (1993) and the Grove Extension Theorem of Bloom and Ésik (2003). In the context of these theories, the extension theorem asserts that if we have unique solutions of certain 'guarded' fixed-point equations, then under certain conditions, the fixed-point operation can be extended in a unique way to provide solutions to all fixed-point equations such that the resulting theory becomes an iteration theory. Possible applications of these results include Process Algebra, where one usually deals with unique fixed points of guarded fixed-point equations (*cf.* Fokkink (2007)).

2. Basic definitions

In this section we review some basic concepts used in this paper – for more details, see Bloom and Ésik (1993). In any category whose objects are the non-negative integers, we will use $f \cdot g$ to denote the composite of the morphisms $f : n \to p$ and $g : p \to q$ in diagrammatical order. We will use $\mathbf{1}_p$ to denote the identity morphism corresponding to object p, and when n is a non-negative integer, we will use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. Thus, $[0]$ is the empty set.

Recall from Bloom and Ésik (1993) that a *(Lawvere) theory* T is a small category with objects the non-negative integers such that each non-negative integer n is the n-fold coproduct of the object 1 with itself. A *scalar morphism* is a morphism $1 \to p$, for some $p \geqslant 0$. We assume that each theory T comes with distinguished coproduct injections $i_n : 1 \to n$, $i \in [n]$, called *distinguished morphisms*, turning n into an n-fold coproduct of object 1 with itself. By the coproduct property, for each finite sequence of scalar morphisms $f_1, \ldots, f_n : 1 \to p$, there is a unique morphism $f : n \to p$ such that $i_n \cdot f = f_i$, for each $i \in [n]$. This unique morphism is denoted by $\langle f_1, \ldots, f_n \rangle$. The operation implicitly defined by the coproduct property is called *tupling*. In particular, when $n = 0$, tupling defines a unique morphism $0_p : 0 \to p$, for each $p \geqslant 0$. Note that $\mathbf{1}_n = \langle 1_n, \ldots, n_n \rangle$ for all non-negative integers n. In addition, we will always assume that $\mathbf{1}_1 = 1_1$, so $\langle f \rangle = f$ for each $f : 1 \to p$. A theory T is said to be *trivial* if $1_2 = 2_2$. In a trivial theory, there is at most one morphism $n \to p$ for each $n, p \geqslant 0$.

Tuplings of distinguished morphisms are called *base morphisms*. For example, 0_n and $\mathbf{1}_n$ are base morphisms. When ρ is a mapping $[n] \to [p]$, there is an associated base morphism $n \to p$, which is the tupling of the distinguished morphisms $(1\rho)_p, \ldots, (n\rho)_p$. A base permutation is a base morphism associated with a bijective mapping. Injective,

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surjective and monotone base morphisms are defined in the same way. Note that in any theory, a base permutation corresponding to the bijection $\pi : [n] \to [n]$ is an isomorphism with inverse the base permutation corresponding to the inverse function of π.

When $f : n \to p$ and $g : m \to p$ in a theory T, we define $\langle f, g \rangle$ to be the morphism $h : n + m \to p$ with $i_{n+m} \cdot h = i_n \cdot f$ and $(n + j)_{n+m} \cdot h = j_m \cdot g$ for all $i \in [n]$ and $j \in [m]$. Moreover, for each $f : n \to p$ and $g : m \to q$, we define

$$f \oplus g = \langle f \cdot \kappa, g \cdot \lambda \rangle : n + m \to p + q,$$

where κ is the base morphism corresponding to the inclusion $[p] \to [p + q]$ and λ is the base morphism corresponding to the translated inclusion $[q] \to [p + q]$ mapping j in $[q]$ to $p + j$ in $[p + q]$, for all $j \in [q]$. Note that the *pairing* operation $\langle f, g \rangle$ and the *separated sum* operation $f \oplus g$ are associative. Moreover,

$$\langle f, 0_p \rangle = f = \langle 0_p, f \rangle$$
$$g \oplus 0_0 = g = 0_0 \oplus g$$

for all $f : n \to p$ and $g : m \to n$.

Also,

$$\langle f, g \rangle \cdot h = \langle f \cdot h, g \cdot h \rangle$$

for all $f : n \to p$, $g : m \to p$ and $h : p \to q$, and

$$(f \oplus g) \cdot \langle h, k \rangle = \langle f \cdot h, g \cdot k \rangle$$

for all $f : n \to p$, $g : m \to q$, $h : p \to r$ and $k : q \to r$.

Finally,

$$(f \oplus g) \cdot (h \oplus k) = (f \cdot h) \oplus (g \cdot k)$$

for all appropriate morphisms f, g, h, k.

A *morphism* $T \to T'$ between theories T and T' is a functor that preserves the objects and the distinguished morphisms. It follows that any theory morphism preserves the pairing, tupling and separated sum operations. Finally, a theory T is a *subtheory* of a theory T' if T is a sub-category of T' and has the same objects and distinguished morphisms as T.

Example 2.1. A basic example of a theory is the *theory of functions over a set A*. In this theory, a morphism $n \to p$ is a function $f : A^p \to A^n$. Note the reversal of the arrow. The composite of morphisms $f : n \to p$ and $g : p \to q$ is their function composition written from right to left, which is a function $A^q \to A^p$. The distinguished morphisms are the projection functions.

Example 2.2. Let $S = (S, +, \cdot, 0, 1)$ be a semiring (Golan 1999). The *theory of matrices* Mat_S *over* S has as morphisms $n \to p$ all $n \times p$ matrices in $S^{n \times p}$. Composition is matrix multiplication defined in the usual way. For each $i \in [p]$, $p \geqslant 0$, the distinguished morphism $i_p : 1 \to p$ is the $1 \times p$ row matrix with a 1 on the ith position whose other entries are all 0. It is known that in each matrix theory, each object n is also the n-fold product of the object 1 with itself. The transposes i_n^T of the distinguished morphisms serve as the projection morphisms $n \to 1$ (Bloom and Ésik 1993; Elgot 1976). The theory Mat_S comes

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with a sum operation $+$ defined on each hom-set $\mathrm{Mat}_S(n,p) = S^{n\times p}$. For each $n, p \geqslant 0$, $(\mathrm{Mat}_S(n,p), +, 0_{n,p})$ is a commutative monoid, where $0_{n,p}$ is the $n \times p$ matrix whose entries are all 0. Moreover, composition distributes over finite sums on both the left and the right. Thus, for each n, $(\mathrm{Mat}_S(n,n), +, \cdot, \mathbf{1}_n, 0_{n,n})$ is itself a semiring, since the product of two $n \times n$ matrices is an $n \times n$ matrix. In particular, $\mathrm{Mat}_S(1,1)$ is isomorphic to S.

Example 2.3. A *ranked alphabet* Σ is a family of pairwise disjoint sets $(\Sigma_n)_n$, where n ranges over the non-negative integers. We assume familiarity with the notion of (total) Σ-*trees* over a set $X_p = \{x_1, \ldots, x_p\}$ of variables, defined as usual – see, for example, Bloom and Ésik (1993). Below we will denote the collection of finite and infinite trees by $T_\Sigma(X_p)$. We say that a tree is *proper* if it is not one of the trees x_i. Σ-trees form a theory $\Sigma\mathrm{TR}$ whose morphisms $n \to p$ are all n-tuples of trees in $T_\Sigma(X_p)$. Composition is defined by substitution for the 'variables' x_i, and for $i \in [p]$ the tree x_i serves as the ith distinguished morphism $1 \to p$. Thus, if $t : 1 \to n$ and $t'_1, \ldots, t'_n : 1 \to p$ in $\Sigma\mathrm{TR}$, then $t \cdot \langle t'_1, \ldots, t'_n \rangle : 1 \to p$ is the tree obtained by substituting a copy of t'_i for each leaf of t labelled x_i, for $i \in [p]$ – see Bloom and Ésik (1993) for details. A tree is said to be *regular* if, up to isomorphism, it has a finite number of subtrees. The subtheory of $\Sigma\mathrm{TR}$ containing only the regular Σ-trees is denoted by $\Sigma\,\mathrm{tr}$, and the subtheory containing only the finite Σ-trees is denoted by $\Sigma\,\mathrm{Term}$. As usual, we identify each letter in Σ_n with a tree in $T_\Sigma(X_n)$. It is known that $\Sigma\,\mathrm{Term}$ is freely generated by Σ in the category of theories. In particular, when Σ is empty, $\Sigma\,\mathrm{Term}$ is an initial theory.

Example 2.4. The theory Θ has as morphisms $n \to p$ all functions $[n] \to [p]$. Composition is defined by function composition written from left to right. For each n and $i \in [n]$, the distinguished morphism $1 \to n$ is the function $[1] \to [n]$ selecting the integer i. When T is a non-trivial theory, the base morphisms form a subtheory of T isomorphic to Θ. Moreover, when Σ is the empty ranked alphabet, the theory $\Sigma\mathrm{TR}$ (or $\Sigma\,\mathrm{Term}$) is isomorphic to Θ.

3. Partial iteration theories

Let T be a theory. A nonempty collection of morphisms I is an *ideal* (Bloom and Ésik 1993) of T if it is closed under tupling, composition with base morphisms on the left, and composition with arbitrary morphisms on the right. Note that every ideal contains the morphisms 0_p, $p \geqslant 0$.

Definition 3.1. A *partial dagger theory* is a theory T equipped with a distinguished ideal $D(T)$ and a partially defined dagger operation

$$^\dagger : T(n, n+p) \to T(n,p),\ n, p \geqslant 0$$

defined on morphisms $n \to n+p$ in $D(T)$.

Let T, T' be partial dagger theories. A *partial dagger theory morphism* $\varphi : T \to T'$ is a theory morphism $T \to T'$ that preserves the distinguished ideal and the dagger operation, that is, $D(T)\varphi \subseteq D(T')$ and $(f^\dagger)\varphi = (f\varphi)^\dagger$ for all $f : n \to n+p$ in $D(T)$ and $n, p \geqslant 0$. We say that T is a *partial sub-dagger theory* of T' if T is a subtheory of T', $D(T) = D(T') \cap T$, and the dagger operation of T is the restriction of the dagger operation of T'.

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In the rest of this paper we will consider partial dagger theories satisfying certain identities that we will define now. For the origins of these identities, see de Bakker and Scott (1969), Bekić (1984), Ésik (1999), Niwinski (1985; 1986), Plotkin (1983) and Wright *et al.* (1976).

Definition 3.2. We say that the partial dagger theory T satisfies:

(1) the *fixed-point identity* if
$$f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$$
for each $f : n \to n+p$ in $D(T)$;

(2) the *left-zero identity* if
$$(0_n \oplus f)^\dagger = f$$
for each $f : n \to p$ in $D(T)$;

(3) the *right-zero identity* if
$$(f \oplus 0_q)^\dagger = f^\dagger \oplus 0_q$$
for each $f : n \to n+p$ in $D(T)$;

(4) the (*base*) *parameter identity* if
$$f^\dagger \cdot g = (f \cdot (\mathbf{1}_n \oplus g))^\dagger$$
for each $f : n \to n+p$ in $D(T)$ and $g : p \to q$ in T (such that g is a base morphism);

(5) the *permutation identity* if
$$(\pi \cdot f \cdot (\pi^{-1} \oplus \mathbf{1}_p))^\dagger = \pi \cdot f^\dagger$$
for each $f : n \to n+p$ in $D(T)$ and base permutation $\pi : n \to n$ with inverse π^{-1};

(6) the *pairing identity* (or *Bekić identity*), if for all $f : n \to n+m+p$ and $g : m \to n+m+p$ in $D(T)$,
$$\langle f, g \rangle^\dagger = \langle f^\dagger \cdot \langle h^\dagger, \mathbf{1}_p \rangle, h^\dagger \rangle,$$
where
$$h = g \cdot \langle f^\dagger, \mathbf{1}_{m+p} \rangle : m \to m+p;$$

(7) the *double-dagger identity* if
$$(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger = (f \cdot \langle f^\dagger, \mathbf{1}_{n+p} \rangle)^\dagger$$
for each $f : n \to n+n+p$ in $D(T)$;

(8) the *composition identity* if
$$f \cdot \langle (g \cdot \langle f, 0_m \oplus \mathbf{1}_p \rangle)^\dagger, \mathbf{1}_p \rangle = (f \cdot \langle g, 0_n \oplus \mathbf{1}_p \rangle)^\dagger$$
for each $f : n \to m+p$ and $g : m \to n+p$ in $D(T)$;

(9) the *simplified composition identity* if
$$(f \cdot g)^\dagger = f \cdot (g \cdot (f \oplus \mathbf{1}_p))^\dagger$$
for each $f : n \to m$ and $g : m \to n+p$ in $D(T)$;

(10) the *group identity* $C(S)$ associated with the finite group $S = ([n], \circ)$ if
$$g_S^\dagger = \tau_n \cdot (g \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger$$

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for each $g : 1 \to n + p$ in $D(T)$, where

$$g_S = \langle g \cdot (\rho_1^S \oplus \mathbf{1}_p), \ldots, g \cdot (\rho_n^S \oplus \mathbf{1}_p) \rangle,$$

τ_n is the unique base morphism $n \to 1$ and

$$\rho_i^S = \langle (i \circ 1)_n, \ldots, (i \circ n)_n \rangle$$

for each $i \in [n]$, where $\circ$ is the group operation of S.

Remark 3.3. Note that the composition identity implies the simplified composition identity and the fixed-point identity implies the left-zero identity. If the fixed-point identity holds, then $f^\dagger$ is in $D(T)$ for each $f : n \to n+p$ in $D(T)$, since $f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$. Then we say that T satisfies:

(11) the *simplified form of the double-dagger identity* if

$$(f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger = f^{\dagger\dagger}$$

for each $f : n \to n + n + p$ in $D(T)$.

If the fixed-point identity holds, the simplified form of the double-dagger identity is equivalent to the double-dagger identity.

The *scalar versions* of the fixed-point, left-zero, right-zero, (base) parameter and double-dagger identities are obtained by taking $n = 1$ in the corresponding identity. The scalar versions of the composition and simplified composition identities are obtained by taking $n = m = 1$ in those identities.

Definition 3.4. A *partial Conway theory* is a partial dagger theory satisfying the fixed-point, right-zero, pairing, double dagger and permutation identities. A *partial iteration theory* is a partial Conway theory satisfying the group identity $C(S)$ for each finite group S. A morphism of partial Conway or iteration theories is a partial dagger theory morphism.

A partial Conway theory T is a *partial sub-Conway theory* of the partial Conway theory T' if T is a partial sub-dagger theory of T'. Similarly, a partial iteration theory T is a *partial subiteration theory* of the partial iteration theory T' if T is a partial sub-dagger theory of T'. The following fact is known from Bloom and Ésik (1993).

Proposition 3.5. Except for the group identities, all of the identities listed above hold in all partial Conway theories.

As shown in Elgot (1973) and Bloom and Ésik (1993), unique solutions to fixed-point equations in algebraic theories give rise to partial iteration theories. We give the details of this result below.

Definition 3.6. A *partial iterative theory* is a theory T with a distinguished ideal $D(T)$ such that for each $f : n \to n+p$ in $D(T)$, the fixed-point equation $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$ associated with f has a unique solution in T.

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A *morphism* $T \to T'$ of partial iterative theories is a theory morphism φ with $D(T)\varphi \subseteq D(T')$. Every partial iterative theory T is a partial dagger theory with dagger operation that maps a morphism $f : n \to n+p$ in $D(T)$ to the unique morphism $f^\dagger : n \to p$ with $f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle$. It is clear that any morphism of iterative theories is a dagger theory morphism.

The following fact follows from the results in Bloom *et al.* (1977).

Remark 3.7. Suppose T is a theory with distinguished ideal $D(T)$. If the equation $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$ has a unique solution for each scalar morphism $f : 1 \to 1+p$ in $D(T)$, where ξ ranges over the set of morphisms $1 \to p$, then T is a partial iterative theory.

By the following result from Bloom and Ésik (1993), the prime examples of partial iteration theories are the partial iterative theories.

Theorem 3.8. Every partial iterative theory is a partial iteration theory.

Definition 3.9. A *dagger theory* is a partial dagger theory T with $D(T) = T$. A *Conway theory* (*iteration theory*) is a dagger theory that is a partial Conway theory (partial iteration theory). A morphism of these theories is a partial dagger theory morphism.

Note that a partial dagger theory T is a dagger theory if and only if $D(T)$ contains $\mathbf{1}_1$, or at least one distinguished morphism. The axioms of Conway theories may be simplified. The following fact is known, *cf.* Bloom and Ésik (1993, Chapter 6).

Proposition 3.10. Suppose T is a dagger theory. Then the following are equivalent:

— T satisfies the left-zero, right-zero, pairing and permutation identities.
— T satisfies the base-parameter, composition and double-dagger identities.
— T satisfies the fixed-point, base-parameter, simplified composition and double-dagger identities.
— T satisfies the scalar versions of the parameter, composition and double-dagger identities, and the pairing identity for $m = 1$.
— T satisfies the scalar versions of the fixed-point, parameter, simplified composition and double-dagger identities, and the pairing identity for $m = 1$.

Example 3.11. Examples of iteration theories are the theories of continuous or monotone functions over complete partial orders equipped with the *least fixed-point operation* as dagger – see Bloom and Ésik (1993) for details.

Example 3.12. Let Σ be a ranked alphabet and let T be the theory ΣTR, or the theory Σ tr. Let the ideal $D(T)$ consist of those morphisms $f : n \to p$ in T whose components $i_n \cdot f$, $i \in [n]$, are proper trees. It is known that for each $f : n \to n+p$ in $D(T)$, the equation

$$\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$$

has a unique solution in the variable $\xi : n \to p$. Using $f^\dagger$ to denote this unique solution, T becomes a partial iterative theory. Moreover, if Σ_0 is not empty, so that there is at least one morphism in $T(1,0)$, then for any choice of a morphism $\perp : 1 \to 0$, the partial dagger operation can be *uniquely* extended to a totally defined dagger operation such that T

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becomes an iteration theory – see Bloom *et al.* (1980a) and Ésik (1982), or Bloom and Ésik (1993).

Example 3.13. Suppose Σ contains a single letter $\perp$ that has rank 0. Then any scalar morphism in $\Sigma\mathrm{TR}$ is either a distinguished morphism or a morphism $\perp_{1,p} = \perp \cdot 0_p$. Given $f : n \to n+p$, we have

$$f^\dagger = f^n \cdot \langle \perp_{n,p}, \mathbf{1}_p \rangle,$$

where

$$\begin{aligned} \perp_{n,p} &= \langle \perp_{1,p}, \ldots, \perp_{1,p} \rangle : n \to p \\ f^0 &= \mathbf{1}_n \oplus 0_p \\ f^{k+1} &= f \cdot \langle f^k, 0_n \oplus \mathbf{1}_p \rangle. \end{aligned}$$

We use $\perp\mathrm{TR}$ to denote this iteration theory. By the previous example, $\perp\mathrm{TR}$ is an initial iteration theory.

Example 3.14. Let Θ' be the theory whose morphisms $n \to p$ are the partial functions $[n] \to [p]$. Composition is function composition and the distinguished morphisms are defined as in the theory Θ. For each $n, p \geqslant 0$, let $\perp_{n,p}$ denote the totally undefined partial function $[n] \to [p]$. For each $\rho : n \to n+p$, define $\rho^\dagger = \rho^n \cdot \langle \perp_{n,p} \mathbf{1}_p \rangle$. Then Θ' is an iteration theory isomorphic to $\perp\mathrm{TR}$. Thus, Θ' is also an initial iteration theory. It is known that Θ' is also an initial Conway theory.

Example 3.15. Each non-trivial Conway theory T contains a subtheory isomorphic to Θ'. Given a partial function $\rho : [n] \to [p]$, we define the corresponding *partial base morphism* in T as the morphism $\langle f_1, \ldots, f_n \rangle$, where for each $i \in [n]$, we have $f_i = j_p$ if $i\rho = j$ is defined, and $f_i = \perp \cdot 0_p = \perp_{1,p}$ if $i\rho$ is not defined. Here, $\perp = \mathbf{1}_1{}^\dagger$.

4. Dagger Extension Theorem

In this section we give a sufficient condition ensuring that the dagger operation of a partial Conway or iteration theory is extendable to a total dagger operation such that the resulting theory becomes a Conway theory, or an iteration theory. Our 'Dagger Extension Theorem' (Theorem 4.4) is a generalisation of the Matrix Extension Theorem found in Bloom and Ésik (1993, pages 323–335), and the extension theorem for grove theories, found in Bloom and Ésik (2003).

In the following, we will make heavy use of the following notation. In any theory, we will use $\pi_{n,s}$, for $n, s \geqslant 0$, to denote the base morphism

$$\langle 0_n \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_s \rangle : s + n \to n + s.$$

Note that $\pi_{n,s}$ corresponds to the permutation $[s+n] \to [n+s]$ mapping each $i \in [s]$ to $n+i$ and each $j \in [s+n]$ with $j > s$ to $j - s$. We will indicate the identities used in a computation as subscripts of the corresponding equality sign, that is, given expressions t, t' denoting morphisms in partial dagger theories, we will write $t =_i t'$ to mean that $t = t'$ follows by application of the ith identity in Definition 3.2 or Remark 3.3.

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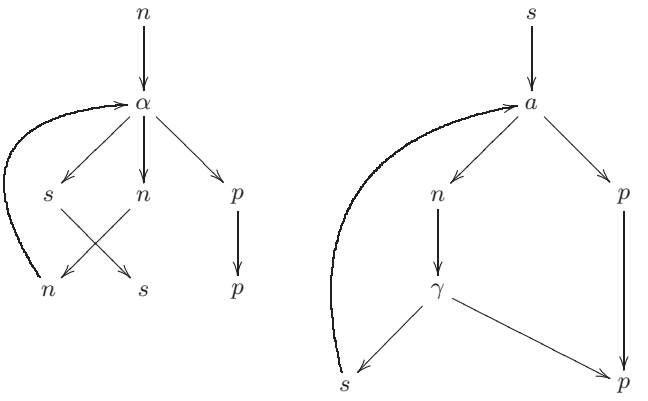


Fig. 1. γ is on the left and c is on the right.

Let T be a partial dagger theory with dagger operation ^{†_D} defined on the morphisms $f : n \to n+p$ in $D(T)$ and let T_0 be a subtheory of T. Suppose T_0 is a dagger theory with dagger operation $^{\dagger_0} : T_0(n, n+p) \to T_0(n, p)$, $n, p \geqslant 0$.

Definition 4.1. A *description* $(\alpha, a) : n \to q$ of weight s consists of a morphism $\alpha : n \to s+q$ in T_0 and a morphism $a : s \to q$ in $D(T)$. We write $|(\alpha, a)|$ for the morphism $\alpha \cdot \langle a, \mathbf{1}_q \rangle$ in T, and say this morphism is the *behaviour* of the description (α, a). Moreover, for a description $(\alpha, a) : n \to n+p$ of weight s, we define $(\alpha, a)^\wedge$ to be the description $(\gamma, c) : n \to p$ of weight s, where

$$\gamma = (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} : n \to s+p$$
$$c = (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} : s \to p,$$

see Figure 1.

Example 4.2. Consider $\{\bot, \sigma, \tau\}$TR, where the arities of $\bot$, σ and τ are $0, 1$ and 1, respectively. Let T_0 be the subtheory $\bot$TR of $\{\bot, \sigma, \tau\}$TR and let $D(\{\bot, \sigma, \tau\}\mathrm{TR})$ be the ideal containing those $(t_1, \ldots, t_n) : n \to p$ in $\bot, \sigma, \tau$TR such that the root of each t_i is labelled by σ.

Then (α, a) is a description $3 \to 5$ of weight 3, where $\alpha = \langle 2_5, \bot, 1_5 \rangle = (x_2, \bot, x_1)$ is $3 \to 8$ and $a = (\sigma(\tau(x_1)), \sigma(x_2), \sigma(x_2))$ is $3 \to 5$. Then $|(\alpha, a)| = \alpha \cdot \langle a, \mathbf{1}_5 \rangle = (\sigma(x_2), \bot, \sigma(\tau(x_1)))$.

Now let $(\gamma, c) = (\alpha, a)^\wedge$. Then $\gamma = (x_5, \bot, x_4)$ and $c = (\sigma(\tau(x_2)), \sigma(\bot), \sigma(\bot))$.

Proposition 4.3. If T is a Conway theory with dagger operation †, and ^{†_0} is the restriction of † to T_0, then for each description $(\alpha, a) : n \to n+p$ of weight s, we have $|(\alpha, a)^\wedge| = |(\alpha, a)|^\dagger$.

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Proof.

$$\begin{aligned}
|(\alpha, a)|^\dagger &= (\alpha \cdot \langle a, \mathbf{1}_{n+p}\rangle)^\dagger \\
&= (\alpha \cdot \langle a, \mathbf{1}_n \oplus 0_p, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&= (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_n \oplus 0_p, a, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&=_{11} (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus \langle a, 0_n \oplus \mathbf{1}_p\rangle))^{\dagger\dagger} \\
&=_4 ((\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^\dagger \cdot \langle a, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&= (\gamma \cdot \langle a, 0_n \oplus \mathbf{1}_p\rangle)^\dagger \\
&=_8 \gamma \cdot \langle (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p\rangle)^\dagger, \mathbf{1}_p\rangle \\
&= \gamma \cdot \langle c, \mathbf{1}_p\rangle = |(\gamma, c)| = |(\alpha, a)^\wedge|
\end{aligned}$$

where $\gamma = (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^\dagger$ and $c = (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p\rangle)^\dagger$. □

We will now give a sufficient condition ensuring that a partially defined dagger operation on the theory T be uniquely extendable to a totally defined dagger operation such that T becomes a Conway theory. Recall that each partial iterative theory T yields a partial dagger theory with a dagger operation that provides unique solutions to fixed-point equations $\xi = f \cdot \langle \xi, \mathbf{1}_p\rangle$ for all $f : n \to n + p$ in $D(T)$. We will use ^{†_D} to denote this operation.

Theorem 4.4. Let T be a partial iterative theory such that T is also a partial dagger theory with the operation ^{†_D} defined on the morphisms $f : n \to n + p$ in $D(T)$. Suppose the following hold:

(1) T_0 is a subtheory of T and a Conway theory with the operation $^{\dagger_0} : T_0(n, n + p) \to T_0(n, p)$, $n, p \geqslant 0$.
(2) Each morphism $n \to p$ in T can be written as $\alpha \cdot \langle a, \mathbf{1}_p\rangle$, where $\alpha : n \to s + p$ is in T_0 and $a : s \to p$ is in $D(T)$.
(3) For all $\alpha : n \to s + n + p$, $\alpha' : n \to r + n + p$ in T_0 and $a : s \to n + p$, $a' : r \to n + p$ in $D(T)$, we have

$$|(\alpha, a)| = |(\alpha', a')| \quad \Longrightarrow \quad |(\alpha, a)^\wedge| = |(\alpha', a')^\wedge|,$$

that is,

$$\alpha \cdot \langle a, \mathbf{1}_{n+p}\rangle = \alpha' \cdot \langle a', \mathbf{1}_{n+p}\rangle \quad \Longrightarrow \quad \gamma \cdot \langle c, \mathbf{1}_p\rangle = \gamma' \cdot \langle c', \mathbf{1}_p\rangle,$$

where

$$\begin{aligned}
(\gamma, c) &= (\alpha, a)^\wedge \\
(\gamma', c') &= (\alpha', a')^\wedge.
\end{aligned}$$

Then the operations ^{†_0} and ^{†_D} can be uniquely extended to a totally defined operation

$$^\dagger : T(n, n + p) \to T(n, p)$$

such that T equipped with † is a Conway theory. Moreover, if T_0 is an iteration theory, T is an iteration theory.

Proof. First note that for morphisms in both T_0 and $D(T)$ the operations ^{†_0} and ^{†_D} coincide since the fixed-point identity holds in T_0. For a morphism $f : n \to n + p$ in T, we

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define $f^\dagger = |(\alpha, a)^\wedge|$, where (α, a) is a description such that $f = |(\alpha, a)|$. By assumption (2), such a description exists, and by assumption (3), the operation † is well defined.

Our aim is to show that T, equipped with the dagger operation † is a Conway theory. To this end, we will prove that T satisfies the following identities:

(a) the fixed-point identity,
(b) the base-parameter identity,
(c) the double-dagger identity,
(d) the simplified composition identity.

We will use the following facts. Let $f : n \to n + p$ be an arbitrary morphism in T with $f = |(\alpha, a)|$, where $\alpha : n \to s + n + p$ in T_0 and $a : s \to n + p$ in $D(T)$. Moreover, let $(\alpha, a)^\wedge = (\gamma, c)$. Then

$$\gamma = \alpha \cdot \langle \mathbf{1}_s \oplus 0_p, \gamma, 0_s \oplus \mathbf{1}_p \rangle. \tag{1}$$

Indeed,

$$\begin{aligned} \gamma &= (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} \\ &=_1 \alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot \langle \gamma, \mathbf{1}_{s+p} \rangle \\ &= \alpha \cdot \langle \mathbf{1}_s \oplus 0_p, \gamma, 0_s \oplus \mathbf{1}_p \rangle. \end{aligned}$$

Moreover,

$$c = a \cdot \langle \gamma \cdot \langle c, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \tag{2}$$

since we have

$$\begin{aligned} c &= (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &=_1 (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle) \cdot \langle c, \mathbf{1}_p \rangle \\ &= a \cdot \langle \gamma \cdot \langle c, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle. \end{aligned}$$

Theorem 3.8 implies the fixed-point identity used in this computation. We now proceed to prove that the fixed-point, parameter, double-dagger and simplified composition identities hold in T.

(a) We will first show that T satisfies the fixed-point identity. Let $f : n \to n + p$ be an arbitrary morphism in T, and let (α, a) be a description of weight s with behaviour f. We calculate as follows:

$$\begin{aligned} f^\dagger &= |(\alpha, a)^\wedge| = \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle \mathbf{1}_s \oplus 0_p, \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot \langle c, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle c, \gamma \cdot \langle c, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle a, \mathbf{1}_{n+p} \rangle \cdot \langle \gamma \cdot \langle c, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= f \cdot \langle f^\dagger, \mathbf{1}_p \rangle. \end{aligned}$$

Here we have applied equation (1) in the third equation and equation (2) in the fifth. We have thus proved that the fixed-point identity holds in T for the operation †.

(b) We will now show that the base-parameter identity holds in T. Let $f : n \to n + p$ be an arbitrary morphism in T and $g : p \to q$ be a base morphism. Let (α, a) be a

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description of weight s with behaviour f. Let $(\gamma, c) = (\alpha, a)^\wedge$. So

$$\gamma = (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} : n \to s + p$$
$$c = (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} : s \to p.$$

Hence

$$f \cdot (\mathbf{1}_n \oplus g) = \alpha \cdot (\mathbf{1}_{s+n} \oplus g) \cdot \langle a \cdot (\mathbf{1}_n \oplus g), \mathbf{1}_{n+q} \rangle,$$

and since g is a base morphism, the morphism $\alpha \cdot (\mathbf{1}_{s+n} \oplus g)$ is in T_0. Thus

$$|(\alpha \cdot (\mathbf{1}_{s+n} \oplus g), a \cdot (\mathbf{1}_n \oplus g))| \; = \; f \cdot (\mathbf{1}_n \oplus g).$$

Defining $(\delta, d) = (\alpha \cdot (\mathbf{1}_{s+n} \oplus g), a \cdot (\mathbf{1}_n \oplus g))^\wedge$, we have

$$\delta = \gamma \cdot (\mathbf{1}_s \oplus g) \tag{3}$$

and

$$d = c \cdot g. \tag{4}$$

Indeed,

$$\begin{aligned} \delta &= (\alpha \cdot (\mathbf{1}_{s+n} \oplus g) \cdot (\pi_{n,s} \oplus \mathbf{1}_q))^{\dagger_0} \\ &= (\alpha \cdot (\pi_{n,s} \oplus g))^{\dagger_0} \\ &= (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\mathbf{1}_{n+s} \oplus g))^{\dagger_0} \\ &=_4 \gamma \cdot (\mathbf{1}_s \oplus g) \end{aligned}$$

and

$$\begin{aligned} d &= (a \cdot (\mathbf{1}_n \oplus g) \cdot \langle \delta, 0_s \oplus \mathbf{1}_q \rangle)^{\dagger_D} \\ &= (a \cdot \langle \gamma \cdot (\mathbf{1}_s \oplus g), 0_s \oplus g \rangle)^{\dagger_D} \\ &= (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot (\mathbf{1}_s \oplus g))^{\dagger_D} \\ &=_4 (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \cdot g \\ &= c \cdot g. \end{aligned}$$

Using these facts, we conclude with the following computation:

$$\begin{aligned} (f \cdot (\mathbf{1}_n \oplus g))^\dagger &= |(\delta, d)| \\ &= \delta \cdot \langle d, \mathbf{1}_q \rangle \\ &= \gamma \cdot (\mathbf{1}_s \oplus g) \cdot \langle d, \mathbf{1}_q \rangle \\ &= \gamma \cdot \langle d, g \rangle \\ &= \gamma \cdot \langle c \cdot g, g \rangle \\ &= \gamma \cdot \langle c, \mathbf{1}_p \rangle \cdot g \\ &= |(\gamma, c)| \cdot g \\ &= f^\dagger \cdot g. \end{aligned}$$

Here we have used equation (3) at the third step of this computation and equation (4) at the fifth. With this we have proved that the base-parameter identity holds in T.

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(c) We will now prove that the (simplified) double-dagger identity holds in T. Let $f : n \to n+n+p$ be an arbitrary morphism in T, and let (α, a) be a description of weight s with behaviour f. Let $(\beta, b) = (\alpha, a)^{\wedge}$ and $(\gamma, c) = (\beta, b)^{\wedge}$. Then, by definition,

$$\beta = (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_{n+p}))^{\dagger_0} : n \to s+n+p,$$
$$b = (a \cdot \langle \beta, 0_s \oplus \mathbf{1}_{n+p} \rangle)^{\dagger_D} : s \to n+p.$$

This yields

$$\gamma = (\alpha \cdot (\langle \pi_{n,s}, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^{\dagger_0}. \tag{5}$$

Indeed,

$$\begin{aligned}
\gamma &= (\beta \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} \\
&=_4 (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_{n+p}) \cdot (\mathbf{1}_n \oplus \pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0 \dagger_0} \\
&= (\alpha \cdot (\langle 0_{n+n} \oplus \mathbf{1}_s, \mathbf{1}_n \oplus 0_{n+s}, 0_n \oplus \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^{\dagger_0 \dagger_0} \\
&=_{11} (\alpha \cdot (\langle \pi_{n,s}, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^{\dagger_0}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
c &= (b \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\
&= ((a \cdot \langle \beta, 0_s \oplus \mathbf{1}_{n+p} \rangle)^{\dagger_D} \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\
&=_4 (a \cdot \langle \beta, 0_s \oplus \mathbf{1}_{n+p} \rangle \cdot (\mathbf{1}_s \oplus \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle))^{\dagger_D \dagger_D} \\
&=_{11} (a \cdot \langle \beta, 0_s \oplus \mathbf{1}_{n+p} \rangle \cdot \langle \mathbf{1}_s \oplus 0_p, \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D},
\end{aligned}$$

and, finally, using equation (1), we conclude that

$$c = (a \cdot \langle \gamma, \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D}. \tag{6}$$

On the other hand,

$$f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) = |(\delta, d)|, \tag{7}$$

since

$$\begin{aligned}
f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) &= \alpha \cdot \langle a, \mathbf{1}_{n+n+p} \rangle \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) \\
&= \alpha \cdot \langle (a \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p)), \langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p \rangle \\
&= \alpha \cdot (\mathbf{1}_s \oplus \langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) \cdot \langle d, \mathbf{1}_{n+p} \rangle \\
&= \delta \cdot \langle d, \mathbf{1}_{n+p} \rangle = |(\delta, d)|,
\end{aligned}$$

where $\delta = \alpha \cdot (\mathbf{1}_s \oplus \langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p)$ and $d = a \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p)$. We define $(\eta, h) = (\delta, d)^{\wedge}$. Then

$$\begin{aligned}
\eta &= (\delta \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} \\
&= (\alpha \cdot (\mathbf{1}_s \oplus \langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) \cdot (\pi_{n,s} \oplus \mathbf{1}_p))^{\dagger_0} \\
&= (\alpha \cdot (\langle \pi_{n,s}, \mathbf{1}_n \oplus 0_s \rangle \oplus \mathbf{1}_p))^{\dagger_0} \\
&= \gamma,
\end{aligned}$$

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where the last computation step uses equation (5). Moreover,

$$\begin{aligned} h &= (d \cdot \langle \eta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= (a \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p) \cdot \langle \eta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= (a \cdot \langle \eta, \eta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= c, \end{aligned}$$

where the last step uses equation (6). We can then conclude the proof of the claim as follows:

$$\begin{aligned} f^{\dagger\dagger} &= |(\alpha, a)^\wedge| \\ &= |(\beta, b)|^\dagger \\ &= |(\beta, b)^\wedge| \\ &= |(\gamma, c)| \\ &= |(\eta, h)| \\ &= |(\delta, d)^\wedge| \\ &= |(\delta, d)|^\dagger \\ &= (f \cdot (\langle \mathbf{1}_n, \mathbf{1}_n \rangle \oplus \mathbf{1}_p))^\dagger, \end{aligned}$$

where we have used Proposition 4.3 in the third and seventh equations and equation (7) in the last one.

(d) We will now show that the simplified composition identity holds in T. Let $f : n \to m$ and $g : m \to n + p$ be arbitrary morphisms in T. Then there exist descriptions $(\alpha, a), (\beta, b)$ of weight s and r, say, with behaviours f and g, respectively.
Consider $(\gamma, c) : n \to n+p$ of weight $s+r$, where $\gamma = \alpha \cdot (\mathbf{1}_s \oplus \beta)$ and $c = \langle a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle, b \rangle$. Then

$$|(\gamma, c)| = |(\alpha, a)| \cdot |(\beta, b)| = f \cdot g.$$

Let

$$(\delta, d) = (\gamma, c)^\wedge : n \to p.$$

Then

$$\begin{aligned} \delta &= (\gamma \cdot (\pi_{n,s+r} \oplus \mathbf{1}_p))^{\dagger_0} : n \to s + r + p \\ d &= (c \cdot \langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle)^{\dagger_D} : s + r \to p. \end{aligned}$$

By the definition of †, we have $|(\delta, d)| = (f \cdot g)^\dagger$. Note that

$$f \oplus \mathbf{1}_p = (\alpha \oplus \mathbf{1}_p) \cdot \langle a \oplus 0_p, \mathbf{1}_{m+p} \rangle$$

and

$$g \cdot (f \oplus \mathbf{1}_p) = \beta \cdot (\mathbf{1}_r \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle x, \mathbf{1}_{m+p} \rangle,$$

where

$$x = \langle b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \oplus 0_p, \mathbf{1}_{m+p} \rangle, a \oplus 0_p \rangle.$$

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By definition, we have $|(\eta, h)| = (g \cdot (f \oplus \mathbf{1}_p))^\dagger$, where

$$\eta = (\beta \cdot (\mathbf{1}_r \oplus \alpha \oplus \mathbf{1}_p) \cdot (\pi_{m,r+s} \oplus \mathbf{1}_p))^{\dagger_0} : m \to r+s+p,$$
$$h = (x \cdot \langle \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle)^{\dagger_D} : r+s \to p,$$

so $(\eta, h) : m \to p$ is of weight $r+s$. Moreover,

$$|(\tau, t)| = |(\alpha, a)| \cdot |(\eta, h)| = f \cdot (g \cdot (f \oplus \mathbf{1}_p))^\dagger,$$

where $\tau = \alpha \cdot (\mathbf{1}_s \oplus \eta)$ and $t = \langle a \cdot \eta \cdot \langle h, \mathbf{1}_p \rangle, h \rangle$, so $(\tau, t) : n \to p$ is of weight $s+r+s$. We now want to prove that

$$|(\tau, t)| = \tau \cdot \langle t, \mathbf{1}_p \rangle = \delta \cdot \langle d, \mathbf{1}_p \rangle = |(\delta, d)|. \tag{8}$$

It will then follow that the simplified composition identity holds in T.
We will first prove that

$$\delta \;=\; \tau \cdot (\langle \mathbf{1}_{s+r}, \mathbf{1}_s \oplus 0_r \rangle \oplus \mathbf{1}_p). \tag{9}$$

We calculate as follows:

$$\begin{aligned}
\delta &= (\alpha \cdot (\mathbf{1}_s \oplus \beta) \cdot (\pi_{n,s+r} \oplus \mathbf{1}_p))^{\dagger_0} \\
&=_9 \alpha \cdot ((\mathbf{1}_s \oplus \beta) \cdot (\pi_{n,s+r} \oplus \mathbf{1}_p) \cdot (\alpha \oplus \mathbf{1}_{s+r+p}))^{\dagger_0} \\
&= \alpha \cdot \langle 0_{s+m} \oplus \mathbf{1}_s \oplus 0_{r+p}, \beta \cdot (\langle 0_{s+m+s} \oplus \mathbf{1}_r, \alpha \oplus 0_{s+r} \rangle \oplus \mathbf{1}_p) \rangle^{\dagger_0} \\
&= \alpha \cdot \langle f', g' \rangle^{\dagger_0} \\
&=_6 \alpha \cdot \langle f'^{\dagger_0} \cdot \langle h'^{\dagger_0}, \mathbf{1}_{s+r+p} \rangle, h'^{\dagger_0} \rangle \\
&= \alpha \cdot \langle \mathbf{1}_s \oplus 0_{r+p}, h'^{\dagger_0} \rangle,
\end{aligned}$$

where

$$\begin{aligned}
f'^{\dagger_0} &= (0_{s+m} \oplus \mathbf{1}_s \oplus 0_{r+p})^{\dagger_0} \\
&=_2 0_m \oplus \mathbf{1}_s \oplus 0_{r+p}
\end{aligned}$$

and

$$\begin{aligned}
h' &= g' \cdot \langle f'^{\dagger_0}, \mathbf{1}_{m+s+r+p} \rangle \\
&= \beta \cdot (\langle 0_{s+m+s} \oplus \mathbf{1}_r, \alpha \oplus 0_{s+r} \rangle \oplus \mathbf{1}_p) \cdot \langle 0_m \oplus \mathbf{1}_s \oplus 0_{r+p}, \mathbf{1}_{m+s+r+p} \rangle \\
&= \beta \cdot (\mathbf{1}_r \oplus \alpha \oplus \mathbf{1}_p) \cdot (\langle 0_{m+s} \oplus \mathbf{1}_r, 0_m \oplus \mathbf{1}_s \oplus 0_r, \mathbf{1}_m \oplus 0_{s+r} \rangle \oplus \mathbf{1}_p).
\end{aligned}$$

From this we have

$$\begin{aligned}
\delta \cdot (\pi_{r,s} \oplus \mathbf{1}_p) &= \alpha \cdot \langle \mathbf{1}_s \oplus 0_{r+p}, h'^{\dagger_0} \rangle \cdot (\pi_{r,s} \oplus \mathbf{1}_p) \\
&= \alpha \cdot \langle 0_r \oplus \mathbf{1}_s \oplus 0_p, h'^{\dagger_0} \cdot (\pi_{r,s} \oplus \mathbf{1}_p) \rangle \\
&=_4 \alpha \cdot \langle 0_r \oplus \mathbf{1}_s \oplus 0_p, \eta \rangle \\
&= \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle 0_r \oplus \mathbf{1}_s \oplus 0_p, \mathbf{1}_{r+s} \oplus 0_p, 0_{r+s} \oplus \mathbf{1}_p \rangle \\
&= \tau \cdot (\langle 0_r \oplus \mathbf{1}_s, \mathbf{1}_{r+s} \rangle \oplus \mathbf{1}_p),
\end{aligned}$$

yielding

$$\begin{aligned}
\delta &= \tau \cdot (\langle 0_r \oplus \mathbf{1}_s, \mathbf{1}_{r+s} \rangle \oplus \mathbf{1}_p) \cdot (\pi_{s,r} \oplus \mathbf{1}_p) \\
&= \tau \cdot (\langle \mathbf{1}_{s+r}, \mathbf{1}_s \oplus 0_r \rangle \oplus \mathbf{1}_p).
\end{aligned}$$

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Hence, we have proved that equation (9) holds. Therefore, in order to prove equation (8), it is enough to show that $t = \langle \mathbf{1}_{s+r}, \mathbf{1}_s \oplus 0_r \rangle \cdot d$, which holds if we can show that:

(i) $(0_r \oplus \mathbf{1}_s) \cdot h = a \cdot \eta \cdot \langle h, \mathbf{1}_p \rangle$;

(ii) $d = \pi_{r,s} \cdot h$.

Note that

$$h = \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D} \rangle. \tag{10}$$

Indeed,

$$\begin{aligned} h &= (x \cdot \langle \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= (\langle b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \oplus 0_p, \mathbf{1}_{m+p} \rangle, a \oplus 0_p \rangle \cdot \langle \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= \langle b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot \eta, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle, a \cdot \eta \rangle^{\dagger_D} \\ &=_6 \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D} \rangle, \end{aligned}$$

where $e = b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot \eta, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle$ and $y = a \cdot \eta \cdot \langle e^{\dagger_D}, \mathbf{1}_{s+p} \rangle$.
We will now prove (i) and (ii).

(i) Since $(0_r \oplus \mathbf{1}_s) \cdot h = y^{\dagger_D}$, we have to prove that $y^{\dagger_D} = a \cdot \eta \cdot \langle h, \mathbf{1}_p \rangle$. Indeed,

$$\begin{aligned} a \cdot \eta \cdot \langle h, \mathbf{1}_p \rangle &= a \cdot \eta \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle \\ &= a \cdot \eta \cdot \langle e^{\dagger_D}, \mathbf{1}_{s+p} \rangle \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle \\ &= y \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle \\ &=_1 y^{\dagger_D}, \end{aligned}$$

which proves (i).

(ii) Since $d = (c \cdot \langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle)^{\dagger_D}$ and T is a partial iterative theory, it is enough to show that $\pi_{r,s} \cdot h$ is the unique solution of $\xi = (c \cdot \langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle) \cdot \langle \xi, \mathbf{1}_p \rangle$, that is, that

$$\langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle \rangle = (c \cdot \langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \tag{11}$$

where we have used equation (10). First note that

$$\langle \delta, 0_{s+r} \oplus \mathbf{1}_p \rangle \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = \langle \delta \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle,$$

which can also be written as

$$\begin{aligned} &\langle \delta \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &\quad = \langle \tau \cdot (\langle \mathbf{1}_{s+r}, \mathbf{1}_s \oplus 0_r \rangle \oplus \mathbf{1}_p) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &\quad = \langle \tau \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &\quad = \langle \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle, \end{aligned}$$

where we have used equation (9) in the computation.
We now define

$$u = \langle \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$$

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and conclude that the right-hand side of equation (11) equals

$$\begin{aligned} c \cdot u &= \langle a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle, b \rangle \cdot u \\ &= \langle a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle \cdot u, b \cdot u \rangle. \end{aligned}$$

Thus, by the coproduct property, equation (11) holds if and only if:
(A) $e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle = b \cdot u$;
(B) $y^{\dagger_D} = a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle \cdot u$.
To prove (A), we calculate as follows:

$$\begin{aligned} e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle &=_4 (b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot \eta, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle \cdot (\mathbf{1}_r \oplus \langle y^{\dagger_D}, \mathbf{1}_p \rangle))^{\dagger_D} \\ &= (b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot z, z, 0_r \oplus \mathbf{1}_p \rangle)^{\dagger_D}, \end{aligned}$$

where $z = \eta \cdot (\mathbf{1}_r \oplus \langle y^{\dagger_D}, \mathbf{1}_p \rangle)$. Thus, using

$$a \cdot z \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = y \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle = y^{\dagger_D}$$

and

$$z \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle = \eta \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle,$$

as a corollary of Theorem 3.8, we have

$$\begin{aligned} e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle &=_1 b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle a \cdot z, z, 0_r \oplus \mathbf{1}_p \rangle \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= b \cdot (\alpha \oplus \mathbf{1}_p) \cdot \langle y^{\dagger_D}, \eta \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= b \cdot \langle \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle y^{\dagger_D}, e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= b \cdot u. \end{aligned}$$

To prove (B), we first note that

$$\begin{aligned} \eta &= (\beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot (\pi_{m,r+s} \oplus \mathbf{1}_p))^{\dagger_0} \\ &=_1 \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot (\pi_{m,r+s} \oplus \mathbf{1}_p) \cdot \langle \eta, \mathbf{1}_{r+s+p} \rangle \\ &= \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_{r+s} \oplus 0_p, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle. \end{aligned}$$

Now, using Theorem 3.8, we calculate as follows:

$$\begin{aligned} y^{\dagger_D} &=_1 (a \cdot \eta \cdot \langle e^{\dagger_D}, \mathbf{1}_{s+p} \rangle) \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle \\ &= a \cdot \eta \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle \\ &= a \cdot \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle \mathbf{1}_{r+s} \oplus 0_p, \eta, 0_{r+s} \oplus \mathbf{1}_p \rangle \cdot t \\ &= a \cdot \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \eta \cdot t, \mathbf{1}_p \rangle \\ &= a \cdot \beta \cdot (\mathbf{1}_s \oplus \alpha \oplus \mathbf{1}_p) \cdot \langle b \cdot u, y^{\dagger_D}, \eta \cdot t, \mathbf{1}_p \rangle \\ &= a \cdot \beta \cdot \langle b \cdot u, \alpha \cdot (\mathbf{1}_s \oplus \eta) \cdot \langle y^{\dagger_D}, t \rangle, \mathbf{1}_p \rangle \\ &= a \cdot \beta \cdot \langle b, \mathbf{1}_{n+p} \rangle \cdot u, \end{aligned}$$

where $t = \langle e^{\dagger_D} \cdot \langle y^{\dagger_D}, \mathbf{1}_p \rangle, y^{\dagger_D}, \mathbf{1}_p \rangle$. Note that we have used (A) for the fifth equation.

With this we have proved that the simplified composition identity holds in T.

So we have now proved that T is a Conway theory with the dagger operation †. We will now prove that whenever T_0 is an iteration theory, T equipped with † is also an

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iteration theory. To do this, we suppose T_0 is an iteration theory, so the group identities hold in T_0. We prove that the group identities hold in T.

We will use the fact that for each morphism $f : n \to n+p$ in $D(T)$ and $h : n \to m$, $g : m \to m+p$ in T, the following implication[†] is valid:

$$f \cdot (h \oplus \mathbf{1}_p) = h \cdot g \implies f^{\dagger_D} = h \cdot g^\dagger. \tag{12}$$

Indeed,

$$\begin{aligned} h \cdot g^\dagger &= h \cdot g \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &= f \cdot (h \oplus \mathbf{1}_p) \cdot \langle g^\dagger, \mathbf{1}_p \rangle \\ &= f \cdot \langle h \cdot g^\dagger, \mathbf{1}_p \rangle, \end{aligned}$$

thus, since T is a partial iterative theory, we obtain $f^{\dagger_D} = h \cdot g^\dagger$.

Let $f : 1 \to n+p$ be an arbitrary morphism in T and (α, a) be a description of weight s with behaviour f. For each base morphism $\rho : n \to k$, we define the description

$$D_\rho = (\alpha \cdot (\mathbf{1}_s \oplus \rho \oplus \mathbf{1}_p), a \cdot (\rho \oplus \mathbf{1}_p)) : 1 \to k+p.$$

Then $|D_\rho| = f \cdot (\rho \oplus \mathbf{1}_p)$. We now consider the unique base morphism $\tau_n : n \to 1$ and the description $D^\wedge_{\tau_n} = (\beta, b) : 1 \to p$ of weight s,

$$\begin{aligned} \beta &= (\alpha \cdot (\mathbf{1}_s \oplus \tau_n \oplus \mathbf{1}_p) \cdot (\pi_{1,s} \oplus \mathbf{1}_p))^{\dagger_0} \\ &= (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \mathbf{1}_{s+p}))^{\dagger_0}, \end{aligned}$$

$$\begin{aligned} b &= (a \cdot (\tau_n \oplus \mathbf{1}_p) \cdot \langle \beta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= (a \cdot \langle \tau_n \cdot \beta, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D}. \end{aligned}$$

So $|D^\wedge_{\tau_n}| = (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger$.

Let $\rho^S_i : n \to n$ for $i \in [n]$ be the base morphisms associated with the group S of order n (see Definition 3.2) and consider the description $(\gamma, c) : n \to n+p$ of weight s given by

$$\begin{aligned} \gamma &= \langle \alpha \cdot (\mathbf{1}_s \oplus \rho^S_1 \oplus \mathbf{1}_p) \cdot (\lambda_1 \oplus \mathbf{1}_p), \ldots, \alpha \cdot (\mathbf{1}_s \oplus \rho^S_n \oplus \mathbf{1}_p) \cdot (\lambda_n \oplus \mathbf{1}_p) \rangle \\ c &= \langle a \cdot (\rho^S_1 \oplus \mathbf{1}_p), \ldots, a \cdot (\rho^S_n \oplus \mathbf{1}_p) \rangle, \end{aligned}$$

where $\lambda_i = (0_{(i-1)\cdot s} \oplus \mathbf{1}_s \oplus 0_{(n-i)\cdot s})$, with $(i-1)\cdot s$ denoting the usual multiplication of the integers $(i-1)$ and s. We have

$$\begin{aligned} |(\gamma, c)| &= \langle |D_{\rho^S_1}|, \ldots, |D_{\rho^S_n}| \rangle \\ &= \langle f \cdot (\rho^S_1 \oplus \mathbf{1}_p), \ldots, f \cdot (\rho^S_n \oplus \mathbf{1}_p) \rangle \\ &= f_S : n \to n+p. \end{aligned}$$

Now consider $(\gamma, c)^\wedge = (\delta, d) : n \to p$ of weight $n \cdot s$. We have

$$\begin{aligned} \delta &= (\gamma \cdot (\pi_{n,n\cdot s} \oplus \mathbf{1}_p))^{\dagger_0} \\ &= \langle \alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\rho^S_1 \oplus \lambda_1 \oplus \mathbf{1}_p), \ldots, \alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\rho^S_n \oplus \lambda_n \oplus \mathbf{1}_p) \rangle^{\dagger_0}, \end{aligned}$$

[†] In the literature this implication is called the *functorial dagger implication.*

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and

$$\begin{aligned} d &= (c \cdot \langle \delta, 0_{n\cdot s} \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= \langle a \cdot (\rho_1^S \oplus \mathbf{1}_p) \cdot \langle \delta, 0_{n\cdot s} \oplus \mathbf{1}_p \rangle, \ldots, a \cdot (\rho_n^S \oplus \mathbf{1}_p) \cdot \langle \delta, 0_{n\cdot s} \oplus \mathbf{1}_p \rangle \rangle^{\dagger_D}. \end{aligned}$$

Also,

$$|(\delta, d)| = f_S^\dagger = \langle f \cdot (\rho_1^S \oplus \mathbf{1}_p), \ldots, f \cdot (\rho_n^S \oplus \mathbf{1}_p) \rangle^\dagger.$$

Now let $(\eta, h) : n \to p$ of weight $n \cdot s$, where

$$\begin{aligned} \eta &= \langle (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \lambda_1 \oplus \mathbf{1}_p))^{\dagger_0}, \ldots, (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \lambda_n \oplus \mathbf{1}_p))^{\dagger_0} \rangle \\ h &= \langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \cdot b. \end{aligned}$$

For each $i \in [n]$, we have

$$\begin{aligned} (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \lambda_i \oplus \mathbf{1}_p))^{\dagger_0} \cdot \langle h, \mathbf{1}_p \rangle & \\ =_4 \ & (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \lambda_i \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus \langle h, \mathbf{1}_p \rangle))^{\dagger_0} \\ = \ & (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \mathbf{1}_{s+p}) \cdot (\mathbf{1}_n \oplus \langle b, \mathbf{1}_p \rangle))^{\dagger_0} \\ =_4 \ & (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\tau_n \oplus \mathbf{1}_{s+p}))^{\dagger_0} \cdot \langle b, \mathbf{1}_p \rangle \\ = \ & \beta \cdot \langle b, \mathbf{1}_p \rangle \\ = \ & |D^\wedge_{\tau_n}|. \end{aligned}$$

Thus,

$$\begin{aligned} |(\eta, h)| &= \langle |D^\wedge_{\tau_n}|, \ldots, |D^\wedge_{\tau_n}| \rangle && (13) \\ &= \tau_n \cdot (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger. && (14) \end{aligned}$$

Now note that

$$\begin{aligned} \delta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p) &=_{10} \tau_n \cdot \beta \\ &=_4 \eta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p). \end{aligned} \qquad (15)$$

Also, we see that

$$\begin{aligned} (c \cdot \langle \delta, 0_{n\cdot s} \oplus \mathbf{1}_p \rangle) \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p) &= \langle a \cdot \langle \rho_1^S \cdot \delta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p), 0_s \oplus \mathbf{1}_p \rangle, \ldots \\ & \qquad a \cdot \langle \rho_n^S \cdot \delta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p), 0_s \oplus \mathbf{1}_p \rangle \rangle \\ &= \langle a \cdot \langle \tau_n \cdot \beta, 0_s \oplus \mathbf{1}_p \rangle, \ldots, a \cdot \langle \tau_n \cdot \beta, 0_s \oplus \mathbf{1}_p \rangle \rangle \\ &= \langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \cdot a \cdot (\tau_n \cdot \beta, 0_s \oplus \mathbf{1}_p), \end{aligned}$$

from which implication (12) yields $d = \langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \cdot b$.

Finally, using equation (15), we have

$$\begin{aligned} f_S^\dagger &= \delta \cdot \langle d, \mathbf{1}_p \rangle \\ &= \delta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p) \cdot \langle b, \mathbf{1}_p \rangle \\ &= \eta \cdot (\langle \mathbf{1}_s, \ldots, \mathbf{1}_s \rangle \oplus \mathbf{1}_p) \cdot \langle b, \mathbf{1}_p \rangle \\ &= \eta \cdot \langle h, \mathbf{1}_p \rangle \\ &= |(\eta, h)| \\ &= \tau_n \cdot (f \cdot (\tau_n \oplus \mathbf{1}_p))^\dagger, \end{aligned}$$

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where in the last step we have used equations (13) and (14).

We have shown that whenever T_0 is an iteration theory, T equipped with the dagger operation † is an iteration theory. It remains to show that † extends ^{†_0} and ^{†_D}.

First we show that † extends ^{†_0}. Consider the description $(\alpha, 0_{n+p}) : n \to n+p$ of weight 0, where α is an arbitrary morphism in T_0. So $|(\alpha, 0_{n+p})| = \alpha$. We will now consider $(\alpha, 0_{n+p})^\wedge = (\alpha^{\dagger_0}, 0_p)$. We have $\alpha^\dagger = |(\alpha, 0_{n+p})^\wedge| = \alpha^{\dagger_0}$, so † extends ^{†_0}. The fact that † extends ^{†_D} follows from the fact that T is a partial iterative theory and that T, equipped with †, satisfies the fixed-point identity – see above.

Finally, as a corollary of Proposition 4.3, we have that the definition of † is forced. Thus, there is at most one desired extension of the operations ^{†_0} and ^{†_D}. We have seen that such an extension exists, so the proof is complete. □

4.1. *Corollaries of the Dagger Extension Theorem*

We now consider some corollaries of the Dagger Extension Theorem (Theorem 4.4). In the first corollary, we replace assumption (3) of the Dagger Extension Theorem by a condition based on the notion of simulation (Bloom and Ésik 1993), which is useful in many applications.

For the rest of this subsection, we assume that T is a partial iterative theory with a subtheory T_0 that is a Conway theory with dagger operation ^{†_0}. Since T is a partial iterative theory, there is a dagger operation ^{†_D} defined on the morphisms $n \to n+p$ in $D(T)$ that provides unique solutions to fixed-point equations $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$ with $f : n \to n+p$.

Definition 4.5. Let $D = (\alpha, a)$ and $E = (\beta, b)$ be descriptions $n \to p$ of weight s and r, respectively, and let $\rho : s \to r$ be a morphism in T_0. We say ρ is a *simulation* $D \to E$, and write

$$D \to^\rho E,$$

when

$$\alpha \cdot (\rho \oplus \mathbf{1}_p) = \beta$$

and

$$a = \rho \cdot b.$$

We define $D \to E$ if there is a morphism $\rho : s \to r$ in T_0 such that $D \to^\rho E$, and use $\leftrightarrow^*$ to denote the least equivalence relation containing $\to$. The relation $\leftrightarrow^*$ is called *simulation equivalence*.

In addition, we write $D \equiv E$ when $|D| = |E|$, that is, when D and E are (behaviourally) *equivalent*.

Lemma 4.6.

(a) Let D, E be descriptions $n \to p$. If $D \to^\rho E$, then $D \equiv E$.
(b) Let D, E be descriptions $n \to n+p$. If $D \to^\rho E$, then $D^\wedge \to^\rho E^\wedge$.

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Proof.

(a) Let $D = (\alpha, a)$ and $E = (\beta, b)$ be descriptions $n \to p$ of weight s and r, respectively. Assume that $D \to^{\rho} E$ holds for a T_0-morphism $\rho : s \to r$. Then

$$\begin{aligned} |D| &= \alpha \cdot \langle a, \mathbf{1}_p \rangle \\ &= \alpha \cdot \langle \rho \cdot b, \mathbf{1}_p \rangle \\ &= \alpha \cdot (\rho \oplus \mathbf{1}_p) \cdot \langle b, \mathbf{1}_p \rangle \\ &= \beta \cdot \langle b, \mathbf{1}_p \rangle \\ &= |E|. \end{aligned}$$

(b) First recall that (12) holds for all morphisms $f : n \to n+p$ and $g : m \to m+p$ in $D(T)$ and any morphism $h : n \to m$. Assume that $D \to^{\rho} E$, where $\rho : s \to r$ is in T_0 and $D = (\alpha, a)$ and $E = (\beta, b)$ are descriptions $n \to n + p$ of weight s and r, respectively. Let $D^{\wedge} = (\gamma, c)$ and $E^{\wedge} = (\delta, d)$. Then

$$\begin{aligned} \gamma \cdot (\rho \oplus \mathbf{1}_p) &=_4 (\alpha \cdot (\pi_{n,s} \oplus \mathbf{1}_p) \cdot (\mathbf{1}_n \oplus \rho \oplus \mathbf{1}_p))^{\dagger_0} \\ &= (\alpha \cdot (\rho \oplus \mathbf{1}_{n+p}) \cdot (\pi_{n,r} \oplus \mathbf{1}_p))^{\dagger_0} \\ &= (\beta \cdot (\pi_{n,r} \oplus \mathbf{1}_p))^{\dagger_0} \\ &= \delta. \end{aligned}$$

Thus, from

$$a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot (\rho \oplus \mathbf{1}_p) = \rho \cdot b \cdot \langle \delta, 0_r \oplus \mathbf{1}_p \rangle,$$

an application of implication (12) yields

$$\begin{aligned} c &= (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= \rho \cdot (b \cdot \langle \delta, 0_r \oplus \mathbf{1}_p \rangle)^{\dagger_D} \\ &= \rho \cdot d. \end{aligned}$$

We conclude that $D^{\wedge} \to^{\rho} E^{\wedge}$. □

We also have the following corollary of the Dagger Extension Theorem.

Corollary 4.7. Let T be a partial iterative theory. Suppose the following hold:

(1) T_0 is a subtheory of T, which is a Conway theory with the operation

$$^{\dagger_0} : T_0(n, n+p) \to T_0(n, p),$$

for $n, p \geqslant 0$.

(2) Each morphism $n \to p$ in T can be written as $\alpha \cdot \langle a, \mathbf{1}_p \rangle$, where $\alpha : n \to s + p$ is in T_0 and $a : s \to p$ is in $D(T)$,

(3) For all descriptions $D, E : n \to p$, $n, p \geqslant 0$, if $D \equiv E$, then $D \leftrightarrow^* E$.

Then ^{†_0} can be uniquely extended to a totally defined operation $^{\dagger} : T(n, n+p) \to T(n, p)$, $n, p \geqslant 0$ such that T equipped with † becomes a Conway theory. Moreover, if T_0 is an iteration theory, then so is T.

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Proof. Let D, E be equivalent descriptions $n \to n+p$. From assumption (3), we have $D \leftrightarrow^* E$. Moreover, from $D \leftrightarrow^* E$ it follows that $D^\wedge \leftrightarrow^* E^\wedge$, as a corollary of Lemma 4.6 (b). Finally, from $D^\wedge \leftrightarrow^* E^\wedge$ Lemma 4.6 (a) yields $D^\wedge \equiv E^\wedge$, and we conclude that assumption (3) of Theorem 4.4 holds. □

Corollary 4.8. Let T be a Conway theory with dagger operation † and $D(T)$ be an ideal in T. Suppose the following hold:

(1) T_0 is a sub-Conway theory of T, which is an iteration theory.
(2) For each $f : n \to n+p$ in $D(T)$, $f^\dagger$ is the unique morphism $n \to p$ in T such that
$$f^\dagger = f \cdot \langle f^\dagger, \mathbf{1}_p \rangle.$$
(3) Each morphism $n \to p$ in T can be written as $\alpha \cdot \langle a, \mathbf{1}_p \rangle$ for some $\alpha : n \to s+p$ in T_0 and $a : s \to p$ in $D(T)$.

Then T is an iteration theory.

Proof. Assumptions (1) and (3) obviously imply assumptions (1) and (2) of Theorem 4.4, respectively. As a corollary of assumption (2), T is a partial iterative theory with the distinguished ideal $D(T)$. Suppose $|(\alpha, a)| = |(\alpha', a')|$, where α, α' and a, a' are as in assumption (3) of Theorem 4.4. Then, using Proposition 4.3 for the first and last equations, we obtain
$$|(\alpha, a)^\wedge| = |(\alpha, a)|^\dagger = |(\alpha', a')|^\dagger = |(\alpha', a')^\wedge|.$$
This establishes assumption (3) of Theorem 4.4, and the result then follows by Theorem 4.4. □

5. Applications

5.1. *Pointed iterative theories*

We say a morphism $f : n \to p$ of a non-trivial theory T is *ideal* if none of the components $i_n \cdot f$ for $i \in [n]$ is a distinguished morphism. An *iterative theory* (Elgot 1973) is a non-trivial partial iterative theory T such that $D(T)$ is the collection of all ideal morphisms. Thus, each iterative theory comes with a partial dagger operation defined on the ideal morphisms $n \to n+p$. In the following, we use † to denote this operation.

In this section we show that the following result from Bloom *et al.* (1980b) and Ésik (1982) is an instance of the Dagger Extension Theorem.

Theorem 5.1. Suppose T is an iterative theory and $\perp : 1 \to 0$. Then there is a unique way of defining a dagger operation on T such that T becomes a Conway theory with $\mathbf{1}_1{}^\dagger = \perp$. Moreover, when equipped with this dagger operation, T is an iteration theory.

Proof. We may assume that T is non-trivial since otherwise the claim is obviously true. Since T is an iterative theory, it is also a partial iterative theory where the distinguished ideal $D(T)$ is the collection of all ideal morphisms. Any scalar morphism in T is either a distinguished morphism or an ideal morphism in $D(T)$.

Consider the least subtheory T_0 of T containing the morphism $\perp$. Then T_0 is isomorphic to the theory Θ' of partial functions (see Example 3.14), which is a Conway theory (in

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fact an iteration theory) in a unique way. Thus, T_0 may be turned into a Conway theory in a unique way. The dagger operation on T_0, denoted ^{†_0}, is defined by $f^{\dagger_0} = f^n \cdot \langle \bot_{n,p}, \mathbf{1}_p \rangle$, for all $f : n \to n+p$ in T_0, where $\bot_{n,p} = \langle \bot \cdot 0_p, \ldots, \bot \cdot 0_p \rangle$. We clearly have $\mathbf{1}_1^{\dagger} = \bot$.

To complete the proof, we need to verify that the partial iterative theory T and the iteration (sub)theory T_0 satisfy the assumptions of Corollary 4.7 – the only non-trivial part is showing that assumption (3) holds, for which we use Lemma 4.6 and Remark 5.5. □

In the following lemmas, we will consider descriptions $(\alpha, a) : n \to p$, where $\alpha : n \to s+p$ is in T_0, isomorphic to the theory Θ'.

Lemma 5.2. For any description $(\alpha, a) : n \to p$ of weight s there is a description $(\beta, b) : n \to p$ of weight $s+1$ such that β is a base morphism and there is a simulation $(\beta, b) \to (\alpha, a)$.

Proof. We identify α with a partial function $[n] \to [s+p]$. Let $b = \langle a, \bot_{1,p} \rangle$ and β be the base morphism corresponding to the function $[n] \to [s+1+p]$ defined by

$$i\beta = \begin{cases} i\alpha & \text{if } i\alpha \in [s] \text{ is defined} \\ i+1 & \text{if } i\alpha > s \text{ is defined} \\ s+1 & \text{if } i\alpha \text{ is undefined.} \end{cases}$$

It is clear that the partial function $\rho : [s+1] \to [s]$ that is undefined on $s+1$ and maps any integer in $[s]$ to itself defines a simulation $(\beta, b) \to (\alpha, a)$. □

We will now consider descriptions (α, a) where α is a base morphism. Let $D = (\alpha, a)$ be such a description $n \to p$ of weight s. We say D is *accessible* if for all $j \in [s]$ there is an $i \in [n]$ such that $i_n \cdot \alpha = j_{s+p}$. Moreover, we say D is *reduced* if for all $i, j \in [s]$, $i_s \cdot a = j_s \cdot a$ implies $i = j$. With these definitions, it is clear that the following lemmas hold.

Lemma 5.3. For every description $D = (\alpha, a) : n \to p$ such that α is a base morphism, there exist an accessible description $D' = (\alpha', a') : n \to p$ and an accessible and reduced description $D'' = (\alpha'', a'') : n \to p$ such that α' and α'' are base morphisms and $D' \to^{\rho} D$ and $D' \to^{\tau} D''$ hold for some injective base morphism ρ and surjective base morphism τ.

Lemma 5.4. Let $D = (\alpha, a)$ and $F = (\beta, b)$ be accessible descriptions such that $D \equiv F$. If F is reduced, then there is a surjective base morphism ρ such that $D \to^{\rho} F$.

Proof. Let s and r denote the weights of D and F, respectively. Since $D \equiv F$, and D is accessible and F is reduced, for each $i \in [s]$ there is a unique $j = i\rho \in [r]$ with $i_s \cdot a = j_r \cdot b$. Since F is accessible, ρ is surjective. Verifying that ρ defines a simulation is then routine. □

Remark 5.5. Note that the above lemmas mean that whenever $(\alpha, a), (\beta, b)$ are descriptions such that $(\alpha, a) \equiv (\beta, b)$, we have $(\alpha, a) \leftrightarrow^* (\beta, b)$.

Indeed, as a corollary of Lemmas 5.2 and 4.6(a), we just need to consider the case when α and β are base morphisms. Then, from Lemma 5.3, there are descriptions (α', a'), (α'', a'') and (β', b') such that

$$(\alpha'', a'') \leftarrow (\alpha', a') \to (\alpha, a) \tag{16}$$

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and

$$(\beta', b') \to (\beta, b), \tag{17}$$

where (α', a'), (β', b') are accessible and (α'', a'') is accessible and reduced.

Assuming that $(\alpha, a) \equiv (\beta, b)$, we have

$$(\alpha'', a'') \equiv (\alpha', a') \equiv (\alpha, a) \equiv (\beta, b) \equiv (\beta', b'),$$

where the first two and final equivalences are from applications of Lemma 4.6(a) to (16) and (17), respectively. Then Lemma 5.4 yields $(\beta', b') \to (\alpha'', a'')$, so $(\alpha, a) \leftrightarrow^* (\beta, b)$.

5.2. *Iteration matrix theories*

In this section we show that the Dagger Extension Theorem (Theorem 4.4) generalises the Matrix Extension Theorem of Bloom and Ésik (1993) – recall Example 2.2.

Let $T = \mathbf{Mat}_S$ be a matrix theory. If T' is a subtheory of T, T' is not necessarily a matrix theory. But when T' contains the zero matrices $0_{n,p}$ and is closed under sum, then T' is a matrix theory, called a *sub-matrix theory* of T. It is easy to see that in this case $S_0 := T'(1,1)$ is a sub-semiring of S, and a matrix is in T' if and only if each of its entries belongs to S_0, so T' is $\mathbf{Mat}_{S_0}$. Conversely, every sub-semiring of S determines a sub-matrix theory of T.

Suppose now that $I = (I(n,p))_{n,p}$ is a collection of morphisms containing the zero morphisms $0_{n,p}$ closed under sum and left and right composition with any morphism in T. Then we say I is a *two-sided ideal* of T. Each two-sided ideal of T determines and is determined by a two-sided ideal of the semiring S (*cf.* Golan (1999)), since if I is a two-sided ideal of T, then $I(1,1)$ is a two-sided ideal of S, and if I_0 is a two-sided ideal of S, then the collection of those matrices all of whose entries belong to I_0 is a two-sided ideal of T. Note that every two-sided ideal of a matrix theory T is an ideal of T as defined at the beginning of Section 3.

Definition 5.6. A *partial Conway matrix theory* is a matrix theory $T = \mathbf{Mat}_S$ that is a partial Conway theory such that $D(T)$ is a two-sided ideal. A *partial iteration matrix theory* is a partial Conway matrix theory that is a partial iteration theory. A *Conway matrix theory* (*iteration matrix theory*) is matrix theory that is a Conway theory (iteration theory).

It is known (Bloom and Ésik 1993) that in a Conway matrix theory $T = \mathbf{Mat}_S$, the dagger operation taking a morphism (that is, matrix) $f : n \to n + p$ to a morphism $f^\dagger : n \to p$ determines and is uniquely determined by a star operation taking a morphism $a : n \to n$ to a morphism $a^* : n \to n$. The two operations are related by the following equations. For all $a : n \to n$ and $b : n \to p$ in T,

$$a^* = \begin{pmatrix} a & \mathbf{1}_n \end{pmatrix}^\dagger$$
$$\begin{pmatrix} a & b \end{pmatrix}^\dagger = a^* b.$$

Also, it is known that the star operation on T determines and is uniquely determined by a star operation $^* : S \to S$. Moreover, S equipped with this star operation is a *Conway*

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semiring, or an *iteration semiring* (Bloom and Ésik 1993; Ésik 1999), if T is an iteration matrix theory.

The Matrix Extension Theorem (Bloom and Ésik 1993) comes in two versions.

Theorem 5.7. Let $T = \mathbf{Mat}_S$ be a matrix theory with a distinguished two-sided ideal I. Suppose the following hold:

(1) $T_0 = \mathbf{Mat}_{S_0}$ is a sub-matrix theory of T, which is a Conway matrix theory equipped with a star operation mapping a morphism $\alpha : n \to n$ in T_0 to a morphism $\alpha^{*_0} : n \to n$ in T_0.
(2) For each $a : n \to n$ in I and $b : n \to p$ in T, there is a unique morphism that solves the equation $\xi = a\xi + b$ in the variable $\xi : n \to p$ in T.
(3) Each morphism $f : n \to p$ can be written as $\alpha + a$, where $\alpha : n \to p$ is in T_0 and $a : n \to p$ is in I.
(4) If $\alpha, \alpha' : n \to p$ in T_0 and $a, a' : n \to p$ in I with $\alpha + a = \alpha' + a'$, then $\alpha = \alpha'$ and $a = a'$.

Then the operation ^{*_0} can be extended in a unique way to a star operation $f \mapsto f^*$ defined on all morphisms $f : n \to n$ in T such that T becomes a Conway matrix theory. Moreover, if T_0 is an iteration matrix theory, then T also becomes an iteration matrix theory.

The second version uses semirings.

Theorem 5.8. Let S be a semiring with a distinguished two-sided ideal I_0. Suppose:

(1) S_0 is a subsemiring of S that is a Conway semiring with star operation ^{*_0}.
(2) For each $a \in I_0$ and $b \in S$, the equation $x = ax + b$ has a unique solution in S.
(3) Each $s \in S$ can be written as $s = x + a$ for some $x \in S_0$ and $a \in I_0$.
(4) For all $x, x' \in S_0$ and $a, a' \in I_0$, if $x + a = x' + a'$, then $x = x'$ and $a = a'$.

Then the operation ^{*_0} can be extended in a unique way to a star operation $^* : S \to S$ such that S becomes a Conway semiring. Moreover, if S_0 is an iteration semiring, then S also becomes an iteration semiring.

Proof of Theorem 5.7. We prove the first version of the Matrix Extension Theorem by showing that the assumptions of this theorem imply the assumptions of the Dagger Extension Theorem with $D(T) := I$ and the following definition of ^{†_D}. Let $f : n \to n + p$ in $D(T)$. Since T is a matrix theory, we may write $f = \begin{pmatrix} a & b \end{pmatrix}$, where $a : n \to n$ and $b : n \to p$ in $D(T)$. By assumption (2) of Theorem 5.7, there is a unique $\xi : n \to n$ in T with $\xi = a\xi + \mathbf{1}_n$. Let a^{*_D} denote this unique solution. Then $a^{*_D}b$ is the unique solution of the equation $\xi = a\xi + b$ and of $\xi = f \cdot \langle \xi, \mathbf{1}_p \rangle$, where ξ now ranges over the morphisms $n \to p$ in T. In other words T is a partial iterative theory. So we define $f^{\dagger_D} := a^{*_D}b$. Now if $f : n \to p$ in T, then by assumption (3) of the theorem, $f = \alpha + a$ for some $\alpha : n \to p$ in T_0 and $a : n \to p$ in $D(T)$. So we may write f as $((\mathbf{1}_n \oplus 0_p) + (0_n \oplus \alpha)) \cdot \langle a, \mathbf{1}_p \rangle$, establishing assumption (2) of Theorem 4.4.

To complete the proof, we show that assumptions (2) and (4) of the theorem imply assumption (3) of Theorem 4.4. Let $(\alpha, a) : n \to n + p$ be a description of weight s. Since T is a matrix theory, we can write $\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix}$ and $a = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$, where $\alpha_0 : n \to s$,

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$\alpha_1 : n \to n$, $\alpha_2 : n \to p$ are in T_0 and $a_1 : s \to n$ and $a_2 : s \to p$ are in $D(T)$. Thus,

$$|(\alpha, a)| = \alpha \cdot \langle a, \mathbf{1}_{n+p} \rangle = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ \mathbf{1}_n & 0 \\ 0 & \mathbf{1}_p \end{pmatrix}$$
$$= \begin{pmatrix} \alpha_0 a_1 + \alpha_1 & \alpha_0 a_2 + \alpha_2 \end{pmatrix}.$$

Let $(\gamma, c) = (\alpha, a)^{\wedge}$. Thus,

$$\gamma = \begin{pmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{pmatrix}^{\dagger_0} = \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \end{pmatrix}.$$

Moreover,

$$\begin{aligned} c &= \left(\begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \\ 0 & \mathbf{1}_p \end{pmatrix} \right)^{\dagger_D} \\ &= \begin{pmatrix} a_1\alpha_1^{*_0}\alpha_0 & a_1\alpha_1^{*_0}\alpha_2 + a_2 \end{pmatrix}^{\dagger_D} \\ &= (a_1\alpha_1^{*_0}\alpha_0)^{*_D}(a_1\alpha_1^{*_0}\alpha_2 + a_2). \end{aligned}$$

Therefore,

$$\begin{aligned} |(\gamma, c)| &= \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \end{pmatrix} \begin{pmatrix} (a_1\alpha_1^{*_0}\alpha_0)^{*_D}(a_1\alpha_1^{*_0}\alpha_2 + a_2) \\ \mathbf{1}_p \end{pmatrix} \\ &= \alpha_1^{*_0}\alpha_0(a_1\alpha_1^{*_0}\alpha_0)^{*_D}(a_1\alpha_1^{*_0}\alpha_2 + a_2) + \alpha_1^{*_0}\alpha_2 \\ &= \alpha_1^{*_0}(\alpha_0 a_1\alpha_1^{*_0})^{*_D}\alpha_0(a_1\alpha_1^{*_0}\alpha_2 + a_2) + \alpha_1^{*_0}\alpha_2, \end{aligned}$$

where the last equation is shown as follows:

$$\begin{aligned} \alpha_0 a_1\alpha_1^{*_0}\alpha_0(a_1\alpha_1^{*_0}\alpha_0)^{*_D} + \alpha_0 &= \alpha_0(a_1\alpha_1^{*_0}\alpha_0(a_1\alpha_1^{*_0}\alpha_0)^{*_D} + \mathbf{1}_p) \\ &= \alpha_0(a_1\alpha_1^{*_0}\alpha_0)^{*_D}, \end{aligned}$$

where we have used assumption (2) of the theorem. Note that $\alpha_0(a_1\alpha_1^{*_0}\alpha_0)^{*_D}$ solves the equation $(\alpha_0 a_1\alpha_1^{*_0})\xi + \alpha_0 = \xi$, therefore assumption (2) yields

$$\alpha_0(a_1\alpha_1^{*_0}\alpha_0)^{*_D} = (\alpha_0 a_1\alpha_1^{*_0})^{*_D}\alpha_0.$$

Now suppose $(\beta, b) : n \to n + p$ is a description of weight r, where

$$\beta = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$$
$$b = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$$

with

$$\beta_0 : n \to r$$
$$\beta_1 : n \to n$$
$$\beta_2 : n \to p$$

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in T_0 and

$$b_1 : r \to n$$
$$b_2 : r \to p$$

in $D(T)$.

Suppose $|(\alpha, a)| = |(\beta, b)|$. Then

$$\begin{aligned} |(\alpha, a)| &= \begin{pmatrix} \alpha_0 a_1 + \alpha_1 & \alpha_0 a_2 + \alpha_2 \end{pmatrix} \\ &= \begin{pmatrix} \beta_0 b_1 + \beta_1 & \beta_0 b_2 + \beta_2 \end{pmatrix} \\ &= |(\beta, b)|. \end{aligned}$$

From this, assumption (4) of the theorem yields

$$\alpha_0 a_1 = \beta_0 b_1, \tag{18}$$
$$\alpha_1 = \beta_1, \tag{19}$$
$$\alpha_0 a_2 = \beta_0 b_2, \tag{20}$$
$$\alpha_2 = \beta_2. \tag{21}$$

Now let $(\delta, d) = (\beta, b)^{\wedge}$. Then

$$\begin{aligned} |(\delta, d)| &= \delta \cdot \langle d, \mathbf{1}_p \rangle \\ &= \beta_1^{*_0} (\beta_0 b_1 \beta_1^{*_0})^{*_D} \beta_0 (b_1 \beta_1^{*_0} \beta_2 + b_2) + \beta_1^{*_0} \beta_2 \\ &= \alpha_1^{*_0} (\alpha_0 a_1 \alpha_1^{*_0})^{*_D} \alpha_0 (a_1 \alpha_1^{*_0} \alpha_2 + a_2) + \alpha_1^{*_0} \alpha_2 \\ &= \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= |(\gamma, c)|, \end{aligned}$$

as a corollary of equations (18), (19), (20) and (21).

We have shown that assumption (4) of the theorem implies assumption (3) of Theorem 4.4, and thus that the Matrix Extension Theorem, Theorem 5.7, follows from the Dagger Extension Theorem. □

Applications of Theorem 5.7 were given in Bloom and Ésik (1993) and Bloom and Ésik (2009). Here we will just mention the following result from Bloom and Ésik (1993).

Corollary 5.9. If S is an iteration semiring, then $S\langle\langle \Delta^* \rangle\rangle$, the semiring of formal power series over an alphabet Δ with coefficients in S is an iteration semiring. The same holds for $S^{rat}\langle\langle \Delta^* \rangle\rangle$, the semiring of rational power series over Δ with coefficients in S.

5.3. *Iteration grove theories*

In this section we will show that the Dagger Extension Theorem proves an extension theorem for grove theories found in Bloom and Ésik (2003).

We will consider theories T equipped with an additive structure, but here, unlike in matrix theories, composition distributes over sums only on the right.

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Definition 5.10. A *grove theory* is a theory T equipped with an additive structure such that each hom-set $T(n,p)$ is a commutative monoid $(T(n,p),+,0_{n,p})$ satisfying

$$\begin{aligned}(f+g)\cdot h &= f\cdot h + g\cdot h\\ 0_{n,p}\cdot h &= 0_{n,q}\end{aligned}$$

for all $f,g : n \to p$ and $h : p \to q$.

Here, we have used the convention that composition has higher precedence than addition. (It was shown in Bloom and Ésik (1993) that the additive structure is uniquely determined by the morphisms $+ := 1_2 + 2_2 : 1 \to 2$ and $\# : 0_{1,0} : 1 \to 0$, for $f+g = +\cdot\langle f,g\rangle$ and $0_{1,p} = \# \cdot 0_p$, for all $p \geqslant 0$ and $f,g : 1 \to p$, moreover,

$$\begin{aligned}i_n\cdot(f+g) &= i_n\cdot f + i_n\cdot g\\ i_n\cdot 0_{n,p} &= 0_{1,p}\end{aligned}$$

for all $f,g : n \to p$, $n,p \geqslant 0$.)

We say that the grove theory T is a *sub-grove theory* of the grove theory T' if T is a subtheory of T' and has the same zero morphisms and sum operation as T'.

Suppose T is a grove theory and T_0 is a sub-grove theory of T. Moreover, suppose T_0 is a matrix theory. Note that if an ideal $D(T)$ is closed under composition with arbitrary morphisms from T_0 on the left, then for all $f,g : n \to p$ in $D(T)$, we have $f+g \in D(T)$ and $0_{n,p} \in D(T)$. We say an ideal $D(T)$ is a T_0-*ideal* if it is closed under composition with arbitrary morphisms from T_0 on the left.

The Grove Extension Theorem (Bloom and Ésik 2003) is as follows.

Theorem 5.11. Let T be a grove theory and T_0 be a sub-grove theory of T, which is a matrix theory. We also assume that:

(1) $D(T)$ is a T_0-ideal.
(2) Every morphism in T can be written uniquely as $\alpha + a$, for some α in T_0 and a in $D(T)$.
(3) For all $\alpha : n \to p$ in T_0 and $f,g : p \to q$ in T, we have
$$\alpha\cdot(f+g) = (\alpha\cdot f) + (\alpha\cdot g).$$
(4) T_0 is a Conway theory with dagger operation $^{\dagger_0} : T_0(n,n+p) \to T_0(n,p)$, $n,p \geqslant 0$.
(5) For every $\alpha : n \to p$ in T_0 and $a : n \to n+p$ in $D(T)$, the fixed-point equation $\xi = ((0_n \oplus \alpha) + a)\cdot\langle \xi, \mathbf{1}_p\rangle$ has a unique solution.

Then there is a unique way to define a total dagger operation † on T extending ^{†_0} such that T becomes a Conway theory. Furthermore, if T_0 is an iteration theory, so is T.

Proof. For every $\alpha : n \to p$ in T_0 and $a : n \to n+p$ in $D(T)$, we use $((0_n \oplus \alpha) + a)^{\dagger_D}$ to denote the unique morphism $n \to p$ in T with

$$((0_n \oplus \alpha) + a)^{\dagger_D} = ((0_n \oplus \alpha) + a)\cdot\langle (0_n \oplus \alpha) + a)^{\dagger_D}, \mathbf{1}_p\rangle.$$

We show that the assumptions in this theorem imply the assumptions of the Dagger Extension Theorem, with the choice of T_0 and $D(T)$ as the notation suggests.

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Assumption (4) of the theorem is the same as assumption (1) of Theorem 4.4. Also, T, equipped with the distinguished ideal $D(T)$ is a partial iterative theory. Moreover, assumption (2) implies that every $n \to p$ morphism in T can be written as

$$\alpha + a = ((0_n \oplus \alpha) + (\mathbf{1}_n \oplus 0_p)) \cdot \langle a, \mathbf{1}_p \rangle$$

for some $\alpha : n \to p$ in T_0 and $a : n \to p$ in $D(T)$. Thus, assumption (2) of the theorem implies assumption (2) of Theorem 4.4.

We now show that assumptions (2), (4) and (5) of the theorem imply assumption (3) of Theorem 4.4. Let $(\alpha, a) : n \to n + p$ be a description of weight s. Since T_0 is a matrix theory, we can write α uniquely as $\alpha = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix}$, where

$$\begin{aligned} \alpha_0 &: n \to s \\ \alpha_1 &: n \to n \\ \alpha_2 &: n \to p \end{aligned}$$

in T_0. Thus,

$$\begin{aligned} |(\alpha, a)| &= \alpha \cdot \langle a, \mathbf{1}_{n+p} \rangle \\ &= \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{pmatrix} \cdot \langle a, \mathbf{1}_{n+p} \rangle \\ &= \alpha_0 a + \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} : n \to n + p. \end{aligned}$$

Let $(\gamma, c) = (\alpha, a)^\wedge$. Then

$$\begin{aligned} \gamma &= \begin{pmatrix} \alpha_1 & \alpha_0 & \alpha_2 \end{pmatrix}^{\dagger_0} \\ &= \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \end{pmatrix} : n \to s + p \end{aligned}$$

and

$$c = (a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle)^{\dagger_D} : s \to s + p.$$

We now define

$$x = a \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle : s \to p.$$

Note that this definition makes sense by assumption (5) in Theorem 5.11.

We now prove that $c = x$. Since

$$a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot \langle x, \mathbf{1}_p \rangle = a \cdot \langle \gamma \cdot \langle x, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle$$

and

$$\begin{aligned} \gamma \cdot \langle x, \mathbf{1}_p \rangle &= \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \end{pmatrix} \cdot \langle a \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle, \mathbf{1}_p \rangle \\ &= \alpha_1^{*_0}\alpha_0 a \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle + \alpha_1^{*}\alpha_2 \\ &= (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2))) \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle \\ &= (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \end{aligned}$$

we obtain $a \cdot \langle \gamma, 0_s \oplus \mathbf{1}_p \rangle \cdot \langle x, \mathbf{1}_p \rangle = x$, so assumption (5) yields $c = x$.

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Moreover, we have

$$\begin{aligned} |(\gamma, c)| &= \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= \begin{pmatrix} \alpha_1^{*_0}\alpha_0 & \alpha_1^{*_0}\alpha_2 \end{pmatrix} \cdot \langle c, \mathbf{1}_p \rangle \\ &= \alpha_1^{*_0}\alpha_0 c + \alpha_1^{*_0}\alpha_2 \\ &= \alpha_1^{*_0}\alpha_0 x + \alpha_1^{*_0}\alpha_2. \end{aligned}$$

Suppose $(\beta, b) : n \to n + p$ is a description of weight r, where $\beta = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 & b_2 \end{pmatrix}$ with

$$\begin{aligned} \beta_0 &: n \to r \\ \beta_1 &: n \to n \\ \beta_2 &: n \to p \end{aligned}$$

in T_0 and

$$b : r \to r + p$$

in $D(T)$. Moreover, suppose $|(\alpha, a)| = |(\beta, b)|$. Then

$$\begin{aligned} |(\alpha, a)| &= \alpha_0 a + \begin{pmatrix} \alpha_1 & \alpha_2 \end{pmatrix} \\ &= \beta_0 b + \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} \\ &= |(\beta, b)|. \end{aligned}$$

So assumption (2) yields

$$\alpha_0 a = \beta_0 b \tag{22}$$

$$\alpha_1 = \beta_1 \tag{23}$$

$$\alpha_2 = \beta_2. \tag{24}$$

Letting $(\delta, d) = (\beta, b)^\wedge$, and using equations (22), (23) and (24), we calculate as follows:

$$\begin{aligned} |(\gamma, c)| &= \gamma \cdot \langle c, \mathbf{1}_p \rangle \\ &= \alpha_1^{*_0}\alpha_0 x + \alpha_1^{*_0}\alpha_2 \\ &= \alpha_1^{*_0}\alpha_0 a \cdot \langle (\alpha_1^{*_0}\alpha_0 a + (0_n \oplus (\alpha_1^{*_0}\alpha_2)))^{\dagger_D}, \mathbf{1}_p \rangle + \alpha_1^{*_0}\alpha_2 \\ &= \beta_1^{*_0}\beta_0 b \cdot \langle (\beta_1^{*_0}\beta_0 b + (0_n \oplus (\beta_1^{*_0}\beta_2)))^{\dagger_D}, \mathbf{1}_p \rangle + \beta_1^{*_0}\beta_2 \\ &= \delta \cdot \langle d, \mathbf{1}_p \rangle \\ &= |(\delta, d)|. \end{aligned}$$

Thus, assumption (3) of Theorem 4.4 holds. □

We say a grove theory that is also a Conway theory (iteration theory) is a *Conway grove theory* (*iteration grove theory*). The following is a corollary of the Grove Extension Theorem – see Bloom and Ésik (2003) and Section 5.2 for the missing definitions. Recall that a formal tree series $1 \to p$ with coefficients in S is a mapping $T_\Sigma(X_p) \to S$ from the set of Σ-trees to a semiring S (*cf.* Ésik and Kuich (2003)).

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Corollary 5.12. The formal tree series over a ranked alphabet with coefficients in a Conway semiring S form a Conway grove theory containing the rational tree series as a sub-Conway grove theory. When S is an iteration semiring, both theories are iteration grove theories.

6. Conclusion

It is known that an identity holds in all iteration theories if and only if it holds in Elgot's iterative theories. In a (partial) iterative theory, the dagger operation provides unique solutions to certain fixed-point equations. We have proved a general extension theorem providing a sufficient condition that the dagger operation on a partial iterative theory T be extendable to a total dagger operation such that T becomes an iteration theory. We have also shown that the general extension theorem implies the extension theorems for pointed iterative theories, matrix theories and grove theories.

It would be interesting to know the conditions under which the dagger operation of a partial Conway or partial iteration theory can be extended to a total dagger operation so that the theory becomes a Conway or an iteration theory.

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