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USING CONTINUED FRACTIONS WITH PRESCRIBED PERIOD FOR UNIVERSAL QUADRATIC FORMS

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Abstract

We study the congruence classes attained by positive integers D with a prescribed period for the continued fraction of $\sqrt{D}$. As an application, we refine the results on large ranks of universal quadratic forms over real quadratic fields by imposing congruence conditions on their discriminants.

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1. Introduction

We consider continued fractions of numbers of the form $\sqrt{D}$ with a given period length. We focus on the possible residue classes of D modulo a positive integer n and then connect our findings with universal quadratic forms over $\mathbb{Z}[\sqrt{D}]$.

In 1770, Lagrange proved the four-square theorem, which states that every positive integer can be written as a sum of four squares. The expression $w^2 + x^2 + y^2 + z^2$ is therefore a universal quadratic form over $\mathbb{Z}$. Universal quadratic forms can, however, be studied more generally over totally real number fields.

Blomer and Kala [1] discovered a connection between universal quadratic forms over rings of integers in number fields (such as $\mathbb{Z}[\sqrt{D}]$) and continued fractions. A key tool is a result of Friesen [2], who studied the representation of $\sqrt{D}$ via continued fractions and determined the conditions under which one can prescribe the coefficients of the continued fraction of $\sqrt{D}$ so that a corresponding D exists. Halter-Koch [3] extended Friesen's results by showing that one can also impose certain divisibility conditions on the coefficients.

Blomer and Kala used these theorems to construct infinitely many D for which universal quadratic forms over $\mathbb{Z}[\sqrt{D}]$ must have arbitrarily many variables. This line of research was further developed by Yatsyna [9], and Kala and Tinková [6], who extended the theorem on large numbers of variables to certain cubic fields; Kala then generalised these results to fields of all degrees divisible by two or three [4]. More

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recently, Kala *et al*. [7] significantly strengthened the initial result [1], proving that almost all quadratic fields require many variables for universal quadratic forms. For more information, see the survey article [5].

In this paper, we refine the results of Friesen and Halter-Koch [2, 3] regarding congruence conditions. The main result is Theorem 3.5, which states that for a continued fraction of $\sqrt{D}$ with an even period length, D can take any value modulo an odd n. We apply this result to universal quadratic forms in Theorems 4.6 and 4.7.

In Section 2, we focus on continued fractions with short periods, explicitly determine the form of D and establish the congruence conditions. In Section 3, we extend this approach by using Friesen's results for general period lengths. We prove that if the continued fraction of $\sqrt{D}$ has an odd period length, then D is not divisible by any prime $p \equiv 3 \pmod 4$ (Theorem 3.2). We also show that when the period length of the continued fraction of $\sqrt{D}$ is even, D can be in any residue class modulo any odd n (Theorem 3.5). These results generalise the results from Section 2. Finally, in Section 4, we connect these results with universal quadratic forms, leading to the following conclusions.

THEOREM 1.1 (Theorem 4.6). *Let m, n, s be positive integers. Then, there exists a nonsquare $D \geq 2$ such that $D \equiv m \pmod n$ and every universal quadratic form over $\mathbb{Z}[\sqrt{D}]$ has at least s variables.*

THEOREM 1.2 (Theorem 4.7). *Let m, n, s, k be positive integers such that n is odd and k even. Then, there exists a nonsquare $D \geq 2$ such that $D \equiv m \pmod n$, the continued fraction of $\sqrt{D}$ has a period length of k and every universal quadratic form over $\mathbb{Z}[\sqrt{D}]$ has at least s variables.*

Interesting open questions are whether Theorems 3.5 and 4.7 can be generalised to even modulus n or to odd period length k. Further, although we have already shown for odd k that D cannot be divisible by any prime $p \equiv 3 \pmod 4$, it may still be interesting to study the other residues modulo such primes. As a small simplification, in the paper, we work only with the continued fraction of $\sqrt{D}$ and, accordingly, with the order $\mathbb{Z}[\sqrt{D}]$. Another interesting open problem is to extend our results to the case of continued fractions of $\frac{1}{2}(1 + \sqrt{D})$ where $D \equiv 1 \pmod 4$, so that one obtains results also for the maximal order $\mathbb{Z}[\frac{1}{2}(1 + \sqrt{D})]$.

2. Continued fractions with short period

Let us recall that for $D \in \mathbb{Z}_{>1}$, which is not a perfect square, the continued fraction expansion of $\sqrt{D}$ is

$$\sqrt{D} = [a_0, \overline{a_1, \ldots, a_k, 2a_0}],$$

where the coefficients a_i are positive integers and $(a_1, \ldots, a_k)$ is a palindrome.

Before discussing general periods using Friesen's theorem, we focus on shorter period lengths. Let us start with the case where the period length is equal to one.

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PROPOSITION 2.1. *Let D and t be positive integers such that $\sqrt{D} = [t, \overline{2t}]$. Then, $D = t^2 + 1$ and D is not divisible by any prime number $p \equiv 3 \pmod 4$.*

PROOF. First, we determine the value of D in terms of t:

$$\sqrt{D} = [t, 2t, 2t, \ldots] = t + \cfrac{1}{2t + \cfrac{1}{2t + \cfrac{1}{\ddots}}}.$$

Let $A = \sqrt{D} - t$. By the periodicity of the continued fraction expansion of the square root, after the last term of the period $2t$, the expression A repeats. Thus,

$$A = \frac{1}{2t + A}.$$

Rearranging gives $A^2 + 2tA = 1$. Substituting $A = \sqrt{D} - t$, we obtain $D = t^2 + 1$. If a prime p divides D, then $t^2 \equiv -1 \pmod p$ and since -1 is not a quadratic residue modulo any prime $p \equiv 3 \pmod 4$, we get our statement. □

Similarly, by explicitly working with periods of length two and three, one can show that for a period of length two, D can take all residues modulo p. In contrast, for a period of length three, D can attain all residues modulo $p = 2$ or $p \equiv 1 \pmod 4$, but it cannot be divisible by a prime $p \equiv 3 \pmod 4$. Both of these results are fairly straightforward exercises whose proofs can be found in our thesis [8].

Let us further illustrate what is happening by working out the case of period four. First, we will establish a general formula for such D.

PROPOSITION 2.2. *Let D, t, u and v be positive integers such that $\sqrt{D} = [t, \overline{u, v, u, 2t}]$. Then,*

$$D = t^2 + \frac{2vut + 2t + v}{u(uv + 2)} \quad \text{and} \quad D = \left(\frac{u(y(uv + 2) - v^2) - v}{2}\right)^2 + y(uv + 2) - v^2 - y$$

for some $y \in \mathbb{Z}^+$. Moreover, assume that u, v, y are positive integers such that u, v are even. Let

$$D' = \left(\frac{u(y(uv + 2) - v^2) - v}{2}\right)^2 + y(uv + 2) - v^2 - y \quad \text{and} \quad t = \frac{u(y(uv + 2) - v^2) - v}{2}.$$

Then, we have the continued fraction $\sqrt{D'} = [t, \overline{u, v, u, 2t}]$.

PROOF. First, we determine the value of D in terms of t, u, v. Let $A = \sqrt{D} - t$. By periodicity, $A = [1, u, v, u, 2t + A]$. This expression can be simplified to

$$A = \frac{2tuv + Auv + v + 2t + A}{2tu^2v + Au^2v + uv + 4tu + 2Au + 1},$$

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that is,

$$2Atu^2v + A^2u^2v + Auv + 4Atu + 2A^2u + A = 2tuv + Auv + v + 2t + A.$$

Substituting $A = \sqrt{D} - t$ gives

$$(\sqrt{D} - t)^2(u^2v + 2u) + (\sqrt{D} - t)(2tu^2v + 4tu) = 2tuv + v + 2t,$$

so that

$$D = t^2 + \frac{2tuv + v + 2t}{u^2v + 2u}.$$

Since D, t, u, v are integers, we must have $u^2v + 2u \mid 2tuv + v + 2t$.

Therefore, $u \mid 2tuv + v + 2t$, that is, there exists $x \in \mathbb{Z}^+$ such that $t = \frac{1}{2}(ux - v)$. Substituting into the original polynomial gives

$$D = \left(\frac{ux - v}{2}\right)^2 + \frac{u^2vx - uv^2 + v + ux - v}{u(uv + 2)} = \left(\frac{ux - v}{2}\right)^2 + \frac{uvx - v^2 + x}{uv + 2}.$$

Now, we must also have $uv + 2 \mid uvx + 2x - v^2 - x$, that is, there exists $y \in \mathbb{Z}^+$ such that $x = y(uv + 2) - v^2$. Substituting gives

$$D = \left(\frac{u(y(uv + 2) - v^2) - v}{2}\right)^2 + y(uv + 2) - v^2 - y.$$

Concerning the 'moreover' part, note that if u, v are both even, then

$$D' = \left(\frac{u(y(uv + 2) - v^2) - v}{2}\right)^2 + y(uv + 2) - v^2 - y$$

is an integer. By back substitution,

$$D' = t^2 + \frac{2tuv + v + 2t}{u^2v + 2u}$$

for some $t \in \mathbb{Z}^+$ which then implies $\sqrt{D'} = [t, u, v, u, 2t, \ldots] = [t, \overline{u, v, u, 2t}]$. □

We can now show that all residues can be attained for a period of four.

THEOREM 2.3. *Let $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$. Then, there exists a positive integer D such that the continued fraction of $\sqrt{D}$ has a period length of four and $D \equiv m \pmod{n}$.*

PROOF. By Proposition 2.2, for any even u, v and

$$D = \left(\frac{u(y(uv + 2) - v^2) - v}{2}\right)^2 + y(uv + 2) - v^2 - y \tag{2.1}$$

for some y, we have $\sqrt{D} = [t, \overline{u, v, u, 2t}]$ for some t. For $u \equiv v \equiv 0 \pmod{2n}$, this gives $D \equiv 2y - y = y \pmod{n}$.

Since we have the freedom to choose y (regardless of the values of u and v), for all $m \in \mathbb{Z}$, we can select y such that $y \equiv m \pmod{n}$. Hence, the polynomial (2.1) attains all residues modulo n. To ensure that the period length is truly four and not shorter (one

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or two), we impose additional conditions on y. Since we have only modular conditions on u, v, y, we can choose such $y > v/u$, so that $2t > v$. Then, $[t, \overline{u, v, u, 2t}]$ has period length exactly four. □

3. Continued fractions with general period length

Now, we work with general period lengths. For $\sqrt{D} = [a_0, \overline{a_1, \dots, a_{k-1}, a_k}]$, we define the convergents p_n and q_n recursively by

$$p_{-1} = 1, \quad p_0 = a_0, \quad p_n = a_n p_{n-1} + p_{n-2},$$
$$q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

In the next theorem, we summarise Friesen's results, which we use to express D in terms of the coefficients of the continued fraction.

THEOREM 3.1 (Friesen [2]). *The equation*

$$\sqrt{D} = [\lfloor \sqrt{D} \rfloor; \overline{a_1, a_2, \dots, a_{k-2} = a_2,\ a_{k-1} = a_1,\ a_k = 2\lfloor \sqrt{D} \rfloor}]$$

has, for any symmetric sequence of positive integers $(a_1, \dots, a_{k-1})$, infinitely many square-free solutions D, whenever q_{k-2} or $(q_{k-2}^2 - (-1)^k)/q_{k-1}$ is even. If both quantities are odd, then there are no solutions D even if the square-free condition is dropped.

Further, $D = \alpha b^2 + \beta b + \gamma$ for any $b \in \mathbb{Z}$ sufficiently large so that $D > 0$, where the coefficients α, β, γ are given as follows.

(1) *If $q_{k-1} \equiv 1 \pmod 2$, then*

$$\alpha = q_{k-1}^2, \quad \beta = 2q_{k-2} - (-1)^k q_{k-2}(q_{k-2}^2 - (-1)^k),$$
$$\gamma = \frac{(\frac{1}{4}q_{k-2}^2 - (-1)^k)(q_{k-2}^2 - (-1)^k)^2}{q_{k-1}^2}.$$

(2) *If $q_{k-1} \equiv 0 \pmod 2$, then*

$$\alpha = \frac{q_{k-1}^2}{4}, \quad \beta = \frac{2q_{k-2} - (-1)^k q_{k-2}(q_{k-2}^2 - (-1)^k)}{2},$$
$$\gamma = \frac{(\frac{1}{4}q_{k-2}^2 - (-1)^k)(q_{k-2}^2 - (-1)^k)^2}{q_{k-1}^2}.$$

The discriminant of the polynomial $D = \alpha b^2 + \beta b + \gamma$ can attain only the values $1, -1, 4$ and -4. Specifically, for odd q_{k-1}, the discriminant equals $4 \cdot (-1)^k$ and for even q_{k-1}, the discriminant equals $(-1)^k$.

We now use these results to prove the statements about the congruence conditions on D for continued fractions of $\sqrt{D}$ with a given period length. We consider the cases of odd and even period lengths separately.

THEOREM 3.2. *Let $\sqrt{D} = [a_0, \overline{a_1, \dots, a_k}]$, where $k \equiv 1 \pmod 2$. Then, $D \not\equiv 0 \pmod p$ for every prime $p \equiv 3 \pmod 4$.*

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PROOF. For the sake of contradiction, assume that there exists such $D \equiv 0 \pmod{p}$. Then, using Theorem 3.1, we can express D as $D = \alpha b^2 + \beta b + \gamma \equiv 0 \pmod{p}$. According to Theorem 3.1, the discriminant of this polynomial is equal to -4 or -1. For the equation to have solutions, it follows that its discriminant has to be congruent to a square modulo p. Thus, $-4 \equiv c^2 \pmod{p}$ or $-1 \equiv c^2 \pmod{p}$. Since 4 and 1 are quadratic residues for every $p \equiv 3 \pmod 4$, then in both cases, -1 has to be a quadratic residue as well, which it is not. This gives the desired contradiction. □

Note that this proof generalises the theorems in the previous section, where we established the same property for $k = 1$ and $k = 3$.

We now continue with the case of an even period length. First, we prove several useful lemmas that will be needed in the main proof. In the statements, α and β are as in Theorem 3.1.

LEMMA 3.3. *Let $\sqrt{D} = [a_0, \overline{a_1, \dots, a_k}]$, where k is even, and let p be an odd prime and $l = \frac{1}{2}k - 1$.*

(a) *If l is odd, choose $a_l \equiv a_{l-2} \equiv a_{l-4} \cdots \equiv a_1 \equiv 0 \pmod{p^a}$.*
(b) *If l is even, choose $a_l \equiv a_{l-2} \equiv a_{l-4} \cdots \equiv a_4 \equiv 0 \pmod{p^a}$, $a_1 \equiv x \not\equiv 0 \pmod{p^a}$ and $a_2 \equiv -1/x \pmod{p^a}$.*

Then, $\alpha \equiv q_{k-1} \equiv 0 \pmod{p^a}$.

PROOF. We start with the well-known property of the convergents q_n that $q_{k-1} = q_l q_{l+1} + q_{l-1} q_l = q_l(q_{l+1} + q_{l-1})$.

(a) For odd l, by choosing $a_l \equiv a_{l-2} \equiv a_{l-4} \equiv \cdots \equiv a_1 \equiv 0 \pmod{p^a}$, we have $q_1 = a_1 \equiv 0 \pmod{p^a}$ and, by induction, if $q_{b-2} \equiv 0 \pmod{p}$, then we have $q_b = a_b q_{b-1} + q_{b-2} \equiv 0 \pmod{p}$ for odd $b \le l$.

(b) For even l, by choosing $a_l \equiv a_{l-2} \equiv a_{l-4} \equiv \cdots \equiv a_4 \equiv 0 \pmod{p^a}$, $a_1 \equiv x \not\equiv 0 \pmod{p^a}$, $a_2 \equiv -1/x \pmod{p^a}$, we obtain $q_2 = a_2 q_1 + q_0 = a_1 a_2 + 1 \equiv 0 \pmod{p^a}$ and, by induction, if $q_{b-2} \equiv 0 \pmod{p}$, then $q_b = a_b q_{b-1} + q_{b-2} \equiv 0 \pmod{p}$ for even $b \le l$.

At the same time, from Theorem 3.1, any set $a_1, a_2, \dots a_l, a_{l+1}$ has a solution if at least one of q_{k-2} and $(q_{k-2}^2 - (-1)^k)/q_{k-1}$ is even. We only have conditions modulo p^a on $a_1, a_2, \dots a_l, a_{l+1}$. It suffices to choose any suitable solution modulo 2. From the Chinese remainder theorem, we obtain conditions on $a_1, a_2, \dots a_l, a_{l+1}$ modulo $2p^a$ and from Theorem 3.1, the given set $a_1, a_2, \dots a_l, a_{l+1}$ has a solution. □

LEMMA 3.4. *Let $\sqrt{D} = [a_0, \overline{a_1, \dots, a_k}]$, where k is even, and let p be an odd prime and $l = \frac{1}{2}k - 1$.*

(a) *If l is odd, choose $a_l \equiv a_{l-2} \equiv a_{l-4} \cdots \equiv a_1 \equiv 0 \pmod{p^a}$.*
(b) *If l is even, choose $a_l \equiv a_{l-2} \equiv a_{l-4} \cdots \equiv a_4 \equiv 0 \pmod{p^a}$, $a_1 \equiv x \not\equiv 0 \pmod{p^a}$ and $a_2 \equiv -1/x \pmod{p^a}$.*

Then, β and p^a are coprime.

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PROOF. From Theorem 3.1, since k is even,

$$\beta = q_{k-2}(3 - q_{k-2}^2) \quad \text{or} \quad \beta = \frac{q_{k-2}(3 - q_{k-2}^2)}{2}.$$

Simultaneously, $q_{k-1} \equiv 0 \pmod{p^a}$. As q_{k-2} and q_{k-1} are coprime, q_{k-2} and p^a are also coprime. It suffices to prove that also $3 - q_{k-2}^2$ and p^a are coprime, so that $p \nmid 3 - q_{k-2}^2$.

(a) For odd l, we choose $a_{k-1} \equiv \cdots a_l \equiv a_{l-2} \equiv a_{l-4} \cdots \equiv a_1 \equiv 0 \pmod{p}$. Then, for any b, we have $q_b = a_b q_{b-1} + q_{b-2} \equiv q_{b-2} \pmod{p}$. By induction, it follows that $q_{k-2} \equiv q_2 = a_2 a_1 + 1 \equiv 1 \pmod{p}$.

(b) For even l, we choose $a_{k-4}, \ldots, a_l \equiv a_{l-2} \equiv a_{l-4} \cdots \equiv a_4 \equiv 0 \pmod{p}$ and $a_{k-2} = a_2 \equiv -1/a_1 \pmod{p}$. Then, for any $b < k-2$, we have $q_b = a_b q_{b-1} + q_{b-2} \equiv q_{b-2} \pmod{p}$. By induction, $q_b \equiv q_1 = a_1 \pmod{p}$. Simultaneously, for $2 < b < k-2$ and b even, we have $q_b = a_b q_{b-1} + q_{b-2} \equiv q_{b-2} \pmod{p}$. By induction, $q_b \equiv q_2 = 0 \pmod{p}$. Further, $q_{k-2} = a_2 q_{k-3} + q_{k-4} \equiv -a_1/a_1 + q_{k-4} \equiv -1 \pmod{p}$.

In both cases, $q_{k-2}^2 \equiv 1 \pmod{p}$. For $p \neq 2$, that means $q_{k-2}^2 \not\equiv 3$ and $p \nmid 3 - q_{k-2}^2$. It follows that p^a and β are coprime. □

We are now ready to present the proof for the case of an even period length.

THEOREM 3.5. *Let m, n, k be positive integers, where n is odd and k is even. Then, there exists a positive integer D such that the continued fraction $\sqrt{D}$ is of the form $\sqrt{D} = [a_0, \overline{a_1, \ldots, a_k}]$ and $D \equiv m \pmod{n}$.*

PROOF. We use the same choice of $a_l, a_{l-2}, a_{l-4}, \ldots$ as in Lemmas 3.3 and 3.4. From Lemma 3.3, we can obtain $\alpha \equiv q_{k-1} \equiv 0 \pmod{p^a}$ for any odd prime p and $a \in \mathbb{Z}^+$ by choosing $a_l, a_{l-2}, a_{l-4}, \ldots$ modulo p^a, where $l = \frac{1}{2}k - 1$. For $n = r_1^{a_1} r_2^{a_2} \cdots r_m^{a_m}$, where $r_1, \ldots, r_m$ are distinct primes, we can choose $a_l, a_{l-2}, a_{l-4}, \ldots$ modulo $r_1^{a_1}, r_2^{a_2}, \ldots, r_m^{a_m}$, and using the Chinese remainder theorem, we then obtain $\alpha \equiv q_{k-1} \equiv 0 \pmod{n}$. At the same time, from Lemma 3.4, whenever $p^a \mid n$, we have $p \nmid \beta$. Thus, n and β are coprime. Then, using Theorem 3.1, $D = \alpha b^2 + \beta b + \gamma \equiv \beta b + \gamma \pmod{n}$. Since n and β are coprime, D with a free variable b attains every value modulo n. □

For odd k, we proved that D cannot be divisible by any prime $p \equiv 3 \pmod{4}$, so the assumption of an even k is necessary.

For even n, using the choice from Lemma 3.3, we cannot ensure that in Case (a), the parity conditions from Friesen in Theorem 3.1 will be satisfied and there might not exist any solution for D. There would be a problem with even n in Lemma 3.4 as well because $3 - q_{k-2}^2 \equiv 2 \equiv 0 \pmod{2}$. The theorem could still hold for even n because there could be a different choice of $a_1, \ldots, a_k$ that satisfies all conditions.

4. Universal quadratic forms

Let us recall the definition of a quadratic form and a universal quadratic form. Throughout this section, let $D \in \mathbb{Z}_{>1}$ be squarefree. As a slight simplification, we work with the order $\mathbb{Z}[\sqrt{D}]$. Of course, this is the full ring of integers when $D \equiv 2, 3 \pmod{4}$.

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DEFINITION 4.1. Let $\alpha = a + b\sqrt{D} \in \mathbb{Z}[\sqrt{D}]$. Then, $\alpha' = a - b\sqrt{D}$ is called its conjugate. We say α is totally positive, denoted $\alpha \succ 0$, if $\alpha > 0$ and $\alpha' > 0$.

DEFINITION 4.2 (Quadratic form). By a quadratic form $Q(x_1, x_2, \ldots, x_r)$ over $\mathbb{Z}[\sqrt{D}]$, we mean

$$\sum_{1 \le i \le j \le r} \alpha_{ij} x_i x_j,$$

where $\alpha_{ij} \in \mathbb{Z}[\sqrt{D}]$ and $x_1, x_2, \ldots, x_r$ are variables; r is the rank of the quadratic form. The quadratic form Q is classical if $2 \mid a_{ij}$ for all $i \neq j$.

DEFINITION 4.3. A totally positive quadratic form is a quadratic form $Q(x_1, x_2, \ldots, x_r)$ over $\mathbb{Z}[\sqrt{D}]$ that satisfies $Q(\gamma_1, \gamma_2, \ldots, \gamma_r) \succ 0$ for all $\gamma_1, \gamma_2, \ldots, \gamma_r \in \mathbb{Z}[\sqrt{D}]$, except in the case $Q(0, 0, \ldots, 0) = 0$.

DEFINITION 4.4 (Universal quadratic form). A universal quadratic form over $\mathbb{Z}[\sqrt{D}]$ is a totally positive quadratic form $Q(x_1, x_2, \ldots, x_r)$ over $\mathbb{Z}[\sqrt{D}]$ such that for every $\alpha \in \mathbb{Z}[\sqrt{D}]$, $\alpha \succ 0$, there exist elements $\gamma_1, \gamma_2, \ldots, \gamma_r \in \mathbb{Z}[\sqrt{D}]$ satisfying $Q(\gamma_1, \gamma_2, \ldots, \gamma_r) = \alpha$.

We also state the result of Kala and Tinková [6] concerning the number of variables required for a universal quadratic form. Note that although [5, 6] state this result for the full ring of integers, it holds with exactly the same proof for the order $\mathbb{Z}[\sqrt{D}]$, even when it is not maximal.

THEOREM 4.5 (Kala and Tinková [5, Theorem 5.5]). *Let* $\sqrt{D} = [a_0, \overline{a_1, \ldots a_k}]$ *and*

$$U = \begin{cases} \max(a_1, a_3, \ldots, a_{k-1}) & \text{if } k \text{ is even}, \\ \sqrt{D} & \text{if } k \text{ is odd}. \end{cases}$$

Let Q be a universal quadratic form over $\mathbb{Z}[\sqrt{D}]$ of rank r.

(a) *If Q is classical, then $r \ge \frac{1}{2}U$.*
(b) *In general, $r \ge \frac{1}{2}\sqrt{U}$ (assuming $U \ge 240$).*

We can now use our results from the previous sections, together with this theorem, to prove that there exists D in a given congruence class such that universal quadratic forms over $\mathbb{Z}[\sqrt{D}]$ must have many variables.

THEOREM 4.6. *Let m, n, s be positive integers. Then, there exists a nonsquare $D \ge 2$ such that $D \equiv m \pmod{n}$ and every universal quadratic form over $\mathbb{Z}[\sqrt{D}]$ has at least s variables.*

PROOF. According to Theorem 2.3, for $\sqrt{D} = [t, \overline{u, v, u, 2t}]$, D can attain every value modulo any $n \in \mathbb{N}$. This imposes only congruence conditions on t, u and v modulo $2n$ (or n), so the coefficients $a_1 = a_3 = u$ can be chosen arbitrarily large. Then, it suffices to choose u such that $\frac{1}{2}\sqrt{u} \ge s$ and simultaneously $\max(a_1 a_3) = u \ge 240$. From

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Theorem 4.5, the number of variables of any universal quadratic form over $Z[\sqrt{D}]$ is at least $r \geq \frac{1}{2}\sqrt{U} = \frac{1}{2}\sqrt{\max(a_1, a_3, \ldots, a_{k-1})} = \frac{1}{2}\sqrt{u} \geq s$. □

The following theorem additionally allows us to prescribe an arbitrary even period length.

THEOREM 4.7. *Let m, n, s, k be positive integers such that n is odd and k even. Then, there exists a nonsquare $D \geq 2$ such that $D \equiv m \pmod{n}$, the continued fraction for $\sqrt{D}$ has a period length of k and every universal quadratic form over $\mathbb{Z}[\sqrt{D}]$ has at least s variables.*

PROOF. From Theorem 3.5, we can choose $a_0, \ldots, a_k$ modulo n such that $D \equiv m \pmod{n}$ for any n. Because these are just modular conditions, we can choose $a_0, \ldots, a_k$ arbitrarily large so that $\frac{1}{2}\sqrt{\max(a_1, a_3, \ldots, a_{k-1})} \geq s$ and, at the same time, $\max(a_0, \ldots, a_k) \geq 240$. Then, by Theorem 4.5, the number of variables of any universal quadratic form over $\mathbb{Z}[\sqrt{D}]$ is at least $r \geq \frac{1}{2}\sqrt{U} = \frac{1}{2}\sqrt{\max(a_1, a_3, \ldots, a_{k-1})} \geq s$. □

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