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Blow-up for the wave equation with nonlinear source and boundary damping terms – CORRIGENDUM

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This note corrects an inaccuracy in the paper "Blow-up for the wave equation with nonlinear source and boundary damping terms" by the authors, appeared in Proc. Roy. Soc. Edinburgh Sect. A **145** (2015), no. 4, 759–778.

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All results in the paper [1] are stated under the assumption that Ω is a bounded open subset of $\mathbb{R}^n$ $(n \geqslant 1)$ with $C^{1,1}$ boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where Γ_0 and Γ_1 are disjoint subsets of Γ. The manifold Γ is equipped with the natural Lebesgue measure σ on the manifold, Γ_0 and Γ_1 are assumed to be measurable with respect to σ and, finally, $\sigma(\Gamma_0)$ is assumed to be positive. The authors recently pointed out that Ω must be connected for most of these results to make sense.

The paper [1] was aimed to relax the assumption $m < \frac{2(n+1)p-4(n-1)}{n(p-2)+4}$, which appears in [3, Theorem 7], to the assumption $m < 1 + p/2$, which appears in [1, Theorems 1.1 and 3.2]. But in [3] the open set Ω was assumed to be connected, while in [1] this assumption is missing. Unfortunately, as we are going to show, the variational setting under which [1, Theorems 1.1 and 3.2] are stated may fail to hold when Ω is disconnected, depending on the location of Γ_0.

We recall that, at p. 761 of [1], we introduced the Hilbert space

$$H^1_{\Gamma_0}(\Omega) = \left\{ u \in H^1(\Omega) : u_{|\Gamma_0} = 0 \right\},$$

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where $u_{|\Gamma_0}$ is intended in the trace sense. Moreover, by recalling the setting of [3], the space $H^1_{\Gamma_0}(\Omega)$ was equipped with the norm $\|\nabla(\cdot)\|_2$, which was asserted to be equivalent to the standard norm of $H^1(\Omega)$, that is to the norm defined by

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_2^2 + \|\nabla u\|_2^2\right)^{1/2} \quad \text{for all } u \in H^1(\Omega).$$

Now the authors realized that this setting is correct only when Ω is connected, i.e. in the original setting of [3], since when Ω is disconnected the bilinear form $(\cdot,\cdot)$ defined on $H^1_{\Gamma_0}(\Omega)$ by

$$(u,v) = \int_\Omega \nabla u \nabla v \quad \text{for all } u,\, v \in H^1_{\Gamma_0}(\Omega),$$

may even fail to be positive definite.

Indeed, let us consider the particular case when $\Omega = \Omega_1 \cup \Omega_2$, Ω_1 and Ω_2 being open and nonempty, $\overline{\Omega_1} \cap \overline{\Omega_2} = \emptyset$, and $\Gamma_0 = \partial\Omega_2$, $\Gamma_1 = \partial\Omega_1$. Let us set the characteristic function χ_{Ω_1}, defined on Ω, that is

$$\chi_{\Omega_1}(x) = \begin{cases} 0 & \text{when } x \in \Omega_2, \\ 1 & \text{when } x \in \Omega_1. \end{cases}$$

Its equivalence class with respect to a.e. equivalence, still denoted (as usual) as χ_{Ω_1}, belongs to $H^1(\Omega)$, and one has $\chi_{\Omega_1|\Gamma_0} \equiv 0$, while $\chi_{\Omega_1|\Gamma_1} \equiv 1$, and hence $\chi_{\Omega_1} \in H^1_{\Gamma_0}(\Omega)$. Since $\nabla\chi_{\Omega_1} \equiv 0$ in Ω, we then have $(\chi_{\Omega_1}, \chi_{\Omega_1}) = 0$, while $\chi_{\Omega_1} \not\equiv 0$.

On the other hand, when Ω is connected, the setting is correct. Indeed, by the Area Formula, see [2, Chapter 9, Theorem 9.27, p. 253], $\sigma(\Gamma_0) = \mathcal{H}^{N-1}(\Gamma_0)$, where $\mathcal{H}^{N-1}$ denotes the Hausdorff measure. Hence, by [4, Chapter 2, Theorem 2.6.16, p. 75], one derives that the capacity $B_{1,2}(\Gamma_0)$ is positive. One then applies [4, Chapter 4, Corollary 4.5.2, p. 195] to conclude that the following Poincaré-type inequality holds: there is a positive constant $c_1 = c_1(\Omega, \Gamma_0)$ such that

$$\|u\|_2 \le c_1 \|\nabla u\|_2$$

for all $u \in H^1(\Omega)$ such that $u_{|\Gamma_0} = 0$. Unfortunately, the authors have not noticed, before the publication of [1], that [4, Chapter 4, Corollary 4.5.2, p. 195] is obtained as a consequence of [4, Chapter 4, Theorem 4.5.1, p. 195], where Ω is assumed to be connected, so this proof is based on the connectedness assumption.

Most of the results of [1] can be misleading when Ω is disconnected. For example, in the particular case $\Omega = \Omega_1 \cup \Omega_2$ considered above, the potential well depth d defined in [1, (1.4)] vanishes. Indeed, we recall that

$$d = \inf_{u \in H^1_{\Gamma_0}(\Omega)\setminus\{0\}} \sup_{\lambda>0} J(\lambda u), \tag{0.1}$$

where the functional J is defined by

$$J(u) = \tfrac{1}{2}\|\nabla u\|_2^2 - \tfrac{1}{p}\|u\|_p^p \quad \text{for all } u \in H^1_{\Gamma_0}(\Omega),$$

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vanishes. An easy calculation shows, as in the proof of [1, Lemma 4.1] that, for any $u \in H^1_{\Gamma_0}(\Omega) \setminus \{0\}$, one has

$$\sup_{\lambda>0} J(\lambda u) = \max_{\lambda>0} J(\lambda u) = \left(\frac{1}{2} - \frac{1}{p}\right) \left(\frac{\|\nabla u\|_2}{\|u\|_p}\right)^{\frac{2p}{p-2}} \geqslant 0.$$

Since we have $\sup_{\lambda>0} J(\lambda \chi_{\Omega_1}) = 0$, we get that the infimum in (0.1) is attained, as a minimum, when $u = \chi_{\Omega_1}$, so $d = 0$. Consequently, the so-called 'bad part of the potential well' defined in [1, (1.5)] does not exist in this case, and [1, Theorem 1.1] is wrong. In the particular case considered above, also [1, Theorem 3.2] cannot be stated. Indeed, the positive constant K_0 defined in formula [1, (2.15)], that is

$$K_0 = \sup_{u \in H^1_{\Gamma_0}(\Omega)\setminus\{0\}} \frac{\int_\Omega F(\cdot, u)}{\|\nabla u\|_2^p},$$

is clearly ill-defined, since $\chi_{\Omega_1} \not\equiv 0$, so we can take it in the supremum, but $\nabla \chi_{\Omega_1} \equiv 0$, so the fraction does not make sense.

Finally, also the local existence of weak solutions of problem [1, (1.1)], which is asserted in [1, Theorem 2.6], cannot be taken for granted when Ω is disconnected. Indeed [1, Remark 2.7], when it asserts that in [3] one can consider also disconnected open sets, is wrong. Indeed, when $\|\nabla(\cdot)\|_2$ is not an equivalent norm in the space $H^1_{\Gamma_0}(\Omega)$, as in the case $\Omega = \Omega_1 \cup \Omega_2$ considered above, the functional I defined in the last line of [3, p. 383] is not coercive, so there is substantial gap in the proof of [1, Theorem 2.6]. The same phenomenon occurs when dealing with [1, Theorem 2.8].

In conclusion, the assumption that 'Ω is a bounded open subset of $\mathbb{R}^n$' must be replaced in paper [1] by 'Ω is a bounded domain (that is, a connected open subset) of $\mathbb{R}^n$'. This assumption appears in three instances, in the Abstract and twice at the beginning of Section 1. Under this new assumption, all the results in paper [1] hold. As correctly pointed out in [1, Remark 2.7], the case in which Ω is disconnected is not of particular interest.

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References

1. A. Fiscella and E. Vitillaro. Blow-up for the wave equation with nonlinear source and boundary damping terms. *Proc. Roy. Soc. Edinburgh Sect. A*, 2015;**145**(4):759–778.
2. G. Leoni. *A first course in Sobolev Spaces*, volume 181 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2017. ISBN 978-1-4704-2921-8.
3. E. Vitillaro. A potential well theory for the wave equation with nonlinear source and boundary damping terms. *Glasg. Math. J.*, **44**(3):375–395, 2002.
4. W. P. Ziemer. *Weakly differentiable functions*, volume 120 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1989. Sobolev Spaces and functions of bounded variation.

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