Defining trigonometric functions via complex sequences

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1. *Introduction*

In the literature we find several different ways of introducing elementary functions. For the exponential function, we mention the following ways of characterising the exponential function:

(a) $\exp(x) = \lim_{n \to +\infty} \left(1 + \frac{x}{n}\right)^n$ [1, 2];

(b) $\exp(x) = \sum_{k=0}^{+\infty} \frac{x^k}{k!}$ [3], also for complex values of x;

(c) $x \to \exp(x)$ is the unique solution to the initial value problem [4]

$$u'(x) = u(x), \qquad u(0) = 1;$$

(d) $x \to \exp(x)$ is the inverse of $\log x = \int_1^x \frac{1}{t}\, dt$ [5];

(e) $x \to \exp(x)$ is the unique continuous function satisfying the functional equation $f(x + y) = f(x)f(y)$ and $f(0) = 1$ [6]; the corresponding definition is done for the logarithm in [7];

(f) Define a^r for rational r, and then use a continuity/density argument [8].

All of them have their advantages and disadvantages. We like (a) and (c), mostly because they have natural interpretations, (a) in the setting of compound interest and (c) being a simple model of many processes in physics and other sciences, but also because they are related to methods and ideas that are (usually) introduced rather early to the students.

One drawback with (c) is that the existence and uniqueness theorems are usually not at hand.

When it comes to introducing the trigonometric function sine (similarly for cosine), we mention:

(i) the geometric definition from a right-angled triangle or via the unit circle [8, 9];

(ii) $\sin x = \frac{\exp(ix) - \exp(-ix)}{2i}$, [3];

(iii) $\sin x = \sum_{k=0}^{+\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$, [10];

(iv) $x \to \sin x$ is the unique solution to the initial value problem

$$u''(x) + u(x) = 0, \qquad u(0) = 0,\ u'(0) = 1\ [4,\ 11];$$

(v) $x \to \sin x$ is the inverse of $\arcsin(x) = \int_0^x \frac{1}{\sqrt{1 - t^2}}\, dt$, continued in a symmetric/periodic manner [12].

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Students seem to be familiar with the first two geometrical definitions from high school. We like (ii), since it connects a trigonometric function with the exponential function, and (iv), since it is an important model appearing in various applications, describing oscillations.

The aim of the present article is twofold. Firstly, we want to define the trigonometric functions sine and cosine in a way that is motivated by the differential equation in (iv), and connected to the exponential function. Secondly, our aim is that the definition should only use very simple concepts, so that it could appear early in the curriculum of a calculus class.

Using the Euler method of numerically solving the differential equations in (c) and (iv), we are naturally led (see Section 2 below for details) to the sequences (E_n) and (S_n):

$$E_n = \left(1 + \frac{x}{n}\right)^n, \qquad S_n = \frac{\left(1 + \frac{ix}{n}\right)^n - \left(1 - \frac{ix}{n}\right)^n}{2i}. \tag{1}$$

We define $\exp(x)$ for real x as the limit of (E_n) as $n \to \infty$. This is pretty standard, so we only sketch a method of doing it in Section 4. Next, the sequence (S_n) suggests that to find a solution of the differential equation (iv) we should study the sequence (E_n), but for complex x. This is done in Section 5, and our main task is to prove convergence. Once that is done, the expected properties follow rather easily.

The idea of defining a complex exponential function via the sequence (E_n) is of course not new, see for example [13, p. 166]. However, we have not seen the motivation from the sequence (S_n) above, and the basic methods of proving convergence before, without discussing the argument of complex numbers or angles. In Section 6 we define the trigonometric functions (via (ii)), and get immediately several of their natural properties from the corresponding properties of the complex exponential function. For the sake of completeness, we also give the usual geometric interpretation of the trigonometric functions on the unit circle, together with a definition of π.

2. *Motivation*

The most basic method to find numerical approximations of differential equations is the Euler method. If we want to approximate the solution to the differential equation

$$u'(t) = f\big(t, u(t)\big), \qquad u(0) = u_0,$$

on the interval $[0, x]$, we first divide the interval into n equally large subintervals, $t_k = kx/n$, $0 \leqslant k \leqslant n$. Here, $h = x/n$ is called the step size. Next, we approximate $u'(t)$ with its difference quotient $\big(u(t + h) - u(t)\big)/h$. Moreover, for $k \geqslant 1$, we let u_k denote the approximate solution at $t = t_k$. The Euler technique gives

$$\begin{cases} u_0 = u(0), \\ u_{k+1} = u_k + hf(t_k, u_k), \quad k \geqslant 0. \end{cases}$$

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Example 1: To find an approximate solution to the differential equation

$$e'(t) = e(t), \qquad e(0) = 1. \tag{2}$$

on the interval $[0, x]$ with step size $h = x/n$, we solve

$$e_{k+1} = e_k + he_k = (1 + h)e_k, \qquad e_0 = 1.$$

Using induction, or some other method, to solve recurrence equations, we find that

$$e_k = (1 + h)^k = \left(1 + \frac{x}{n}\right)^k.$$

In particular, the solution at the endpoint x is approximated as

$$e(x) \approx e_n = \left(1 + \frac{x}{n}\right)^n. \tag{3}$$

In Figure 1, we have plotted the approximative sequence $(e_k)_{k=1}^n$ to (2) in the case $n = 10$ (left) and $n = 100$ (right), in the interval [0, 1]. We have also plotted the exact solution of (2), and we see that as n increases the approximate solution gets closer to the exact solution. This motivates the definition of the sequence (E_n) in (1).

Example 2: The Euler method works also on systems of differential equations. Thus, if we want to solve the differential equation

$$s''(t) + s(t) = 0, \qquad s(0) = 0, s'(0) = 1, \tag{4}$$

then it is appropriate to introduce $c(t) = s'(t)$. Then $c'(t) = -s(t)$, and $c(t)$ will necessarily satisfy

$$c''(t) + c(t) = 0, \qquad c(0) = 1, c'(0) = 0. \tag{5}$$

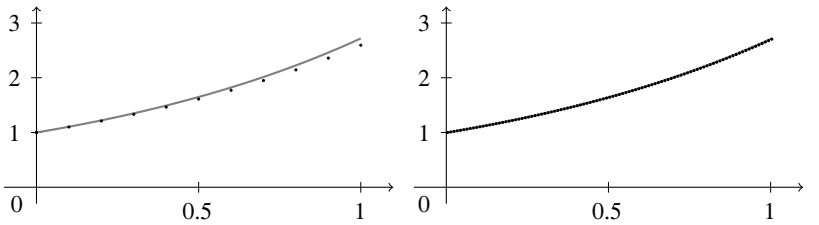


FIGURE 1: The approximation of the solution to $e'(t) = e(t)$, $e(0) = 1$ on the interval [0,1], with ten subintervals. The exact solution is plotted in grey. Right: The same, but with 100 subintervals.

Moreover, we get the first order system

$$\begin{bmatrix} s'(t) \\ c'(t) \end{bmatrix} = \begin{bmatrix} c(t) \\ -s(t) \end{bmatrix}.$$

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This time, with s_k and c_k, the approximate solutions at t_k, we get

$$\begin{bmatrix} s_{k+1} \\ c_{k+1} \end{bmatrix} = \begin{bmatrix} s_k \\ c_k \end{bmatrix} + h\begin{bmatrix} c_k \\ -s_k \end{bmatrix} = \begin{bmatrix} 1 & h \\ -h & 1 \end{bmatrix}\begin{bmatrix} s_k \\ c_k \end{bmatrix}, \qquad \begin{bmatrix} s_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The matrix above has eigenvalues $1 \pm hi$ with eigenvectors $(\mp i,\ 1)$ respectively. This means that we can find constants a and b such that

$$\begin{bmatrix} s_k \\ c_k \end{bmatrix} = a(1 + hi)^k\begin{bmatrix} -i \\ 1 \end{bmatrix} + b(1 - hi)^k\begin{bmatrix} i \\ 1 \end{bmatrix}.$$

Inserting $k = 0$,

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\begin{bmatrix} -i \\ 1 \end{bmatrix} + b\begin{bmatrix} i \\ 1 \end{bmatrix},$$

so $a = b = \frac{1}{2}$. Thus, we can write s_k and c_k as

$$s_k = \frac{(1 + hi)^k - (1 - hi)^k}{2i}, \qquad c_k = \frac{(1 + hi)^k + (1 - hi)^k}{2}.$$

For $k = n$, we get

$$s(x) \approx s_n = \frac{\left(1 + \frac{xi}{n}\right)^n - \left(1 - \frac{xi}{n}\right)^n}{2i} \tag{6}$$

and

$$c(x) \approx c_n = \frac{\left(1 + \frac{xi}{n}\right)^n + \left(1 - \frac{xi}{n}\right)^n}{2i}. \tag{7}$$

In Figure 2 we have plotted the approximative sequences $(s_k)_{k=0}^n$ of (4) in the case $n = 100$ (left) and $n = 1000$ (right), in the interval $[0,\ 10]$. We have also plotted the exact solution of (4) in grey, and we see that as n increases the approximate solution gets closer to the real solution. This motivates the study of the sequence (S_n) in (1), and thus, the study of the sequence (E_n) for complex x.

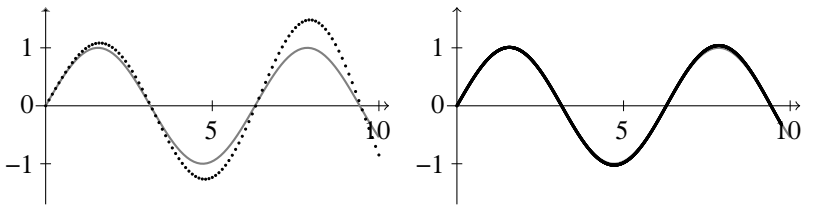


FIGURE 2: The approximation of the solution of $s''(t) + s(t) = 0$, with initial conditions $s(0) = 0, s'(0) = 1$, in the interval $[0,\ 10]$, using 100 subintervals. The exact solution is plotted in grey. Right: The same, but with 1000 subintervals.

Motivated by these examples, we continue to define the exponential function, first for real arguments, and then for complex ones. Then we

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define sine and cosine in terms of the complex exponential function, and verify that they indeed satisfy the corresponding initial value problem. Before doing so, we mention some tools that we believe the reader/student will have met before, and that we will use below.

3. *Prerequisites*

Here we list some results that we will use, and believe that the university student is, or at least could be, familiar with.

Axiom 1: A bounded monotone sequence converges.

Result 1: Let $a_k \geqslant 0$ and let there be a constant C such that $\sum_{k=1}^{n} a_k \leqslant C$ for all $n \in \mathbb{N}$. Then $\lim_{n \to +\infty} \sum_{k=1}^{n} a_k$ exists.

Result 2: Let $a_k \in \mathbb{C}$, and let there be a constant C such that $\sum_{k=1}^{n} |a_k| \leqslant C$ for all $n \in \mathbb{N}$. Then $\lim_{n \to +\infty} \sum_{k=1}^{n} a_k$ exists.

Result 3 (The Bernoulli inequality): Let $x > -1$ and $n \in \mathbb{N}$. Then

$$(1 + x)^n \geqslant 1 + nx.$$

Result 4 (The arithmetic-geometric mean inequality): Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ be non-negative real numbers. Then

$$(a_1 a_2 \ldots a_n)^{1/n} \leqslant \frac{a_1 + a_2 + \ldots + a_n}{n}.$$

This inequality is easily proved. One nice approach that fits this presentation is given in [14].

4. *The (real) exponential function*

The exponential function for real arguments has been discussed in many places before. We give some details of one way to define it, from which many of its properties follows directly.

Theorem 1: Let $x \in \mathbb{R}$. Then, if $n > |x|$,

$$\left(1 + \frac{x}{n}\right)^n \leqslant \left(1 + \frac{x}{n+1}\right)^{n+1} \leqslant \frac{1}{\left(1 - \frac{x}{n+1}\right)^{n+1}} \leqslant \frac{1}{\left(1 - \frac{x}{n}\right)^n}.$$

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Proof: The first inequality follows from the arithmetic-geometric mean inequality

$$1 \times \left(1 + \frac{x}{n}\right)^n \leqslant \left(\frac{1 + n\left(1 + \frac{x}{n}\right)}{n + 1}\right)^{n+1} = \left(1 + \frac{x}{n+1}\right)^{n-1}. \qquad (8)$$

The second one follows by writing

$$\left(1 + \frac{x}{n+1}\right)^{n+1}\left(1 - \frac{x}{n+1}\right)^{n+1} = \left(1 - \frac{x^2}{(n+1)^2}\right)^{n+1} \leqslant 1.$$

The third one follows from (8) by replacing x with $-x$.

We conclude that for all $x \in \mathbb{R}$, the limit $\lim_{n \to +\infty}\left(1 + \frac{x}{n}\right)^n$ exists, by Axiom 1 and Theorem 1, and make the following definition.

Definition: We define the real exponential function, $\exp : \mathbb{R} \to \mathbb{R}$, as

$$\exp(x) = \lim_{n \to +\infty}\left(1 + \frac{x}{n}\right)^n.$$

From $\left(1 - \frac{x^2}{n^2}\right)^n = \left(1 + \frac{x}{n}\right)^n\left(1 - \frac{x}{n}\right)^n$ and the Bernoulli inequality, valid if $|x| < n$,

$$1 \geqslant \left(1 - \frac{x^2}{n^2}\right)^n \geqslant 1 - \frac{x^2}{n}.$$

It follows that

$$\exp(x)\exp(-x) = 1,$$

and in particular that $\exp(x) \neq 0$. It also follows that

$$\exp(x) = \frac{1}{\exp(-x)} = \lim_{n \to +\infty}\left(1 - \frac{x}{n}\right)^{-n}.$$

Next, we note that $\exp(x) > 0$ for all $x \in \mathbb{R}$, since (if $n > |x|$)

$$\exp(x) \geqslant \left(1 + \frac{x}{n}\right)^n \geqslant 0.$$

The Bernoulli inequality also implies that $\exp x \exp y = \exp(x + y)$, via the two inequalities (again, we assume that n is sufficiently large)

$$\frac{\left(1 + \frac{x}{n}\right)^n\left(1 + \frac{y}{n}\right)^n}{\left(1 + \frac{x+y}{n}\right)^n} = \left[1 + \frac{1}{n}\frac{xy}{n + (x+y)}\right]^n > 1 + \frac{xy}{n + x + y}$$

and (the equality here follows from the one in the previous formula, just by changing n to $-n$)

$$\frac{\left(1 - \frac{x}{n}\right)^{-n}\left(1 - \frac{y}{n}\right)^{-n}}{\left(1 - \frac{x+y}{n}\right)^{-n}} = \frac{1}{\left[1 + \frac{1}{n}\frac{xy}{n-(x+y)}\right]^n} < \frac{1}{1 + \frac{xy}{n-(x+y)}}.$$

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Also, for $n > |x|$,

$$\exp x \geqslant \left(1 + \frac{x}{n}\right)^n \geqslant 1 + x,$$

and since the left-hand side and right-hand side are independent of n, the inequality $\exp x \geqslant 1 + x$ follows. In the same manner we get differentiability of exp. Indeed, if $n \in \mathbb{N}$ and $0 < |h| < 1$,

$$h \leqslant \left(1 + \frac{h}{n}\right)^n - 1 \leqslant \exp h - 1 \leqslant \left(1 - \frac{h}{n}\right)^{-n} - 1 \leqslant \frac{1}{1-h} - 1 = \frac{h}{1-h},$$

so the value of $(\exp h - 1)/h$ is between 1 and $1/(1-h)$, and thus

$$\lim_{h \to 0} \frac{\exp h - 1}{h} = 1.$$

It follows directly that the derivative of $\exp x$ is $\exp x$,

$$\lim_{h \to 0} \frac{\exp(x+h) - \exp x}{h} = \exp x \lim_{h \to 0} \frac{\exp h - 1}{h} = \exp x.$$

One conclusion from this is that the exponential function is monotonically increasing, and differentiating once more, we find that it is convex. Moreover, it satisfies the initial value problem (2).

We conclude this section by noting that since the exponential function is monotonically increasing it is invertible, and we define the logarithm by $y = \log x$ if, and only if, $x = \exp y$. The usual properties of log follow easily, and we do not give the details, but mention that at this point it is a good idea to define a^b for $a > 0$ and $b \in \mathbb{R}$ as $a^b = \exp(b \log a)$.

5. *The complex exponential function*

We will base the definition of the complex exponential function on the following limit.

Theorem 2: Let $z \in \mathbb{C}$. Then the limit

$$\lim_{n \to +\infty} \left(1 + \frac{z}{n}\right)^n$$

exists.

We emphasise again, that for purely imaginary z the existence of the limit above is easily found if we are familiar with both the absolute value and the argument of a complex number (see [13]). We assume that only the absolute value is known. We divide some calculations needed in the proof of the theorem into two lemmas.

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Lemma 1: Let $0 \leqslant j < k$ be integers. Then

$$\binom{k}{j}\frac{1}{k^j} - \binom{k-1}{j}\frac{1}{(k-1)^j} \geqslant 0.$$

Proof: The stated inequality is equivalent to

$$\frac{(k-1)^j}{k^j} \geqslant \frac{\binom{k-1}{j}}{\binom{k}{j}}.$$

Now, the left-hand side equals $\left(1 - \frac{1}{k}\right)^j$ and the right-hand side equals $1 - \frac{j}{k}$. The inequality thus follows from the Bernoulli inequality.

Lemma 2: Let $z \in \mathbb{C}$ and let $k \geqslant 2$ be an integer. Then

$$\left|\left(1 + \frac{z}{k}\right)^k - \left(1 + \frac{z}{k-1}\right)^{k-1}\right| \leqslant \left(1 + \frac{|z|}{k}\right)^k - \left(1 + \frac{|z|}{k-1}\right)^{k-1}.$$

Proof: We write, using the binomial theorem,

$$\left(1 + \frac{z}{k}\right)^k - \left(1 + \frac{z}{k-1}\right)^{k-1} = \frac{z^k}{k^k} + \sum_{j=2}^{k-1} z^j\left[\binom{k}{j}\frac{1}{k^j} - \binom{k-1}{j}\frac{1}{(k-1)^j}\right].$$

Here, we have used that the first two terms of the sum (corresponding to $j = 0$ and $j = 1$) cancel. The expression in brackets is non-negative according to Lemma 1, so the triangle inequality gives

$$\left|\left(1 + \frac{z}{k}\right)^k - \left(1 + \frac{z}{k-1}\right)^{k-1}\right| \leqslant \frac{|z|^k}{k^k} + \sum_{j=2}^{k-1} |z|^j\left[\binom{k}{j}\frac{1}{k^j} - \binom{k-1}{j}\frac{1}{(k-1)^j}\right].$$

Using the binomial theorem again, we find that the right-hand side equals

$$\left(1 + \frac{|z|}{k}\right)^k - \left(1 + \frac{|z|}{k-1}\right)^{k-1}.$$

We are now ready to give a proof of Theorem 2.

Proof of Theorem 2: Let $z \in \mathbb{C}$ be fixed. We use a telescoping sum,

$$\left(1 + \frac{z}{n}\right)^n = 1 + z + \sum_{k=2}^{n}\left[\left(1 + \frac{z}{k}\right)^k - \left(1 + \frac{z}{k-1}\right)^{k-1}\right].$$

Using Lemma 2 on the terms in the brackets, we find that

$$\sum_{k=2}^{n}\left|\left[\left(1 + \frac{z}{k}\right)^k - \left(1 + \frac{z}{k-1}\right)^{k-1}\right]\right| \leqslant \sum_{k=2}^{n}\left(\left(1 + \frac{|z|}{k}\right)^k - \left(1 + \frac{|z|}{k-1}\right)^{k-1}\right)$$

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$$= \left(1 + \frac{|z|}{n}\right)^n - 1 - |z|$$

$$\leqslant \exp|z| - 1 - |z|.$$

Thus, according to Result 2, the limit

$$\lim_{n \to +\infty} \sum_{k=2}^{n} \left[\left(1 + \frac{z}{k}\right)^k - \left(1 + \frac{z}{k-1}\right)^{k-1}\right]$$

exists, which implies that the sequence $n \to \left(1 + \frac{z}{n}\right)^n$ converges.

Remark: The above calculation also give the inequality

$$\left|\left(1 + \frac{z}{n}\right)^n - 1 - z\right| \leqslant \left(1 + \frac{|z|}{n}\right)^n - 1 - |z| \tag{9}$$

for all $n \in \mathbb{N}$.

We extend the exponential function so that it is defined on $\mathbb{C}$.

Definition: For $z \in \mathbb{C}$, we let

$$\exp z = \lim_{n \to +\infty} \left(1 + \frac{z}{n}\right)^n.$$

Before we turn to the trigonometric functions, we give some properties of the complex exponential function.

Theorem 3: Let $x \in \mathbb{R}$, $z \in \mathbb{C}$ and $w \in \mathbb{C}$. Then

(a) $|\exp(ix)| = 1$,

(b) $\overline{\exp z} = \exp \bar{z}$,

(c) $\exp z \exp w = \exp(z + w)$ (in particular $\exp z \exp(-z) = 1$),

(d) $(\exp z)^n = \exp(nz)$,

(e) The mapping $x \to \exp(ix)$ is differentiable, with $\mathrm{D}\exp(ix) = i\exp(ix)$.

Proof:

(a) We estimate the absolute value,

$$1 \leqslant \left|\left(1 + \frac{ix}{n}\right)^n\right| = \left(1 + \frac{x^2}{n^2}\right)^{n/2} = \left[\left(1 + \frac{x^2}{n^2}\right)^{n^2}\right]^{1/(2n)}$$

$$\leqslant \left(\exp\left(x^2\right)\right)^{1/(2n)} = \exp\left(\frac{x^2}{2n}\right),$$

and the right-hand side converges to $\exp 0 = 1$ as $n \to +\infty$, so the property follows from the sandwich theorem for sequences.

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(b) This follows directly from the fact that the conjugate of $\left(1 + \frac{z}{n}\right)^n$ is $\left(1 + \frac{\bar{z}}{n}\right)^n$.

(c) We use the binomial theorem, and write

$$\left(1 + \frac{z}{n}\right)^n \left(1 + \frac{w}{n}\right)^n = \left(1 + \frac{z + w}{n} + \frac{zw}{n^2}\right)^n$$

$$= \left(1 + \frac{z+w}{n}\right)^n + \sum_{k=1}^{n} \binom{n}{k} \left(1 + \frac{z+w}{n}\right)^{n-k} \left(\frac{zw}{n^2}\right)^k.$$

Our task is to show that the sum converges to zero as $n \to +\infty$. Indeed, we factor out $\frac{1}{n}$, and get, as $n \to +\infty$,

$$\left|\sum_{k=1}^{n} \binom{n}{k} \left(1 + \frac{z + w}{n}\right)^{n-k} \left(\frac{zw}{n^2}\right)^k\right| \leqslant \frac{1}{n} \sum_{k=1}^{n} \left(1 + \frac{|z + w|}{n}\right)^{n-k} \left(\frac{|zw|}{n}\right)^k$$

$$\leqslant \frac{1}{n} \left(1 + \frac{|z + w| + |zw|}{n}\right)^n$$

$$\leqslant \frac{1}{n} \exp\left(|z + w| + |zw|\right) \to 0.$$

(d) This follows immediately from the last item upon induction.

(e) Let $h \neq 0$ be real. We write the difference quotient

$$\frac{\exp(i(x + h)) - \exp(ix)}{h} = \exp(ix) \frac{\exp(ih) - 1}{h}.$$

We are done if we can show that the quotient has limit i as $h \to 0$. We use (9) with $z = ih$, to get

$$\left|\frac{\left(1 + \frac{ih}{n}\right)^n - 1}{h} - i\right| \leqslant \frac{1}{|h|}\left[\left(1 + \frac{|h|}{n}\right)^n - 1 - |h|\right].$$

As $n \to +\infty$,

$$\left|\frac{\exp(ih) - 1}{h} - i\right| \leqslant \frac{1}{|h|}\left(\exp|h| - 1 - |h|\right) = \frac{\exp|h| - 1}{|h|} - 1,$$

which tends to zero as h tends to zero.

Remark: With a similar argument we could show that $z \to \exp z$ is complex differentiable, and hence holomorphic on all of $\mathbb{C}$, but that lies outside the scope of our aim.

6. *The trigonometric functions*

According to our plans, we now define the trigonometric functions in terms of the complex exponential function.

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Definition: For $x \in \mathbb{R}$, we let

$$\cos x = \frac{\exp(ix) + \exp(-ix)}{2}, \qquad \sin x = \frac{\exp(ix) - \exp(-ix)}{2i}.$$

We first note that $\cos 0 = 1$ and $\sin 0 = 0$, and given the definition of $\exp z$ for complex z the definitions above agree with the suggestion we got from solving the corresponding differential equations with the Euler method, (6) and (7). It is a good exercise for the students to verify all the properties listed below.

Theorem 4: Let x and y be real numbers. Then

(a) $\cos x \in \mathbb{R}$ and $\sin x \in \mathbb{R}$,

(b) $\mathrm{D} \cos x = -\sin x$,

(c) $\mathrm{D} \sin x = \cos x$,

(d) $x \to \sin x$ is a solution of (4),

(e) $x \to \cos x$ is a solution of (5),

(f) $\mathrm{Re}\big(\exp(ix)\big) = \cos x$,

(g) $\mathrm{Im}\big(\exp(ix)\big) = \sin x$,

(h) $\exp(ix) = \cos x + i \sin x$,

(i) $\cos^2 x + \sin^2 x = 1$,

(j) $\cos(x + y) = \cos x \cos y - \sin x \sin y$,

(k) $\sin(x + y) = \sin x \cos y + \cos x \sin y$,

(l) $x - \frac{x^3}{6} \leqslant \sin x \leqslant x$, if $x \geqslant 0$,

(m) $1 - \frac{x^2}{2} \leqslant \cos x \leqslant 1 - \frac{x^2}{2} + \frac{x^4}{24}$.

Proof: All of these properties except the last two follow directly from the corresponding properties of the exponential function, given in Theorem 3.

From (i) it follows that $1 - \cos x \geqslant 0$. By differentiating $x - \sin x$ and evaluating at $x = 0$, we find that the right-hand inequality in (l) is valid for $x \geqslant 0$. In the same manner, since $\mathrm{D}\big(\cos x - \big(1 - \frac{1}{2}x^2\big)\big) = x - \sin x$, we find that the left-hand inequality in (m) is valid for $x \geqslant 0$. Now, $\mathrm{D}\big(\sin x - \big(x - \frac{1}{6}x^3\big)\big) = \cos x - \big(1 - \frac{1}{2}x^2\big)$ implies the left-hand inequality in (l) for $x \geqslant 0$, and finally, $\mathrm{D}\big(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cos x\big) = \sin x - \big(x - \frac{x^3}{6}\big)$ implies the right-hand inequality in (m) for $x \geqslant 0$. We finally comment that

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(m) is valid for all $x \in \mathbb{R}$, since the functions involved are even, while the inequalities in (l) are reversed for $x < 0$ since the functions are odd.

The following lemma shows that cosine eventually has a positive zero. Our proof is inspired by [10].

Theorem 5: The function $x \to \cos x$ has a positive zero.

Proof: According to Theorem 4(m), we find that $\cos 2 \leqslant 1 - \frac{2^2}{2} + \frac{2^4}{24} = -\frac{1}{3}$. Since $\cos 0 = 1$, and $x \to \cos x$ is continuous, the existence of a positive zero follows from the intermediate value theorem.

We denote by $\frac{\pi}{2}$ the smallest positive zero of $x \to \cos x$, and we stress that this is in fact a definition of the constant π.

Theorem 6: Both sine and cosine are periodic, with smallest positive period 2π.

Proof: We note that it also follows that $\sin \frac{\pi}{2} > 0$ and, thus by (i) in Theorem 4 that $\sin \frac{\pi}{2} = 1$. Now, the addition formulas give

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x, \qquad \text{and} \qquad \sin\left(x - \frac{\pi}{2}\right) = -\cos x.$$

These formulas completely determine the values of cosine and sine in the interval $\left[\frac{\pi}{2}, \pi\right]$, which in turn gives the values in $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$, and

$$\cos\left(x - 4\tfrac{\pi}{2}\right) = \sin\left(x - 3\tfrac{\pi}{2}\right) = -\cos\left(x - 2\tfrac{\pi}{2}\right) = -\sin\left(x - \tfrac{\pi}{2}\right) = \cos x,$$

so 2π is a period for cosine. Likewise for sine. Since cosine is decreasing from 1 to 0 and sine is increasing from 0 to 1 in the interval $\left[0, \frac{\pi}{2}\right]$ we can study the sign changes in the intervals $\left[0, \frac{\pi}{2}\right]$, $\left[\frac{\pi}{2}, \pi\right]$, $\left[\pi, \frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2}, 2\pi\right]$, and it follows that the smallest period of sine and cosine is 2π.

Theorem 7: If $z \in \mathbb{C}$, with $|z| = 1$, then there exists a unique $t \in [0, 2\pi)$ such that $z = \exp(it) = \cos t + i \sin t$.

Proof: We complete the proof of existence in the case $\operatorname{Re} z \geqslant 0$ and $\operatorname{Im} z \geqslant 0$. The other cases are similar.

Since $0 \leqslant \operatorname{Re} z \leqslant 1$ it follows that there exists a unique $t \in \left[0, \frac{\pi}{2}\right]$ such that $\cos t = \operatorname{Re} z$. But since $(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = 1$ and $\cos^2 t + \sin^2 t = 1$ and $\sin t \geqslant 0$ for $t \in \left[0, \frac{\pi}{2}\right]$ it follows that $\operatorname{Im} z = \sin t$, so $z = \cos t + i \sin t = \exp(it)$.

The uniqueness follows by the signs of sine and cosine in the four different quadrants.

Next, we consider the complex number $\exp(iT)$, where $T \in \left[0, \frac{\pi}{2}\right]$. This corresponds to a point on the unit circle, in the first quadrant. A

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parametrization of the part of the unit circle connecting $(1, 0)$ with $\exp(iT)$ is given by $t \to \exp(it)$, $0 \leqslant t \leqslant T$. This curve is simple by Theorem 7.

Theorem 8: The arc along the unit circle from the point $(1, 0)$ to the point $\exp(iT)$ has length T.

The proof usually requires integrals which we at this point do not handle. Thus, let us explain what we mean by arc length. Let $n \in \mathbb{N}$, and let $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_n = T$ be a partition of the interval $[0, T]$. A polygon with corners in the points $\exp(it_k)$ will then have length

$$\ell_n = \sum_{k=1}^{n} |\exp(it_k) - \exp(it_{k-1})|.$$

We define the length ℓ of the arc described above (see Figure 3) to be the least upper bound of all sums of the type ℓ_n, where the number of dividing points of the partitions as well as the lengths of the intervals may vary.

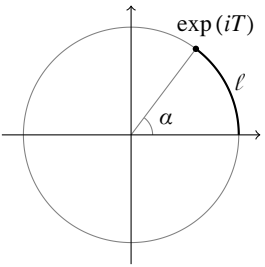


FIGURE 3: The unit circle

Proof of Theorem 8: First, since

$$\exp(it_k) - \exp(it_{k-1}) = \exp\left(i\,\frac{t_k + t_{k-1}}{2}\right)\left[\exp\left(i\,\frac{t_k - t_{k-1}}{2}\right) - \exp\left(-i\,\frac{t_k - t_{k-1}}{2}\right)\right]$$

we find, by (a) in Theorem 3 and (l) in Theorem 4,

$$|\exp(it_k) - \exp(it_{k-1})| = 2\sin\left(\frac{t_k - t_{k-1}}{2}\right) \leqslant t_k - t_{k-1}.$$

Thus, for all $n \in \mathbb{N}$,

$$\ell_n \leqslant \sum_{k=1}^{n} (t_k - t_{k-1}) = T.$$

On the other hand, by (l) in Theorem 4 again, with $t = \frac{1}{2}(t_k - t_{k-1})$,

$$t_k - t_{k-1} - \frac{1}{24}(t_k - t_{k-1})^3 \leqslant 2\sin\left(\frac{t_k - t_{k-1}}{2}\right),$$

so, by summing,

$$T - \frac{1}{24}\sum_{k=1}^{n} (t_k - t_{k-1})^3 \leqslant \ell_n.$$

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If we add a point to a partition, we will, by the triangle inequality increase the length of the polygon train. Thus, given $\varepsilon > 0$, the least upper bound will be attained when the lengths of the subintervals are all less than ε. Since, for such a partition,

$$0 \leqslant \sum_{k=1}^{n} (t_k - t_{k-1})^3 \leqslant \varepsilon^2 \sum_{k=1}^{n} (t_k - t_{k-1}) = \varepsilon^2 T,$$

we find that

$$T\left(1 - \frac{\varepsilon^2}{24}\right) \leqslant \ell_n \leqslant T.$$

But ε was arbitrary, so $\ell = T$.

Finally, for $0 \leqslant T < 2\pi$ we define the angle α (in radians) as $\alpha = \ell = T$, according to Figure 3. The students should now feel that they are on safe grounds again. We comment that this might be a good place to discuss complex numbers in polar form, $z = |z| \dfrac{z}{|z|} = |z| \exp(i\alpha)$.

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The answers to the *Nemo* page from November 2015 on conic sections were:

1.	Ian McEwan	The Innocent
2.	Keith Douglas	How to Kill
3.	Emily Dickinson	Let us Play Yesterday
4.	Coventry Patmore	Deliciae Sapientiae de Amore
5.	George Canning and John Frere	The Loves of the Triangles Canto II
6.	Thomas Hardy	Far from the Madding Crowd Chapter 15

Congratulations to Henry Ricardo and Nigel Derby (who mentioned that he uses the last quote in class to explain asymptotes to his students) for identifying all these quotations. Also apologies to Lawrence Smallman for omitting his name from the list of correct solvers in November 2015.

The theme this time is mathematics as solace. The quotations on page 33 are to be identified by reference to author and work. Solutions are invited to the Editor by 31st May 2016.

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