ON THE THOM ISOMORPHISM THEOREM

By W. H. COCKCROFT

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1. *Introduction.* In what follows (E, p, B) will always denote a fibering in the sense of Serre,† with arc-wise connected base B, projection $p: E \to B$, and fibre $F = p^{-1}(b)$, $b \in B$. Coefficients will always be taken in a commutative ring A with a unit element, and the cohomology will be singular (cubical), as in (3).

Assuming that the local system formed by $H^k(F; A)$ over B is trivial, Serre (3) observed that one could obtain from such a fibering, by means of standard spectral arguments, a Gysin sequence, provided only that the fibre F had the same cohomology over A as a k-sphere, $k \geqslant 1$. In particular, one does not require the fibering to have the full structure of a sphere bundle in order to obtain such a sequence.

The object of this note is to show that by using the Moore (relative) spectral sequence‡ associated with (E, p, B) in case $E_0 \subset E$ is such that $(E_0, p|E_0, B)$ is a fibering, one can obtain from the fibering the Thom isomorphism,§ $\phi: H^i(B; A) \cong H^{i+n}(E, E_0; A)$ for all $i \geqslant 0$, provided only that the fibering has sufficient cohomological properties of an n-dimensional vector bundle relative to its subsphere bundle. Just as in the Gysin sequence argument one can relax the hypotheses originally required, so also in this case one does not require the fibering to have the full structure of a vector bundle in order to obtain the isomorphism in the form described by Thom.

Specifically, then, let E_0 be a subspace of the total space E of the Serre fibering (E, p, B) such that $(E_0, p|E_0, B)$ is again a fibering with fibre $F_0 = (p|E_0)^{-1}(b)$, $b \in B$. Assume $H^i(F, F_0; A) = 0$, $i \neq n$, and that $H^n(F, F_0; A) \cong A$. Let $j: (F, F_0) \to (E, E_0)$ be the inclusion map inducing $j^*: H^*(E, E_0; A) \to H^*(F, F_0; A)$, and let $\mathbf{u}_b$ be the image of $\mathbf{1} \in A$ under the isomorphism $H^n(F, F_0; A) \cong A$. I prove the following result.

Theorem. *$H^k(E, E_0; A) = 0$ for $k < n$, and, for any $b \in B$, there exists a unique class $\mathbf{u} \in H^n(E, E_0; A)$ such that $j^*\mathbf{u} = \mathbf{u}_b$, and moreover such that if the projection $p: E \to B$ induces $p^*: H^*(B; A) \to H^*(E; A)$, then the correspondence $\mathbf{x} \to p^*\mathbf{x} \cup \mathbf{u}$, defined for all $\mathbf{x} \in H^*(B; A)$, defines an isomorphism $\phi: H^i(B; A) \cong H^{i+n}(E, E_0; A)$, for all $i \geqslant 0$.*

† As defined by him, for example, in (3).

‡ Cf. (2). The spectral sequence which I shall use is the second of Moore's sequences, which involves the relativization of the 'fibre' terms of the Serre sequence, not the first, more usual, sequence which involves the relativization of the 'base' terms of the Serre sequence.

§ Cf. (4), and also (1) where Milnor observes that the isomorphism can be obtained by spectral sequence methods. I do not know if the methods which I use below are those Milnor intended to imply could be used. The Moore spectral sequence makes it obvious that such an isomorphism exists; that it takes the cup-product form of the Thom isomorphism is not perhaps so obvious, and is not the consequence of what are usually called 'standard' spectral sequence arguments.

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One might of course hope to prove the result by paraphrasing the proof given by Serre (3) of the existence of the Gysin sequence for a fibering with cohomology spheres as fibre. I do not see how one could do this. In particular, in order to obtain the cup-product form of the Thom isomorphism one would require (at the point in Serre's proof where he assumes the existence of a unit element $\mathbf{1} \in H^0(F; A)$) that there should be a unit element $1 \in H^0(F, F_0; A)$ in the circumstances of the Thom theorem, and there is, of course, no such element.‖

2. *Proof of theorem.* I recall the spectral sequence of Moore (2). Given a fibering (E, p, B) as above, with fibre F, and a subspace E_0 of E such that $(E_0, p|E_0, B)$ is a fibering with fibre $F_0 = E_0 \cap F$, then the injection of the (cubical) relative cochains $C^*(E, E_0; A)$ into the absolute cochains $C^*(E; A)$ induces, from the filtration defined by Serre (3) on $C^*(E; A)$, a filtration of $C^*(E, E_0; A)$. There results a spectral sequence $\{E_r^*\}$ with $E_2^* \cong H^*(B; H^*(F, F_0; A))$ and E_∞^* the graded module associated with $H^*(E, E_0; A)$ suitably filtered. The isomorphism $H^*(B; H^*(F, F_0; A)) \cong E_2^*$ is precisely the isomorphism induced on the relative groups by the Serre isomorphism defined in (3) for the E_2^* term of the absolute cohomology spectral sequence.

Under the conditions of the theorem as stated in § 1 above, viz. when $H^i(F, F_0; A) = 0$ $i \neq n$, this Moore spectral sequence is trivial, and one deduces immediately that for $m < n$, $H^m(E, E_0; A) = 0$ as required. On the other hand, for $m \geqslant n$ one has

$$H^m(E, E_0; A) = E_\infty^{m-n,n} = \ldots = E_2^{m-n,n} \cong H^{m-n}(B; H^n(F, F_0; A)).$$

Of course, in case $m = n$, after identifying $H^0(B; H^n(F, F_0; A))$ with $H^n(F, F_0; A)$, this isomorphism is precisely $j^*: H^n(E, E_0; A) \to H^n(F, F_0; A)$, induced by the inclusion map $j: (F, F_0) \to (E, E_0)$ as above. Given a generator $\mathbf{u}_b$ of $H^n(F, F_0; A)$ we have therefore, as required, a unique element $\mathbf{u} \in H^n(E, E_0; A)$ such that $j^*\mathbf{u} = \mathbf{u}_b$. One last deduction is also immediate: since $H^n(F, F_0; A) = A$, we have

$$H^i(B; A) = H^i(B; H^n(F, F_0; A)) \cong E_2^{i,n} = H^{i+n}(E, E_0; A).$$

Thus, once given the Moore spectral sequence, the crux of the proof of the theorem consists simply in verifying that this last isomorphism, which evaluates the E_2^* term of the spectral sequence, is given by the cup-product formula of the Thom isomorphism. Now we can identify $C^*(B; H^n(F, F_0; A))$ with $C^*(B; A) \otimes H^n(F, F_0; A)$. We may therefore define a homomorphism

$$\lambda: C^i(B; H^n(F, F_0; A)) \to C^{i+n}(E, E_0; A)$$

by setting $\lambda(b^i \otimes \mathbf{u}_b) = p^\sharp b^i \cup u$, where u is some cocycle in the class $\mathbf{u}$, $b^i \in C^i(B; A)$, and $p^\sharp$ is induced by the projection $p: E \to B$. If d_X $(X = B, E)$ denotes the coboundary operator in X, then it is trivial that λ is a cochain mapping with respect to the operators $d_B \otimes 1$ and d_E. Moreover, of course, it induces in cohomology the same homomorphism as

$$\mu: C^i(B; H^*(F, F_0; A)) \to C^{i+n}(E, E_0; A),$$

† My attention was first drawn to this point and its attendant difficulties by F. P. Peterson, in a conversation about the proof of the Thom isomorphism theorem.

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defined by $\mu(b^i \otimes \mathbf{u}_b) = (-1)^{ni} u \cup p^\sharp b^i$. I claim that μ induces precisely the Moore–Serre isomorphism $H^*(B; H^*(F, F_0; A)) \cong E_2^*$, in the circumstances under consideration. For let v be an $(i+n)$-dimensional cube of filtration $s \leqslant i$ in E. Considering $C^*(E, E_0; A)$ always as a submodule of $C^*(E; A)$, and using the familiar notation of Serre ((3), chap. II, §§ 1, 4), we have

$$\begin{aligned}(-1)^{ni}(u \cup p^\sharp b^i)(v) &= (-1)^{ni} \Sigma \rho_{H,K} u(\lambda_K^0 v) . b^i(p\lambda_H^1 v) \\ &= (-1)^{ni} \Sigma \rho_{H,K} b^i(p^\sharp \lambda_H^1 v) . u(\lambda_K^0 v).\end{aligned}$$

Now, since v is of filtration $s \leqslant i$, $p^\sharp \lambda_H^1 v$ is degenerate unless $s = i$, and even then it is degenerate unless $H = (i+1, i+2, \ldots, i+n)$. Thus there is only one non-zero term on the right of this last equality sign; the others all involve cochains evaluated on degenerate cubes. In fact, still using Serre's notation, viz. $v \to Bv \otimes Fv$ for the chain map which evaluates the E_2^* term in the absolute case,

$$\begin{aligned}(-1)^{ni}(u \cup p^\sharp b^i)(v) &= b^i(Bv) . u(Fv) \\ &= b^i(Bv) . u_b(Fv),\end{aligned}$$

where $u_b \in \mathbf{u}_b$ is the restriction of the cocycle u to cubes of the fibre. All in all then, it follows that λ induces in cohomology the same homomorphism as the map

$$\mu : C^*(B; H^*(F, F_0; A)) \to C^*(E, E_0; A)$$

which, from the immediately preceding argument, evidently induces in cohomology precisely the same isomorphism as the Serre chain map $v \to Bv \otimes Fv$. Thus, λ itself induces the Moore–Serre isomorphism

$$H^i(B; A) = H^i(B; H^n(F, F_0; A)) \cong E_2 = H^{i+n}(E, E_0; A),$$

which is therefore given in the circumstances of the theorem by $\mathbf{x} \to p^*\mathbf{x} \cup \mathbf{u}$, for all $\mathbf{x} \in H^*(B; H^*(F, F_0; A)) = H^*(B; A)$, as required to complete the proof of the result.

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