THE EVALUATION OF THE BASIC CONFLUENT HYPERGEOMETRIC FUNCTIONS

By L. J. SLATER

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ABSTRACT. This paper contains all those formulae which are necessary for the numerical evaluation of the basic confluent hypergeometric functions, including asymptotic formulae, and three skeleton tables which can be used as a basis for the study of the numerical behaviour of these functions over a wide field.

1. *Introduction.* The functions

$$ {}_1\Phi_1(q^a;\, q^b;\, y) = \sum_{n=0}^{\infty} \frac{(q^a)_n\, y^n}{(q^b)_n\, (q)_n} \tag{1·1} $$

and

$$ {}_1\Phi_1(q^a;\, q^b;\, q^{\frac{1}{2}n(n+1)}y) = \sum_{n=0}^{\infty} \frac{(q^a)_n y^n q^{\frac{1}{2}n(n+1)}}{(q^b)_n\, (q)_n}, \tag{1·2} $$

where

$$ (q^a)_n = (1-q^a)(1-q^{a+1}) \ldots (1-q^{a+n-1}), $$

are the basic analogues of the confluent hypergeometric function, but there does not seem to have been any previous attempt at the numerical evaluation of these functions. The functions are of four variables and their tabulation is of greatest interest for values of q near unity. For these values of q, the functions assume values near those of ${}_1F_1(a;\, b;\, x)$, where $y = (1-q)\,x$. The series (1·1) and (1·2) converge for all values, real and complex, of q, a, b and x; and, for $|\,q\,| < 1$, (1·2) converges more rapidly than (1·1).

The two ${}_1\Phi_1$ functions contain as special cases the basic analogues of many functions, such as the basic trigonometric and exponential functions, the basic Bessel and Weber functions, and various results of importance in the theory of numbers (see Hahn (2) for details). In particular, for $|\,q\,| < 1$, since

$$ {}_1\Phi_0(q^a;\, ;\, x) = \prod_{n=0}^{\infty} \frac{(1-xq^{a+n})}{(1-xq^n)}, \tag{1·3} $$

we have

$$ \lim_{a\to+\infty} {}_1\Phi_0(q^a;\, ;\, x) = {}_0\Phi_0(;\, ;x) = \prod_{n=0}^{\infty} 1/(1-xq^n) \tag{1·4} $$

and

$$ \lim_{a\to-\infty} {}_1\Phi_0(q^a;\, ;\, xq^{-a}) = {}_0\Phi_0(;\, ;-q^{\frac{1}{2}n(n-1)}x) = \prod_{n=0}^{\infty} (1-xq^n). \tag{1·5} $$

The function ${}_2\Phi_1(q^a, q^b;\, q^c;\, x)$ satisfies the q-difference equation

$$ x(\vartheta+a)(\vartheta+b)z = (\vartheta)(\vartheta+c-1)z, \tag{1·6} $$

where $q^{\vartheta}z(x) = q^{x(d/dx)}z(x) = z(qx)$ and

$$ (\vartheta+a)z = \{q^a z(qx) - z(x)\}/(q-1) $$

(see Jackson (3) for full definitions). So, since

$$ \lim_{q\to1}\lim_{b\to+\infty} {}_1\Phi_1(q^a, q^{b+1};\, q^c;\, (1-q)\,x) = \lim_{q\to1} {}_1\Phi_1(q^a;\, q^c;\, (1-q)\,x) = {}_1F_1(a;\, c;\, x), \tag{1·7} $$

(1·1) satisfies the equation

$$ x(\vartheta+a)z = (\vartheta)(\vartheta+b-1)z. \tag{1·8} $$

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Similarly,

$$\lim_{q\to 1}\lim_{b\to -\infty} {}_1\Phi_1(q^a, q^{b+1}; q^c; (1-q)x/q^b) = \lim_{q\to 1} {}_1\Phi_1(q^a; q^c; q^{\frac{1}{2}(n+1)}(1-q)x) = {}_1F_1(a; c; x), \tag{1·9}$$

and so (1·2) satisfies the equation

$$x(\vartheta + a)q^{\vartheta} z = (\vartheta)(\vartheta + b - 1)z. \tag{1·10}$$

The order of proceeding to the limit is not interchangeable.

Corresponding to $${}_1F_1(a; b; -x) = e^{-x}\,{}_1F_1(b-a; b; x),$$

we now have

$${}_1\Phi_1(q^a; q^c; q^{\frac{1}{2}(n+1)}x) = \prod_{n=0}^{\infty}(1 + xq^{1+a-c+n})\,{}_1\Phi_1(q^{c-a}; q^c; -xq^{1+a-c}) \tag{1·11}$$

and

$${}_1\Phi_1(q^a; q^c; x) = \prod_{n=0}^{\infty}\{1/(1-xq^n)\} \times {}_1\Phi_1(q^{c-a}; q^c; -xq^{\frac{1}{2}(n+1)}q^{a-1}). \tag{1·12}$$

In order to prove (1·11) we see that the coefficient of x^n on the right is

$$\begin{aligned}\sum_{s=0}^{n} &\frac{(-q^{(a-c)})^s(-x)^n q^{\frac{1}{2}s(s+1)}(q^{c-a})_{n-s} q^{(1+a-c)(n-s)}}{(q)_s (q)_{n-s}(q^c)_{n-s}} \\ &= \frac{q^{(1+a-c)n}(-x)^n (q^{c-a})_n}{(q)_n (q^c)_n} \sum_{s=0}^{n} \frac{q^{(a+n)s}(q^{1-c-n})_s (q^{-n})_s}{(q)_s (q^{1+a-c-n})_s} \\ &= \frac{(q^a)_n x^n q^{\frac{1}{2}n(n+1)}}{(q^c)_n (q)_n},\end{aligned}$$

when this series is summed by the basic analogue of Gauss's theorem (see Bailey (1), §8·4), and this is the coefficient of x^n on the left of (1·11). The relation

$$(q^a)_{N-n} = (q^a)_N q^{\frac{1}{2}n(n+1)}/\{(q^{1-N-a})_n (-q^{a+N})^n\}$$

has been used frequently in the reduction. (1·12) can be proved in a similar way.

Alternatively, these results can be proved by letting $b \to \pm\infty$ in the result

$${}_2\Phi_1(q^a, q^b; q^c; x) = {}_1\Phi_0(q^{a+b-c}; ; x)\,{}_2\Phi_1(q^{c-a}, q^{c-b}; q^c; xq^{a+b-c}). \tag{1·13}$$

2. *Recurrence relations.* The function (1·1) satisfies the six recurrence relations

$$\begin{aligned}{}_1\Phi_1(q^{a-2}; q^b; x) = \{&(q^{a+b-1} - q^b)\,{}_1\Phi_1(q^a; q^b; x) \\ &+ (q^{1+b} - q^{a+b-1} + q^b - q^a + q^{2a-2}x)\,{}_1\Phi_1(q^{a-1}; q^b; x)\}/(q^{b+1} - q^a),\end{aligned} \tag{2·1}$$

$$\begin{aligned}{}_1\Phi_1(q^{a+1}; q^b; x) = \{&(q^{1+b} - q^{1+a})\,{}_1\Phi_1(q^{a-1}; q^b; x) \\ &+ (q^{a+b} - q^{1+b} - q^b + q^{a+1} - q^{2a}x)\,{}_1\Phi_1(q^a; q^b; x)\}/(q^{b+a} - q^b),\end{aligned} \tag{2·2}$$

$$\begin{aligned}{}_1\Phi_1(q^a; q^{b-1}; x) = \{&(q^{b-a}(1-q^{a-1}) + q^{a-1}x)\,{}_1\Phi_1(q^a; q^b; x) \\ &+ (1 - q^{b-a})\,{}_1\Phi_1(q^{a-1}; q^b; x)\}/(1 - q^{b-1}),\end{aligned} \tag{2·3}$$

$$\begin{aligned}{}_1\Phi_1(q^a; q^{b-1}; x) = \{&q^{b-1}(1-q^a)\,{}_1\Phi_1(q^{a+1}; q^b; x) \\ &+ (q^a - q^{b-1})\,{}_1\Phi_1(q^a; q^b; x)\}/q^a(1 - q^{b-1}),\end{aligned} \tag{2·4}$$

$${}_1\Phi_1(q^a; q^{b+1}; x) = (1-q^b)\{{}_1\Phi_1(q^a; q^b; x) - {}_1\Phi_1(q^{a-1}; q^b; x)\,q^{1-a}\}/x, \tag{2·5}$$

and

$$\begin{aligned}{}_1\Phi_1(q^a; q^{b+1}; x) = \{&(1-q^b)q^b(1-q^a)\,{}_1\Phi_1(q^{a+1}; q^b; x) \\ &- (1-q^b)(q^b - q^{a+b} + q^{2a}x)\,{}_1\Phi_1(q^a; q^b; x)\}/(q^b - q^a)q^a x.\end{aligned} \tag{2·6}$$

By the use of these and similar recurrence relations derived from them, the tables can be extended over a wide range of values of a and b.

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For a simple proof of (2·1), let us assume a relation of the form

$$ {}_1\Phi_1(q^{a-2};\, q^b;\, x) = A\,{}_1\Phi_1(q^a;\, q^b;\, x) + (Bx+C)\,{}_1\Phi_1(q^{a-1};\, q^b;\, x), $$

and equate the coefficients of x^n. Then we can solve the resulting equations for A, B and C. The proofs of (2·3) and (2·5) follow in the same way, and the remaining three relations (2·2), (2·4) and (2·6) can then be deduced directly from the preceding three results.

Six similar recurrence relations for the function (1·2) follow from (2·1)–(2·6) by making the substitution (1·12) in the six previous results. These recurrences are:

$$ {}_1\Phi_1(q^{a+2};\, q^b;\, xq^{\frac{1}{2}(n+1)-2}) = \{(q^{b-a-1}-1)\,{}_1\Phi_1(q^a;\, q^b;\, xq^{\frac{1}{2}(n+1)}) + (q-q^{b-a-1}+1-q^{-a}-q^{-a-1}x)\,{}_1\Phi_1(q^{a+1};\, q^b;\, xq^{\frac{1}{2}(n+1)-1})\}/(q-q^{-a}), \quad (2\cdot7) $$

$$ {}_1\Phi_1(q^{a-1};\, q^b;\, xq^{\frac{1}{2}(n+1)+1}) = \{(q-q^{1-a})\,{}_1\Phi_1(q^{1+a};\, q^b;\, xq^{\frac{1}{2}(n+1)-1}) + (q^{b-a}-q-1+q^{1-a}+q^{1-a}x)\,{}_1\Phi_1(q^a;\, q^b;\, xq^{\frac{1}{2}(n+1)})\}/(q^{b-a}-1), \quad (2\cdot8) $$

$$ {}_1\Phi_1(q^{a-1};\, q^{b-1};\, xq^{\frac{1}{2}(n+1)}) = \{(q^a-q^{b-1}-x)\,{}_1\Phi_1(q^a;\, q^b;\, xq^{\frac{1}{2}(n+1)}) + (1-q^a)\,{}_1\Phi_1(q^{1+a};\, q^b;\, xq^{\frac{1}{2}(n+1)-1})\}/(1-q^{b-1}), \quad (2\cdot9) $$

$$ {}_1\Phi_1(q^{a-1};\, q^{b-1};\, xq^{\frac{1}{2}(n+1)}) = \{(q^{a-1}-q^{b-1})\,{}_1\Phi_1(q^{a-1};\, q^b;\, xq^{\frac{1}{2}(n+1)+1}) + (1-q^{a-1})\,{}_1\Phi_1(q^a;\, q^b;\, xq^{\frac{1}{2}(n+1)})\}/(1-q^{b-1}), \quad (2\cdot10) $$

$$ {}_1\Phi_1(q^{a+1};\, q^{b+1};\, xq^{\frac{1}{2}(n+1)}) = (q^b-1)\{{}_1\Phi_1(q^a;\, q^b;\, xq^{\frac{1}{2}(n+1)}) - {}_1\Phi_1(q^{a+1};\, q^b;\, xq^{\frac{1}{2}(n+1)-1})\}/x, \quad (2\cdot11) $$

$$ {}_1\Phi_1(q^{a+1};\, q^{b+1};\, xq^{\frac{1}{2}(n+1)}) = (1-q^b)\{(1-q^{b-a})\,{}_1\Phi_1(q^{a-1};\, q^b;\, xq^{\frac{1}{2}(n+1)+1}) - (1-q^{b-a}-q^{1-a}x)\,{}_1\Phi_1(q^a;\, q^b;\, xq^{\frac{1}{2}(n+1)})\}/(q^{1-a}-q)\,x. \quad (2\cdot12) $$

3. *Asymptotic formulae.* The asymptotic expansions of (1·1) and (1·2) are

$$ {}_1\Phi_1(q^a;\, q^b;\, q^{\frac{1}{2}(n+1)}x) \sim \prod_{n=0}^{\infty} \frac{(1+q^{1+a-b+n}x)\,(1+q^{b-a+n}/x)\,(1-q^{1-b+n})}{(1-q^{1+a-b+n})\,(1+q^{b+n}/x)\,(1+xq^{1-b+n})}\,{}_2\Phi_0(q^a, q^{1+a-b};\,;\,-q^{b-a}/x) + \prod_{n=0}^{\infty} \frac{(1+xq^{1+n})\,(1+q^n/x)\,(1-q^{a+n})}{(1-q^{b+n})\,(1+xq^{1+a-b+n})\,(1+q^{b-a+n}/x)}\,{}_2\Phi_0(q^{1-a}, q^{b-a};\,;\,q^{a-\frac{1}{2}(n+1)}/x) \quad (3\cdot1) $$

and

$$ {}_1\Phi_1(q^a;\, q^b;\, x) \sim \prod_{n=0}^{\infty} \frac{(1-q^{2a-1+n}/x)\,(1-q^{1-b+n})}{(1-q^{1-a+n})\,(1-q^{1+b-a+n}/x)\,(1-xq^{a-b+n})}\,{}_2\Phi_0(q^{1-a}, q^{b-a};\,;\,q/x) + \prod_{n=0}^{\infty} \frac{(1-xq^{a+n})\,(1-q^{1-a+n}/x)\,(1-q^{b-a+n})}{(1-q^n x)\,(1-q^{b+n})\,(1-xq^n)\,(1-q^{1+n}/x)}\,{}_2\Phi_0(q^{1+a-b}, q^a;\,;\,-q^{1-2a+b}/xq^{\frac{1}{2}(n+1)}). \quad (3\cdot2) $$

These can be deduced from an asymptotic extension, which I hope to publish shortly, of the general theorem on the transformation of basic series (Slater (4)). If in this result we take $K = L = M = 1$, $N = 2$, we can show that

$$ \frac{1}{2\pi i}\int_{-\pi i/t}^{\pi i/t} \prod_{n=0}^{\infty} \frac{(1-q^{1-x+s+n})\,(1-q^{x-s+n})}{(1-q^{a+s+n})\,(1-q^{1-b-s+n})\,(1-q^{-s+n})}\,ds = \prod_{n=0}^{\infty} \frac{(1-q^{1-x+n})\,(1-q^{x+n})}{(1-q^{a+n})\,(1-q^{1-b+n})\,(1-q^{n+1})}\,{}_1\Phi_1(q^a;\, q^b;\, -q^{\frac{1}{2}(n+1)+x+b-1}) + \prod_{n=0}^{\infty} \frac{(1-q^{2-b-x+n})\,(1-q^{x-1+b+n})}{(1-q^{1+a-b+n})\,(1-q^{b-1+n})\,(1-q^{n+1})}\,{}_1\Phi_1(q^{1+a-b};\, q^{2-b};\, -q^{\frac{1}{2}(n+1)+x}) \sim \prod_{n=0}^{\infty} \frac{(1-q^{x+a+n})\,(1-q^{1-x-a+n})}{(1-q^{1+a-b+n})\,(1-q^{a+n})\,(1-q^{n+1})}\,{}_2\Phi_0(q^{1+a-b}, q^a;\,;\,q^{1-x-a}), \quad (3\cdot3) $$

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Table 1. $q = 0{\cdot}9$

a	$b = 0{\cdot}2$	$b = 0{\cdot}4$	$b = 0{\cdot}6$	$b = 0{\cdot}8$	$b = 1{\cdot}0$
			$y = (1-q)\,x = 0{\cdot}1$		
0·0	1·00000 00	1·00000 00	1·00000 00	1·00000 00	1·00000 00
0·2	2·40978 10	1·67321 45	1·43416 83	1·31783 73	1·24980 35
0·4	3·97257 70	2·40978 10	1·90416 73	1·65890 09	1·51590 40
0·6	5·68866 33	3·20955 29	2·40978 10	2·02297 20	1·79809 71
0·8	7·55788 65	4·07222 00	2·95070 83	2·40978 10	2·09614 53
1·0	9·57968 65	4·99731 80	3·52656 79	2·81901 08	2·40978 10
1·2	11·75311 7	5·98423 68	4·13690 37	3·25029 98	2·73870 79
1·4	14·07686 7	7·03222 98	4·78118 93	3·70324 56	3·08260 37
1·6	16·54928 4	8·14042 28	5·45883 37	4·17740 75	3·44112 22
1·8	19·16839 5	9·30782 26	6·16918 53	4·67231 02	3·81389 54
2·0	21·93192 6	10·53332 6	6·91153 76	5·18744 67	4·20053 58
			$y = (1-q)\,x = 0{\cdot}2$		
0·0	1·00000 00	1·00000 00	1·00000 00	1·00000 00	1·00000 00
0·2	5·59139 48	3·07966 61	2·28438 59	1·90709 64	1·69179 48
0·4	11·24019 3	5·59139 48	3·81086 20	2·97016 69	2·49267 23
0·6	18·01101 0	8·55893 50	5·59139 48	4·19614 06	3·40698 22
0·8	25·96462 2	12·00447 3	7·63708 40	5·59139 48	4·43869 50
1·0	35·15775 7	15·94856 4	9·95813 24	7·16174 19	5·59139 48
1·2	45·64292 2	20·41008 2	12·56382 4	8·91241 82	6·86827 48
1·4	57·46826 5	25·40617 3	15·46250 7	10·84807 8	8·27213 48
1·6	70·67748 1	30·95223 3	18·66158 8	12·97279 2	9·80538 20
1·8	85·30973 4	37·06188 9	22·16752 5	15·29004 5	11·47003 3
2·0	101·39962	43·74699 6	25·98582 9	17·80274 4	13·26771 7
			$y = (1-q)\,x = 0{\cdot}3$		
0·0	1·00000 00	1·00000 00	1·00000 00	1·00000 00	11·00000 00
0·2	12·54210 5	5·98851 91	3·95854 89	3·01747 27	2·49248 98
0·4	27·99816 4	12·54210 5	7·77753 03	5·58013 67	4·36072 21
0·6	47·81131 8	20·82635 2	12·54210 5	8·73852 13	6·63734 05
0·8	72·42097 7	31·00474 9	18·33600 5	12·54210 5	9·35418 30
1·0	102·26000	43·23767 8	25·24103 0	17·03903 2	12·54210 5
1·2	137·75208	57·68146 4	33·33658 7	22·27584 8	16·23081 9
1·4	179·30925	74·48750 7	42·69926 2	28·29725 9	20·44874 6
1·6	227·32969	93·80148 9	53·40243 6	35·14592 0	25·22288 3
1·8	282·19558	115·76264	65·51594 0	42·86224 1	30·57869 2
2·0	344·27135	140·50313	79·10574 5	59·03116 4	36·53999 6
			$y = (1-q)\,x = 0{\cdot}4$		
0·0	1·00000 0	1·00000 0	1·00000 0	1·00000 0	1·00000 00
0·2	27·28863 6	11·91591 4	7·24390 5	5·12146 8	3·96121 98
0·4	64·96160 4	27·28863 6	15·89122 6	10·73928 8	7·93735 91
0·6	115·83584 8	47·79419 3	27·28863 6	18·05884 5	13·06107 7
0·8	181·77713	74·12400 7	41·78935 6	27·28863 6	19·46656 6
1·0	264·68654	106·97996	59·75065 7	38·63882 1	27·28863 6
1·2	366·48711	147·06954	81·53142 5	52·31980 6	36·66182 4
1·4	489·11062	195·10102	107·48977	68·54087 8	47·71953 0
1·6	634·48485	251·77893	137·98078	87·50889 0	60·59319 9
1·8	804·52112	317·79950	173·35425	109·42699 7	75·41154 2
2·0	1001·1026	393·84664	213·95270	134·49353	92·29981 0
			$y = (1-q)\,x = 0{\cdot}5$		
0·0	1·00000 0	1·00000 0	1·00000 0	1·00000 0	1·00000 0
0·2	57·75399 8	23·79069 3	13·63499 4	9·10142 6	6·66666 7
0·4	143·60410	57·75399 8	32·18759 3	20·82602 2	14·75294 0
0·6	264·38892	105·04379	57·75399 8	36·81891 4	25·67307 6
0·8	426·27572	167·92641	91·48338 8	57·75399 8	39·85833 9
1·0	635·71795	248·76503	134·56995	84·32926 8	57·75399 8
1·2	899·41084	350·00331	188·24463	117·26199	79·81627 6
1·4	1224·2453	474·14867	253·76677	157·28390	106·50925
1·6	1617·2614	623·75535	332·41561	205·13620	138·30181
1·8	2085·6009	801·40735	425·48179	261·56479	175·66457
2·0	2636·4605	1009·7016	534·25915	327·31549	219·06692

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Table 1. (*contd.*)

a	$b = 0{\cdot}2$	$b = 0{\cdot}4$	$b = 0{\cdot}6$	$b = 0{\cdot}8$	$b = 1{\cdot}0$
			$y = (1-q)\,x = 0{\cdot}6$		
0·0	1·00000	1·00000	1·00000 0	1·00000 0	1·00000 0
0·2	119·18450	47·16358	25·92327 4	16·58256 3	11·64247 2
0·4	305·87768	119·18450	64·32077 5	40·28921 4	27·63234 5
0·6	577·30828	222·99323	119·18450	73·86664 4	50·08296 8
0·8	951·03524	364·98254	193·72820	119·18450	80·18209 6
1·0	1445·8467	551·96994	291·36800	178·22398	119·18450
1·2	2081·6469	791·15576	415·70117	253·06528	168·40403
1·4	2879·3316	1090·0776	570·48318	345·87431	229·20506
1·6	3860·6554	1456·5628	759·60362	458·88869	302·99372
1·8	5048·0922	1898·6765	987·06088	594·40321	391·20875
2·0	6464·6908	2424·6713	1256·9366	754·75522	495·31232
			$y = (1-q)\,x = 0{\cdot}7$		
0·0	1·00000	1·00000	1·00000	1·00000 0	1·00000 0
0·2	240·32533	92·36631	49·24075	30·51747 4	20·74511 2
0·4	631·88930	240·32533	126·54627	77·31765 4	51·71655 5
0·6	1216·6125	459·69165	240·32533	145·69010	96·62739 5
0·8	2039·7017	766·77399	398·70614	240·32533	158·43476
1·0	3150·4646	1179·3047	610·49947	366·29263	240·32533
1·2	4602·0775	1716·3533	885·15401	529·01359	345·69829
1·4	6451·3093	2398·2242	1232·7046	734·23188	478·14597
1·6	8758·2046	3246·3423	1663·7152	987·97992	641·43283
1·8	11585·742	4283·1266	2189·2155	1296·5421	839·47248
2·0	14999·456	5531·8571	2820·6344	1666·4183	1076·3038
			$y = (1-q)\,x = 0{\cdot}8$		
0·0	1·00000	1·00000	1·00000	1·00000	1·00000 0
0·2	474·35998	178·30596	92·87997	56·19794	37·26909 4
0·4	1271·46298	474·35998	244·96449	146·75317	96·23260 4
0·6	2488·4381	923·65714	474·35998	282·49112	184·05787
0·8	4233·4381	1564·8795	800·18765	474·35998	307·59601
1·0	6626·4160	2440·8043	1243·5321	734·39850	474·35998
1·2	9798·7506	3598·1604	1827·3666	1075·6901	692·49456
1·4	13892·744	5087·4368	2576·4542	1512·3056	970·73919
1·6	19060·973	6962·6510	3517·2306	2059·2337	1318·38380
1·8	25465·534	9281·0688	4677·6660	2732·3016	1745·2196
2·0	33277·205	12102·913	6087·1142	3548·0890	2261·4839
			$y = (1-q)\,x = 0{\cdot}9$		
0·0	1·0000	1·00000	1·00000	1·00000	1·00000
0·2	917·9834	339·03444	173·42392	102·98359	66·99131
0·4	2499·5541	917·98343	466·60169	275·10284	177·51975
0·6	4959·5783	1813·9628	917·98343	538·70615	345·88907
0·8	8542·5835	3113·7346	1570·1291	917·98343	587·13029
1·0	13523·793	4914·6869	2470·6839	1439·9440	917·98343
1·2	20208·766	7324·6786	3672·2890	2134·3591	1356·8576
1·4	28932·662	10461·753	5232·4304	3033·6704	1923·7710
1·6	40059·137	14453·731	7213·2302	4172·8663	2640·2715
1·8	53978·911	19437·682	9681·1820	5589·3291	3529·3397
2·0	71108·017	25559·301	12706·840	7322·6558	4615·2770
			$y = (1-q)\,x = 1{\cdot}0$		
0·0	1·0000	1·00000	1·00000	1·00000	1·00000
0·2	1744·1685	634·96471	320·05780	187·21872	119·92330
0·4	4812·0314	1744·1685	874·91751	509·04913	324·13887
0·6	9660·1159	3489·5121	1744·1685	1010·9770	641·18182
0·8	16816·764	6057·0870	3018·4510	1744·1685	1102·6370
1·0	26884·436	9658·5587	4800·6067	2766·5369	1744·1685
1·2	40540·076	14531·225	7205·6754	4142·7227	2605·4981
1·4	58534·528	20937·750	10360·737	5943·9903	3730·3321
1·6	81691·043	29165·589	14404·609	8248·0435	5166·2409
1·8	1 10902·91	39526·106	19487·397	11138·764	6964·4911
2·0	1 47130·27	52353·420	25769·919	14705·877	9179·8366

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Table 2. $q = 0{\cdot}99$

a	$b = 0{\cdot}2$	$b = 0{\cdot}4$	$b = 0{\cdot}6$	$b = 0{\cdot}8$	$b = 1{\cdot}0$
			$y = (1-q)\ x = 0{\cdot}01$		
0·0	1·00000 00	1·00000 00	1·00000 00	1·00000 00	1·00000 00
0·2	2·68464 43	1·78376 71	1·49441 57	1·35501 96	1·27424 57
0·4	4·65632 57	2·68464 43	2·05435 76	1·75218 89	1·57791 25
0·6	6·93970 82	3·71192 05	2·68464 43	2·19439 29	1·91288 12
0·8	9·56076 28	4·87534 85	3·39032 06	2·68464 43	2·28111 15
1·0	12·54681 3	6·18515 81	4·17666 49	3·22608 73	2·68464 43
1·2	15·92657 8	7·65207 26	5·04919 50	3·82200 17	3·12560 34
1·4	19·73022 0	9·28732 32	6·01367 69	4·47580 65	3·60619 84
1·6	23·98939 2	11·10266 4	7·07612 98	5·19106 29	4·12872 63
1·8	28·73728 4	13·11039 1	8·24283 54	5·97147 98	4·69557 42
2·0	34·00867 2	15·32335 5	9·52034 44	6·82091 64	5·30922 15
			$y = (1-q)\ x = 0{\cdot}02$		
0·0	1·00000 00	1·00000 00	1·00000 00	1·00000 00	1·00000 00
0·2	7·17236 77	3·67225 07	2·58849 27	2·08563 83	1·80458 61
0·4	15·46304 3	7·17236 77	4·62295 29	3·44867 45	2·79711 22
0·6	26·24462 1	11·63677 3	7·17236 77	5·12947 30	4·00335 55
0·8	39·93212 8	17·21666 1	10·31283 9	7·17236 77	5·45151 34
1·0	56·98640 5	24·07913 3	14·12810 6	9·62594 20	7·17236 77
1·2	77·91770 4	32·40839 1	18·71009 6	12·54332 3	9·19945 64
1·4	103·28948	42·40699 5	24·15950 4	15·98248 9	11·56925 4
1·6	133·72236	54·29719 4	30·58640 1	20·00659 9	14·32136 1
1·8	169·89844	68·32232 1	38·11087 9	24·68432 9	17·49870 4
2·0	212·56565	84·74826 6	46·86371 5	30·09023 1	21·14773 9
			$y = (1-q)\ x = 0{\cdot}03$		
0·0	1·00000 0	1·00000 00	1·00000 00	1·00000 00	1·00000 00
0·2	19·07018 3	8·32326 28	5·10742 95	3·66777 97	2·89099 25
0·4	46·26834 3	19·07018 3	10·99134 9	7·40349 40	5·48338 34
0·6	84·98511 7	34·09085 9	19·07018 3	12·44591 5	8·92616 38
0·8	138·02396	54·37608 6	29·82876 7	19·07018 3	13·39011 9
1·0	208·65256	81·07425 9	43·82605 1	27·59188 0	19·07018 3
1·2	300·65914	115·50981	61·70347 5	38·37144 4	26·18799 3
1·4	418·41374	159·20328	84·19406 0	51·81895 8	34·99464 1
1·6	566·93521	213·89313	112·13228	68·39932 3	45·77365 8
1·8	751·96394	281·55943	146·46473	88·63785 5	58·84421 1
2·0	980·04098	364·44952	188·26168	113·12631	74·56457 0
			$y = (1-q)\ x = 0{\cdot}04$		
0·0	1·00000 0	1·00000 0	1·00000 0	1·00000 00	1·00000 00
0·2	50·46502 8	19·92417 1	11·06199 1	7·22239 83	5·21767 59
0·4	132·09654	50·46502 8	26·94207 7	16·82872 2	11·59056 1
0·6	256·90186	96·42880 9	50·46502 8	30·83479 2	20·73792 5
0·8	438·40576	162·46142	83·84395 6	50·46502 8	33·40177 8
1·0	693·07218	254·18475	129·74144	77·18427 9	50·46502 8
1·2	1040·7801	378·35005	191·33804	112·73341	72·97174 3
1·4	1505·3590	543·00985	272·40897	159·16900	102·14966
1·6	2115·1896	757·71048	377·40960	218·90750	139·43520
1·8	2903·8759	1033·7068	511·57066	294·77424	186·50114
2·0	3910·9967	1384·2013	681·00398	390·05785	245·28722
			$y = (1-q)\ x = 0{\cdot}05$		
0·0	1·00000	1·00000 0	1·00000 0	1·00000 0	1·00000 0
0·2	132·92168	49·06350 5	25·38181 0	15·41723 6	10·36727 9
0·4	367·40519	132·92168	67·11540 3	39·61808 6	25·78484 4
0·6	747·11903	266·94799	132·92168	77·25593 2	49·42995 8
0·8	1326·7423	469·43946	231·30042	132·92168	84·02158 1
1·0	2175·4698	763·41292	372·88154	212·32390	132·92168
1·2	3379·9222	1177·5269	570·83047	322·49608	200·25137
1·4	5047·5137	1747·1447	841·31416	472·03406	291·02366
1·6	7310·3336	2515·5566	1204·0350	671·36741	411·29467
1·8	10329·607	3535·3797	1682·8409	933·06851	568·33535
2·0	14300·800	4870·1555	2306·4196	1272·2035	770·82612

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Table 2. (*contd.*)

a	$b = 0{\cdot}2$	$b = 0{\cdot}4$	$b = 0{\cdot}6$	$b = 0{\cdot}8$	$b = 1{\cdot}0$
			$y = (1-q)\ x = 0{\cdot}06$		
0·0	1·00000	1·00000	1·00000	1·00000 0	1·00000 0
0·2	348·49492	122·51248	60·19887	34·64821 8	22·03930 8
0·4	1004·6200	348·49492	168·56161	95·23445 0	59·28416 3
0·6	2118·5558	727·99938	348·49492	194·66747	119·67957
0·8	3888·8394	1325·8866	629·43774	348·49492	212·23621
1·0	6575·5819	2226·6462	1049·5204	576·74128	348·49492
1·2	10515·055	3538·9420	1657·5314	904·89813	543·07342
1·4	16137·011	5400·9909	2515·2376	1365·0883	814·30748
1·6	23985·161	7986·9023	3700·1104	1997·4296	1184·9993
1·8	34741·265	11514·113	5308·5130	2851·6248	1683·2879
2·0	49253·368	16252·079	7459·4169	3988·8107	2343·6584
			$y = (1-q)\ x = 0{\cdot}07$		
0·0	1·00000	1·00000	1·00000	1·00000	1·00000
0·2	909·53116	307·88733	145·43531	80·32821	48·94920
0·4	2712·9128	909·53116	424·92799	231·75112	139·18035
0·6	5898·9284	1962·6493	909·53116	491·72452	292·52344
0·8	11137·264	3681·6996	1694·5241	909·53116	536·96765
1·0	19336·096	6354·9573	2907·2889	1550·7036	909·53116
1·2	31701·678	10364·132	4715·6717	2501·1716	1458·5190
1·4	49818·329	16208·155	7338·0865	3872·2567	2246·2337
1·6	75742·112	24531·858	11055·662	5806·6600	3352·2147
1·8	1 12114·05	36160·380	16226·766	8485·5990	4877·0882
2·0	1 62295·71	52140·233	23304·299	12137·284	6947·1277
			$y = (1-q)\ x = 0{\cdot}08$		
0·0	1·00000	1·00000	1·00000	1·00000	1·00000
0·2	2363·1096	775·68158	354·96046	189·72536	111·74596
0·4	7253·4142	2363·1096	1072·4846	567·98730	331·09640
0·6	16189·786	5241·5236	2363·1096	1242·7140	718·98886
0·8	31326·961	10085·952	4520·9028	2363·1096	1358·5785
1·0	55659·792	17829·857	7950·4477	4133·4072	2363·1096
1·2	93278·051	29742·147	13199·196	6828·6893	3884·4108
1·4	1 49683·48	47522·674	20997·450	10814·379	6123·3267
1·6	2 32181·59	73419·894	32307·476	16570·105	9342·4527
1·8	3 50362·75	1 10374·90	48383·449	24718·749	13881·591
2·0	5 16689·65	1 62196·82	70844·248	36061·625	20176·419
			$y = (1-q)\ x = 0{\cdot}09$		
0·0	1·0000	1·0000	1·00000	1·00000	1·00000
0·2	6112·5079	1954·8771	871·14253	453·14159	259·54937
0·4	19230·293	6112·5079	2706·0003	1397·5653	794·26831
0·6	43917·555	13884·939	6112·5079	3138·4208	1772·6297
0·8	86833·475	27320·454	11967·001	6112·5079	3433·8520
1·0	1 57470·74	49325·131	21506·922	10933·675	6112·5079
1·2	2 69100·70	83944·517	36447·523	18449·132	10268·364
1·4	4 39974·05	1 36720·30	59129·177	29810·296	16523·808
1·6	6 94834·33	2 15138·53	92701·947	46561·299	25710·458
1·8	10 66812·2	3 29189·02	1 41355·25	70748·804	38926·844
2·0	15 99780·7	4 92059·01	2 10601·94	1 05057·470	57609·355
			$y = (1-q)\ x = 0{\cdot}10$		
0·0	1·000	1·0000	1·0000	1·0000	1·00000
0·2	15741·565	4922·1819	2143·9140	1089·6165	609·52546
0·4	50605·801	15741·565	6818·8877	3445·4773	1915·4076
0·6	1 17941·95	36516·278	15741·565	7914·0022	4376·4997
0·8	2 37723·26	73287·378	31453·927	15741·565	8664·4121
1·0	4 39083·40	1 34828·06	57631·318	28722·249	15741·565
1·2	7 63632·66	2 33618·96	99481·348	49387·862	26960·631
1·4	12 69753·1	3 87107·42	1 64257·30	81251·762	44191·732
1·6	20 38113·5	6 19319·63	2 61913·52	1 29119·18	69983·869
1·8	31 78693·9	9 62908·54	4 05935·59	1 99479·28	1 07768·29
2·0	48 39670·3	14 61736·9	6 14384·93	3 00997·04	1 62113·06

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Table 3. $q = 0{\cdot}9$

a	$b = 0{\cdot}2$	$b = 0{\cdot}4$	$b = 0{\cdot}6$	$b = 0{\cdot}8$	$b = 1{\cdot}0$
			$y = (1-q)\,x = 0{\cdot}1$		
0·0	1·00000 00	1·00000 00	1·00000 00	1·00000 00	1·00000 00
0·2	2·79441 77	1·84243 09	1·53601 44	1·38814 29	1·30234 78
0·4	4·85515 88	2·79441 77	2·13371 30	1·81615 65	1·63263 07
0·6	7·19061 70	3·85882 93	2·79441 77	2·28472 47	1·99121 93
0·8	9·80833 35	5·03819 72	3·51926 76	2·79441 77	2·37840 85
1·0	12·71499 04	6·33471 44	4·30921 98	3·34569 37	2·79441 77
1·2	15·91640 83	7·75023 74	5·16505 15	3·93890 03	3·23939 28
1·4	19·41754 7	9·28628 83	6·08736 16	4·57427 62	3·71340 75
1·6	23·22251 6	10·94406 0	7·07657 55	5·25195 44	4·21646 62
1·8	27·33457 9	12·72442 2	8·13294 71	5·97196 42	4·74850 46
2·0	31·75617 6	14·62792 5	9·25656 50	6·73423 55	5·30939 37
			$y = (1-q)\,x = 0{\cdot}2$		
0·0	1·00000 00	1·00000 00	1·00000 00	1·00000 00	1·00000 00
0·2	8·30161 10	4·17938 70	2·89945 76	2·30439 16	1·97138 74
0·4	17·97340 30	8·30161 10	5·31489 94	3·93440 52	3·16638 41
0·6	30·30783 85	13·47502 5	8·30161 10	5·92260 47	4·60588 46
0·8	45·60177 9	19·80911 3	11·91520 9	8·30161 10	6·31073 41
1·0	64·15468 6	27·41383 2	16·21131 8	11·10391 3	8·30161 10
1·2	86·26681 7	36·39896 7	21·24524 3	14·36168 6	10·59891 2
1·4	112·23747	46·87350 4	27·07166 5	18·10661 4	13·22264 4
1·6	142·36330	58·94502 2	33·74434 4	22·36972 3	16·19232 1
1·8	176·93664	72·71910 8	41·31583 0	27·18122 1	19·52686 4
2·0	216·24393	88·29880 0	49·83720 0	32·57034 5	23·24451 3
			$y = (1-q)\,x = 0{\cdot}3$		
0·0	1·00000 0	1·00000 0	1·00000 00	1·00000 0	1·00000 00
0·2	26·50851 3	11·34814 8	6·80154 6	4·76412 9	3·66496 82
0·4	64·61742 7	26·50851 3	15·14049 2	10·07612 4	7·36052 44
0·6	118·00821	47·45824 4	26·50851 3	17·22228 0	12·26869 0
0·8	189·58406	75·25047 5	41·43320 1	26·50851 3	18·58317 8
1·0	282·45878	111·01007	60·47582 9	38·25901 3	26·50851 3
1·2	399·94359	155·92862	84·22878 3	52·81472 6	36·25904 1
1·4	545·53179	211·25885	113·31268	70·53165 1	48·05785 7
1·6	722·88181	278·30831	148·37325	91·77903 7	62·13565 2
1·8	935·79847	358·43270	190·07793	116·93742	78·72948 2
2·0	1188·2134	453·02873	239·11239	146·39662	98·08151 2
			$y = (1-q)\,x = 0{\cdot}4$		
0·0	1·00000 0	1·00000 0	1·00000 0	1·00000 0	1·00000 0
0·2	92·34265 7	35·82535 4	19·40721 9	12·29854 6	8·59520 9
0·4	242·79929	92·34265 7	48·82847 3	30·08104 6	20·36548 8
0·6	469·99482	176·81568	92·34265 7	56·10163 7	37·40371 4
0·8	794·10319	296·38289	153·44371	92·34265 7	60·93995 6
1·0	1237·9069	459·07463	236·04719	141·01744	92·34265 7
1·2	1826·8290	673·82077	344·49242	204·57065	133·11835
1·4	2588·9368	950·44882	483·54009	285·67618	184·90990
1·6	3554·9169	1299·6723	658·36530	387·23255	249·49330
1·8	4758·0231	1733·0699	874·54634	512·35598	328·77308
2·0	6233·9961	2263·0553	1138·0489	664·37110	424·77631
			$y = (1-q)\,x = 0{\cdot}5$		
0·0	1·00000	1·00000	1·00000 0	1·00000 0	1·00000 0
0·2	358·35272	130·32887	65·94241 1	38·91439 6	25·27666 3
0·4	994·82654	358·35272	179·23056	104·32118	66·67557 0
0·6	2015·8661	721·56570	358·35272	206·94444	131·11576
0·8	3548·3425	1263·7112	624·19210	358·35272	225·61333
1·0	5742·2278	2036·3252	1001·2710	572·08588	358·35272
1·2	8772·0964	3099·2206	1517·9661	863·76642	538·75111
1·4	12838·426	4520·9019	2206·6919	1251·1928	780·09439
1·6	18168·670	6378·9039	3104·0470	1754·4130	1086·6468
1·8	25018·085	8760·0448	4250·9219	2395·7768	1479·5454
2·0	33670·284	11760·591	5692·5646	3199·9657	1971·3725

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Table 3. (*contd.*)

a	$b=0{\cdot}2$	$b=0{\cdot}4$	$b=0{\cdot}6$	$b=0{\cdot}8$	$b=1{\cdot}0$
			$y=(1-q)\,x=0{\cdot}6$		
0·0	1·0000	1·00000	1·00000	1·00000	1·00000 0
0·2	1598·7448	554·85236	267·37349	149·95902	92·38076 4
0·4	4629·6405	1598·7448	766·00929	426·79372	260·91996
0·6	9739·9640	3350·6261	1598·7448	886·75460	539·45547
0·8	17743·062	6083·9585	2892·9687	1598·7448	968·81493
1·0	29640·515	10134·605	4804·7242	2646·9425	1598·7448
1·2	46642·956	15907·583	7521·7434	4132·3827	2488·8132
1·4	70190·181	23883·564	11266·357	6174·4696	3709·2699
1·6	1 01970·13	34624·992	16298·223	8912·3888	5341·8473
1·8	1 43936·42	48781·707	22916·826	12506·394	7480·4851
2·0	1 98324·09	67095·977	31463·696	17138·945	10231·970
			$y=(1-q)\,x=0{\cdot}7$		
0·0	1·0000	1·0000	1·0000	1·00000	1·00000
0·2	8635·8998	2888·7314	1340·5626	723·36602	428·25871
0·4	25887·826	8635·8998	3995·6857	2148·9924	1267·6482
0·6	56218·098	18710·501	8635·8998	4632·5852	2725·1170
0·8	1 05473·03	35032·066	16134·745	8635·8998	5068·1636
1·0	1 81121·60	60048·232	27603·947	14745·456	8635·8998
1·2	2 92502·47	96814·204	44428·824	23690·859	13849·434
1·4	4 51076·77	1 49073·82	68304·359	36363·426	21222·416
1·6	6 70682·00	2 21341·02	1 01271·34	53834·831	31371·560
1·8	9 67783·32	3 18980·12	1 45751·97	77375·407	45026·968
2·0	13 61717·6	4 48283·77	2 04584·29	1 08471·84	63042·093
			$y=(1-q)\,x=0{\cdot}8$		
0·0	1·000	1·000	1·0000	1·0000	1·0000
0·2	62359·816	20230·091	9101·7638	4759·6299	2729·5985
0·4	1 92505·33	62359·816	28013·406	14625·326	8372·8849
0·6	4 29662·14	1 39004·40	62359·816	32511·227	18585·015
0·8	8 27137·00	2 67283·21	1 19762·85	62359·816	35601·416
1·0	14 55327·8	4 69777·43	2 10263·79	1 09358·65	62359·816
1·2	24 04988·3	7 75566·6	3 46779·56	1 80173·62	1 02630·91
1·4	37 90712·7	12 21333·5	5 45588·68	2 83196·98	1 61157·27
1·6	57 54592·1	18 52523·9	8 26840·25	4 28807·34	2 43798·62
1·8	84 70000·1	27 24543·2	12 15079·6	6 29637·38	3 57681·39
2·0	121 45444	39 03977·2	17 39782·5	9 00845·70	5 11350·69
			$y=(1-q)\,x=0{\cdot}9$		
0·0	1·00	1·00	1·00	1·00	1·000
0·2	7 77563·84	2 45681·30	1 07641·60	54807·19	30598·036
0·4	24 62436·7	7 77563·84	3 40461·84	1 73236·47	96648·703
0·6	56 30213·0	17 76862·0	7 77563·84	3 95410·18	2 20463·52
0·8	110 89222	34 97899·5	15 29891·3	7 77563·84	4 33292·16
1·0	199 39484	62 86585·6	27 48251·6	13 96094·5	7 77563·84
1·2	336 39049	106 01172	46 32334·3	23 52107·4	13 09400·2
1·4	540 77252	170 35192	74 40659·1	37 76437·8	21 01389·4
1·6	836 54493	263 42436	115 01400	58 35098·4	32 45599·1
1·8	1253 6762	394 63617	172 23954	87 35114·3	48 56795·6
2·0	1829 0002	575 54563	251 11178	127 30666	70 75834·3

where $q=e^{-t}$, and, if we take $K=2$, $L=M=N=1$, we find

$$\frac{1}{2\pi i}\int_{-\pi i/t}^{\pi i/t}\prod_{n=0}^{\infty}\frac{(1-q^{b+s+n})(1-q^{1-x+s+n})(1-q^{x-s+n})}{(1-q^{a+s+n})(1-q^{-s+n})}\,ds$$

$$=\prod_{n=0}^{\infty}\frac{(1-q^{b+n})(1-q^{1-x+n})(1-q^{x+n})}{(1-q^{a+n})(1-q^{n+1})}\,{}_1\Phi_1(q^a;\,q^b;\,q^x)$$

$$\sim\prod_{n=0}^{\infty}\frac{(1-q^{b-a+n})(1-q^{1-x-a+n})(1-q^{x+a+n})}{(1-q^{a+n})(1-q^{1+n})}\,{}_2\Phi_0(q^a,q^{1+a-b};\,;\,-q^{1+b-2a-x-\frac{1}{2}(n+1)}), \tag{3·4}$$

from which (3·1) and (3·2) follow quickly with the help of (1·11) and (1·12).

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4. *Tables of the functions.* These three tables were prepared by me using the electronic computer EDSAC in the Cambridge University Mathematical Laboratory, and, being too fragmentary for adequate differencing, they were checked by double runs. The series, particularly for the larger values of a and b, are very slowly convergent, and it was observed that in some cases as many as three hundred terms were calculated by the machine before arriving at a result.

Table 1 is of the function (1·2) over the range

$$a = 0{\cdot}0(0{\cdot}2)+2{\cdot}0, \quad b = +0{\cdot}2(0{\cdot}2)+1{\cdot}0, \quad y = +0{\cdot}1(0{\cdot}1)+1{\cdot}0, \quad q = +0{\cdot}9.$$

Table 2 is of the same function over the range

$$a = 0{\cdot}0(0{\cdot}2)+2{\cdot}0, \quad b = +0{\cdot}2(0{\cdot}2)+1{\cdot}0, \quad y = +0{\cdot}01(0{\cdot}01)+0{\cdot}10, \quad q = +0{\cdot}99,$$

and Table 3 is of the function (1·1) over the range

$$a = 0{\cdot}0(0{\cdot}2)+2{\cdot}0, \quad b = +0{\cdot}2(0{\cdot}2)+1{\cdot}0, \quad y = +0{\cdot}1(0{\cdot}1)+0{\cdot}9, \quad q = +0{\cdot}9.$$

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Newnham College
Cambridge

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