Math. Proc. Camb. Phil. Soc. (1984), **95**, 223
Printed in Great Britain

On groups with real valued length function

By A. H. M. HOARE
Department of Mathematics, University of Birmingham

(*Received* 17 *March* 1983; *revised* 26 *October* 1983)

Groups with length functions were introduced by Lyndon [5], and those with integer valued length were shown by Chiswell [2] to be equivalent to groups acting on trees, investigated by Serre [6]. Watatani [7] showed that for a group with integer valued length function the edges of the tree form an orthonormal set in a Hilbert space on which the group acts as a group of isometries.

In this paper we show that if G is a group with a real valued length function l then l is negative definite. Moreover there is a naturally defined Hilbert space on which G acts as a group of isometries. We also show a connection with property FA of Serre and with property T of Kazhdan.

1. Let G be a group, with identity e, having a normalized real valued length function l, that is a real function on G satisfying

$$\text{A } 1'. \quad l(e) = 0,$$

$$\text{A } 2. \quad l(x^{-1}) = l(x),$$

and

$$\text{A } 4. \quad d(x,y) > d(y,z) \text{ implies } d(x,z) = d(y,z)$$

where

$$2d(x,y) = l(x) + l(y) - l(xy^{-1}).$$

It follows easily that

$$l(x) \geqslant d(x,y) = d(y,x) \geqslant 0. \tag{1}$$

Let $S = \{(x,m) \mid x \in F, m \in \mathbb{R}, 0 \leqslant m \leqslant l(x)\}$, and for (x,m) and (y,n) in S define $(x,m) \sim (y,n)$ if $d(x,y) \geqslant m = n$. This is an equivalence relation on S, transitivity being a consequence of A 4. We denote the equivalence class of (x,m) by $[x,m]$ or sometimes by x_m. Note that $x_0 = e_0 = [e,0]$ for all x in G. (For notational convenience we have slightly changed the definitions given by Chiswell [2].)

Let V be the complex vector space with basis $B = \{x_m | x \in G, 0 \leqslant m \leqslant l(x)\}$, and let E be the subspace of all linear combinations of elements of B such that the sum of the coefficients is zero. Then E has basis $B' = \{x'_m = x_m - e_0 | x \in G, 0 < m \leqslant l(x)\}$. In order to define the inner product on E we need the following Lemma.

Lemma. *Let D be a non-negative real symmetric $n \times n$ matrix such that*

(i) *$d_{ij} > d_{ik}$ implies $d_{kj} = d_{ik}$.*

Then $\sum_{i,j} d_{ij} z_i \bar{z}_j \geqslant 0$ for all complex $z_1, \ldots, z_n$. Moreover, if

(ii) *$d_{ii} > 0$ for $i = 1, \ldots, n$, and*

(iii) *The three numbers d_{ii}, d_{ij}, d_{jj} are not all equal for $1 \leqslant i \neq j \leqslant n$,*

then $\sum_{i,j} d_{ij} z_i \bar{z}_j > 0$ for complex $z_1, \ldots, z_n$ not all zero.

Proof. True for $n = 1$. Suppose $n \geqslant 2$. Let $a = \min\{d_{ij}: 1 \leqslant i,j \leqslant n\}$. By (i), $d_{ii} \geqslant d_{ij}$, so renumbering if necessary we may assume that

$$d_{11} \geqslant d_{12} \geqslant \ldots \geqslant d_{1n} = a.$$

https://doi.org/10.1017/S0305004100061508 Published online by Cambridge University Press

Let r be the least positive integer such that $d_{1,r+1} = \ldots = d_{1n}$. Then $r \leqslant n-1$, and moreover we have, directly if $r = 1$ and by applying (i) to d_{1i} and d_{1j} if $r \geqslant 2$, that $d_{ij} = a = d_{ji}$ for $i = 1, \ldots, r$ and $j = r+1, \ldots, n$. Therefore $\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij} z_i \bar{z}_j$ is equal to

$$\sum_{j=1}^{n}\sum_{i=1}^{n} a z_i \bar{z}_j + \sum_{i=1}^{r}\sum_{j=1}^{r} (d_{ij}-a) z_i \bar{z}_j + \sum_{i=r+1}^{n}\sum_{j=r+1}^{n} (d_{ij}-a) z_i \bar{z}_j. \tag{2}$$

The first sum is non-negative since $a \geqslant 0$, and the second and third are non-negative by induction, which proves the first part of the Lemma.

For the second part, we observe that $d_{ii} \geqslant a$, and by conditions (i) and (iii) at most one equality holds. If no d_{ii} is equal to a, then by induction at least one of the last two sums in (2) is positive unless $z_1 = \ldots = z_r = z_{r+1} = \ldots = z_n = 0$. If some $d_{ii} = a$, then we could have renumbered so that $i = 1$, and thus $r = 1$ and $a > 0$, in which case the first sum in (2) is positive unless $\Sigma_{i=1}^{n} z_i = 0$, the second is zero, and the third positive by induction unless $z_2 = z_3 = \ldots = z_n = 0$. This completes the proof of the Lemma.

A function $\phi: G \to \mathbb{C}$ is *negative definite* [1] if $\phi(e) \geqslant 0$, $\phi(g^{-1}) = \overline{\phi(g)}$, and

$$\sum_{i,j=1}^{n} \phi(g_i g_j^{-1}) z_i \bar{z}_j \leqslant 0,$$

for all $g_1, \ldots, g_n \in G$ and $z_1, \ldots, z_n \in \mathbb{C}$ with $\sum_{i=1}^{n} z_i = 0$, $n \in \mathbb{N}$. We have the following Corollary to the first part of the Lemma.

COROLLARY 1. *Any real valued length function on a group G is negative definite.*

Proof. From A 1′ and A 2 we have $l(e) \geqslant 0$ and $l(x^{-1}) = \overline{l(x)}$, for all $x \in G$. Suppose $g_1, \ldots, g_n \in G$ and $z_1, \ldots, z_n \in \mathbb{C}$ with $\sum_{i=1}^{n} z_i = 0$, then

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} l(g_i g_j^{-1}) z_i \bar{z}_j &= \sum_i \sum_j (l(g_i) + l(g_j) - 2d(g_i, g_j)) z_i \bar{z}_j \\ &= -2 \sum_{i,j} d(g_i, g_j) z_i \bar{z}_i \quad (\text{since } \sum_i z_i = 0) \\ &\leqslant 0\end{aligned}$$

since $D = (d(g_i, g_j))$ satisfies condition (i) of the Lemma.

COROLLARY 2. *A separable locally compact group with unbounded continuous real valued length function does not have property T of Kazhdan.*

Proof. By a result of Akemann and Walter [1] a separable locally compact group with a continuous unbounded negative definite function does not have property T.

A group G has property FA (Serre [6]) if whenever G acts on a tree (without inversions) it has a fixed point (fixed vertex). The following result, which has also been observed independently by Chiswell, connects this with length functions on G.

PROPOSITION. *G has property FA of Serre if and only if it has no unbounded integer valued length function.*

Proof. Let G act on a tree T. If P_0 is a vertex of T let $l(g)$ be the length of the reduced path from P_0 to $g(P_0)$. This is a length function, [2], so if G has no unbounded length function then $G(P_0)$ spans a subtree T' of finite diameter. In this case the tree T' has a 'centre' which is either a vertex fixed by G or an edge whose 'mid-point' is fixed by all elements of G. Conversely, suppose G has property FA and let l be a length function on G. Then $2l$ is a length function on G satisfying condition $C0$ of [5], i.e. $d(x, y)$ is always an integer, and by [2], theorem 1, there is a tree T on which G acts and

https://doi.org/10.1017/S0305004100061508 Published online by Cambridge University Press

a vertex P_0 such that $2l(g)$ is the length of the reduced path from P_0 to gP_0. Moreover it is easily seen that G acts on T without inversions. If P_1 is one of the fixed points and d the length of the reduced path from P_0 to P_1, then d is also the length of the reduced path from gP_0 to P_1, so $2l(g) \leqslant 2d$ for all $g \in G$.

COROLLARY (Watatani [7]). *For a countable discrete group, property T of Kazhdan implies property FA of Serre.*

Proof. If G does not have property FA, then it has an unbounded integer valued length function, which is continuous since G is discrete. G is separable and locally compact, so by Corollary 2 does not have property T of Kazhdan.

Watatani has observed (private communication) that it follows from Corollary 1 above and results in Delorme [3] that if G has a real valued length function then it has both a unitary representation ρ on a Hilbert space H and a 1-cocycle $b: G \to H$ such that $b(gh) = b(g) + \rho(g)\, b(h)$ and $2l(g) = \|b(g)\|^2$. He also conjectured some connection with the tree-like space defined by Chiswell [2]. We now use Lemma 1 and some results from [2] to give an explicit description of the Hilbert space H and of some invariant subspaces.

THEOREM 1. *The function*

$$\langle x'_m | y'_n \rangle = \min\{m, n, d(x, y)\}$$

defined on the basis B' of E extends to an inner product on E.

Proof. By the extension to E we have $\langle u|v\rangle = \langle v|u\rangle$ and

$$\langle \alpha u + \beta v | w\rangle = \alpha\langle u|w\rangle + \beta\langle v|w\rangle.$$

It remains to show that $\langle u|u\rangle \geqslant 0$ with equality only if $u = 0$.

Let u be expressed as a linear combination of basis elements by

$$u = \sum_{i=1}^{n} z_i([g_i, m_i] - [e, 0]), \quad z_i \in \mathbb{C};$$

then $m_i > 0$ and $[g_i, m_i] \neq [g_j, m_j]$. Let $d_{ij} = \min\{m_i, m_j, d(g_i, g_j)\}$. Then the matrix $D = (d_{ij})$ is non-negative real and symmetric, so the theorem is proved if it satisfies conditions (i), (ii) and (iii) of the Lemma.

Now $d_{ii} = \min\{m_i, l(g_i)\} = m_i > 0$, so (ii) holds. Moreover $d_{ii} = d_{ij} = d_{jj}, i \neq j$ implies $m_i = m_j = \min\{m_i, m_j, d(g_i, g_j)\}$, that is $d(g_i, g_j) \geqslant m_i = m_j$ contradicting

$$[g_i, m_i] \neq [g_j, m_j].$$

Thus (iii) holds.

To prove (i), we observe that by A 4 the triple $(d(g_i, g_j), d(g_j, g_k), d(g_k, g_i))$ has the property that two of its terms are equal and the third no smaller. The triple

$$(\min\{m_i, m_j\}, \min\{m_j, m_k\}, \min\{m_k, m_i\})$$

also has this property. Moreover, given any family of triples (a_i, b_i, c_i), $i \in I$, with this property, it is easily seen that $(\inf a_i, \inf b_i, \inf c_i)$ is a triple with the same property. Applying this to the two triples above gives (i), and thus completes the proof of the Theorem.

2. We now use the action of g on B, for all $g \in G$, given by

$$g[x, m] = \begin{cases} [g^{-1}, l(g) - m] & \text{if} \quad d(g, x) \geqslant m, \\ [xg^{-1}, l(g) + m - 2d(g, x)] & \text{if} \quad d(g, x) < m. \end{cases}$$

https://doi.org/10.1017/S0305004100061508 Published online by Cambridge University Press

This definition, with the change of notation mentioned above, is also due to Chiswell [2]. He proved that *this gives a well-defined group action on* B, so G acts on V by extension, and E is clearly an invariant subspace.

THEOREM 2. *G acts as a group of isometries on E.*

Proof. It suffices to show that $\langle gx'_m|gy'_n\rangle = \langle x'_m|y'_n\rangle$ for all x'_m and y'_n in B'. Chiswell defined a metric δ on B, which in our notation can be written as

$$\delta(x_m, y_n) = m + n - 2\min\{m, n, d(x, y)\},$$

and showed it to be G-invariant ([2], lemma 8).

From the definitions we have

$$2\langle x'_m|y'_n\rangle = \delta(x_m, e_0) + \delta(y_n, e_0) - \delta(x_m, y_n). \tag{3}$$

Moreover, $x'_m = x_m - e_0$, so

$$gx'_m = gx_m - ge_0 = (gx_m)' - (ge_0)',$$

and similarly for gy'_n, with the convention that $e'_0 = 0$. Therefore

$$\langle gx'_m|gy'_n\rangle = \langle (gx_m)'|(gy_n)'\rangle - \langle (ge_0)'|(gy_n)'\rangle - \langle (gx_m)')|(ge_0)'\rangle + \langle (ge_0)'|(ge_0)'\rangle.$$

Applying the formula (3), we get

$$2\langle gx'_m|gy'_n\rangle = -\delta(gx_m, gy_n) + \delta(ge_0, gy_n) + \delta(gx_m, ge_0) - \delta(ge_0, ge_0),$$

all other terms cancelling out. Using the G-invariance of δ and $\delta(ge_0, ge_0) = 0$, we have

$$\langle x'_m|y'_n\rangle = \langle gx'_m|gy'_n\rangle.$$

Remark 1. It can be seen that $\delta(x_m, y_n) = \|x'_m - y'_n\|^2$ where the norm is defined from the inner product.

THEOREM 3. *If G has a real valued length function, then there exists a unitary representation ρ of G on a Hilbert space H and a map $b\colon G \to H$ such that*

$$b(gh) = b(g) + \rho(g)\, b(h) \quad \textit{and} \quad l(gh^{-1}) = \|b(g) - b(h)\|^2.$$

Proof. Let H be the completion of E, and for each $g \in G$ let $\rho(g)$ be the extension of the action on E above. Let $b(g) = [g^{-1}, l(g)] - [e, 0] \in B'$. Then

$$\langle b(g)|b(h)\rangle = d(g, h),$$

and so

$$\|b(g) - b(h)\|^2 = l(gh^{-1}).$$

Moreover

$$\begin{aligned} b(g) + \rho(g)\, b(h) &= [g^{-1}, l(g)] - [e, 0] + g[h^{-1}, l(h)] - g[e, 0] \\ &= [h^{-1}g^{-1}, l(gh)] - [e, 0] \\ &= b(gh), \end{aligned}$$

since $g[e, 0] = [g^{-1}, l(g)]$ and $g[h^{-1}, l(h)] = [h^{-1}g^{-1}, l(gh)]$.

Remark 2. If G has any topology which makes l continuous, then b is also continuous since $\|b(g) - b(h)\|^2 = l(gh^{-1})$.

Remark 3. H has the following G invariant subspaces.

https://doi.org/10.1017/S0305004100061508 Published online by Cambridge University Press

(i) Let $B_1 = \{[x, l(x)],\ x \in G\} \subseteq B$. Since $g[x, l(x)] = [xg^{-1}, l(xg^{-1})]$, by defining B_1', E_1, and H_1 in the same way as B', E and H we have that Theorem 3 holds for G acting on H_1 with the same b. The elements of length zero in G form a subgroup, A say. If x and y belong to the same coset, i.e. $l(xy^{-1}) = 0$, then from (1) and the definition of $d(x, y)$ we have $l(x) = l(y) = d(x, y)$, so $[x, l(x)] = [y, l(y)]$. Conversely, if this holds then x and y belong to the same coset. Thus B_1 is in one-one correspondence with the set of cosets of A in G.

(ii) Let $B_2 = \{[x, d(x, y)]$ all $x, y \in G\}$; then $B_1 \subseteq B_2 \subseteq B$. Moreover, using A4,

$$g[x, d(x, y)] = \begin{cases} [g^{-1}, d(g^{-1}, yg^{-1})] & \text{if} \quad d(x, g) > d(x, y) \\ [g^{-1}, d(g^{-1}, xg^{-1})] & \text{if} \quad d(x, g) = d(x, y) \\ [xg^{-1}, d(xg^{-1}, yg^{-1})] & \text{if} \quad d(x, g) < d(x, y). \end{cases}$$

Thus we have another subspace $H_2 \supset H_1$ on which G acts.

(iii) If $d(x, y)$ is always an integer let $B_3 = \{[x, m], m = 0, 1, \ldots, l(x)\}$. Then B_3 is G-invariant and $B_3 \supset B_2$. The space E_3 is isomorphic to the space defined by Watatani [7] via the boundary map of the cellular chain complex (with coefficients in $\mathbb{C}$) of the tree defined by Chiswell [2].

Remark 4. Imrich [4] has shown that B, with the metric δ above, is a 'tree-like' space in which any two points are connected by a unique shortest path. Conversely, given a group G acting on such a metric space, then each point x in the metric space defines a length function on G by $l_x(g) = \delta(x, gx)$.

REFERENCES

[1] C. A. Akemann and M. Walter. Unbounded negative definite functions. *Canad. J. Math.* **23** (1981), 862–871.

[2] I. M. Chiswell. Abstract length functions in groups. *Math. Proc. Cambridge Philos. Soc.* **80** (1976), 451–463.

[3] P. Delorme. 1-Cohomologie des représentations unitaires. *Bull Soc. Math. France* **105** (1977), 281–336.

[4] W. Imrich. On metric properties of tree-like spaces. *Beiträge zur Graphentheorie und deren Anwendungen, Oberhof (DDR)* (ed. Sektion MARÖK der Technischen Hochschule Ilmenau), pp. 129–156.

[5] R. C. Lyndon. Length functions in groups. *Math. Scand.* **12** (1963), 209–234.

[6] J.-P. Serre. *Trees* (Springer-Verlag, 1980).

[7] Y. Watatani. Property T of Kazhdan implies property FA of Serre. *Math. Japon.* **27** (1982), 99–103.

https://doi.org/10.1017/S0305004100061508 Published online by Cambridge University Press