An Involution in the Complex Variable. By Mr J. H. GRACE, Peterhouse.

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1. Suppose that f_1, f_2, f_3 are three quadratics in the complex variable z with complex coefficients, say

$$f_1 = l_1 z^2 + m_1 z + n_1,$$

and consider the equation

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots (1{\cdot}1)$$

the α's being real quantities. It determines two values ζ, ζ' of z and thus two points P, P' in the complex plane. Clearly when P is given the ratios $\alpha_1 : \alpha_2 : \alpha_3$ are in general determinate and accordingly P' also. This is the involutory transformation I propose to discuss.

There are (it will be proved) four neutral points, i.e. four positions of P for which P' is indeterminate, and calling them A, B, C, D the construction for P' is as follows:

Let the circles PBC, PAD meet in L; PCA, PBD in M; PAB, PCD in N. Then the circles MAB, MCD, NCA, NBD meet in a point L'; the circles NBC, NAD, LAB, LCD meet in a point M'; the circles LCA, LBD, MBC, MAD meet in a point N'; and finally the six circles

$$L'BC, L'AD, M'CA, M'BD, N'AB, N'CD$$

meet in the point P'.

The complete system forms a configuration in that it bears the same relation to each of the eight points P, P'; L, L'; M, M'; N, N' and starting from any one of a pair in place of P we end with the other.

There is no difficulty in seeing that P and L' are a conjugate pair in the ordinary involution of which (B, C) and (A, D) are also conjugate pairs and so on: from this fact alone the existence of the configuration may be established but not its connection with the transformation above.

The construction may be put more concisely thus: suppose that we write

$$L = \alpha(P), \quad M = \beta(P), \quad N = \gamma(P),$$

and regard α, β, γ as operators, then we have an Abelian group

$$1, \alpha, \beta, \gamma, \beta\gamma, \gamma\alpha, \alpha\beta, \alpha\beta\gamma,$$

which operating on P (or any of the eight points mentioned) gives the whole system.

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2. I now pass to the proof.

If a value of z is to make the other value z' given by (1·1) indeterminate it must make the ratios

$$f_1 : f_2 : f_3$$

real.

Now when f_1/f_2 is real it is equal to the conjugate quantity, i.e.

$$f_1/f_2 = \bar{f}_1/\bar{f}_2; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2{\cdot}1)$$

for a moment regard z, $\bar{z}$ as quasi-Cartesian coordinates, the equation (2·1) shows that the locus of the point z is a bicircular quartic passing through the four points given by

$$f_1 = 0, \quad \bar{f}_1 = 0,$$

and the four given by

$$f_2 = 0, \quad \bar{f}_2 = 0.$$

Similarly when f_1/f_3 is real the locus of z is a second bicircular quartic; the two meet in eight points of which those given by

$$f_1 = 0, \quad \bar{f}_1 = 0$$

are to be rejected and the remaining four are the neutral points A, B, C, D mentioned above.

It is true that they may be imaginary in pairs; I shall nevertheless argue as though they were real, the deductions being geometrical will still be valid when they are not*.

3. Suppose that $z = d$ corresponds to the point D, that a, b, c are arbitrary and introduce the system of quadratics

$$\frac{p}{z-a} + \frac{q}{z-b} + \frac{r}{z-c} + \frac{s}{z-d} = 0, \quad \ldots\ldots\ldots (3{\cdot}1)$$

when p, q, r, s are real quantities satisfying

$$p + q + r + s = 0.$$

Apply to the original quadratics

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$$

and to (3·1) a linear transformation in z which changes $z = d$ into $z = \infty$.

They become

$$\alpha_1 f_1' + \alpha_2 f_2' + \alpha_3 f_3' = 0, \quad \ldots\ldots\ldots\ldots\ldots (3{\cdot}2)$$

$$\frac{p}{z-a'} + \frac{q}{z-b'} + \frac{r}{z-c'} = 0, \quad \ldots\ldots\ldots\ldots\ldots (3{\cdot}3)$$

where a', b', c' are the transforms of a, b, c.

* The disadvantage is natural to the method. Complex variable geometry is tantamount to geometry on a quadric surface, the position of a point being fixed by the parameters z, $\bar{z}$ (now quite independent quantities) of the generators through it. § 1 would thus open with two sets of quadratics, one in z, the other in $\bar{z}$; the quantities α would be the same for both. Logically there is a gain and there is a loss of flexibility.

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Since $f_1' : f_2' : f_3'$ are real for $z = \infty$ the ratios $l_1' : l_2' : l_3'$ are real and so without loss of generality we may take these quantities to be real.

The points given by (3·2) are clearly the real foci of a conic belonging to a tangential net

$$\alpha_1 \Sigma_1 + \alpha_2 \Sigma_2 + \alpha_3 \Sigma_3 = 0;$$

they are thus conjugate points for a pencil of rectangular hyperbolas, i.e. isogonal conjugates for a fixed triangle, and so the foci of a conic inscribed in that triangle.

If now a, b, c are chosen so that a', b', c' represent the corners of this triangle the foci of the inscribed conics are given by

$$\frac{\lambda}{z-a'} + \frac{\mu}{z-b'} + \frac{\nu}{z-c'} = 0,$$

where λ, μ, ν are real.

The two systems (3·2), (3·3) are therefore identical *in toto* and consequently the original system

$$\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0$$

is identical with

$$\frac{p}{z-a} + \frac{q}{z-b} + \frac{r}{z-c} + \frac{s}{z-d} = 0, \quad \ldots\ldots\ldots(3{\cdot}4)$$

which henceforth replaces it. Clearly now a, b, c represent the other neutral points A, B, C.

I recall that p, q, r, s are real and that

$$p + q + r + s = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3{\cdot}5)$$

For a moment introduce rectangular Cartesian coordinates, and let $A = 0$ be the tangential equation of the point A; then the equation (3·4) gives the two real finite foci of a class cubic

$$\frac{p}{A} + \frac{q}{B} + \frac{r}{C} + \frac{s}{D} = 0.$$

This cubic touches the six joins of the four points A, B, C, D and, in virtue of (3·5), the line at infinity.

Conjugate points such as P and P' (§ 1) are thus the real finite foci of such a curve of class three *.

4. The equation (3·4) may be written

$$p(a-d)(z-b)(z-c) + q(b-d)(z-c)(z-a) + r(c-d)(z-a)(z-b) = 0,$$

and so has the covariant property, i.e. the relation of a conjugate pair like P, P' to the points A, B, C, D is undisturbed by a linear transformation and so too by an inversion.

* In the original form P, P' are the real foci of a member of a net of curves of class three, all of which touch the line at infinity. The reduction corresponds exactly to that of the case of conics given above.

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If the point P' is at infinity the corresponding curve of class three touches the line at infinity twice and the point K conjugate to the point at infinity is thus the single finite focus of the cubic which touches the six joins of the four points A, B, C, D and also has the line at infinity for double tangent. This brings the geometry into line with the Clifford-Miquel chain of theorems* which I briefly recall.

Four lines form four triangles and their four circumcircles meet in a point, viz. the focus of the parabola touching the lines.

Given five lines, the foci of the parabolas that touch them in sets of four lie on a circle—the Miquel circle.

Given six lines, the Miquel circles derived from them in sets of five meet in a point, viz. the focus of the curve of class three that touches the six lines and the line at infinity twice.

Applying this construction in detail to the case in which the six lines are the joins of the four points A, B, C, D we arrive at the point K and on inversion there follows at once the construction of § 1.

5. It is easy to see *ab initio* that there are four self-corresponding points in the transformation. I denote these by A', B', C', D'.

The points P, A, B, C, D, L, M, N (cf. § 1) will lie on a bicircular quartic and (using elliptic parameters whose sum for four concyclic points on this curve is zero) it is easy to see that the points L', M', N' also lie on the curve.

The parameters of A, B, C, D must be of the type

$$\theta + \omega_1, \quad \theta + \omega_2, \quad \theta + \omega_3, \quad \theta,$$

and then for two conjugate points such as P and P' we have

$$\pi + \pi' + 2\theta = 0.$$

The four points A', B', C', D' thus lie on the curve and have parameters

$$-\theta + \omega_1, \quad -\theta + \omega_2, \quad -\theta + \omega_3, \quad -\theta.$$

All configurations such as that described in § 1 thus lie on one or other of a pencil of bicircular quartics, namely those through the eight points

$$A, B, C, D, A', B', C', D'.$$

It is curious that, as is visible above, the relation between these two tetrads of points is reciprocal. One might pursue the line of thought indicated in this paragraph, but I content myself with the brief sketch.

* Clifford, *Collected Papers*, pp. 38–54.

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6. The transformation under discussion has an intimate connection with an elementary theorem due to Mr G. T. Bennett, viz.:

If A, B, C, D are four points in a plane there exists a point K at which the four circles BCD, CDA, DAB and ABC subtend equal angles. The point K is the same as that described above, namely the conjugate to the point at infinity.

To prove this I follow an argument I owe to Mr Bennett. Through two points P, P' two circles can be drawn to touch a given circle—their angle of intersection is called the angle subtended by the given circle at the point-pair P, P'. The inverted form of the above theorem then runs thus:

If A, B, C, D, P are five points in a plane there exists a unique point P' such that the four circles BCD, etc. subtend equal angles at the pair P, P'.

And it is to be proved that this holds when P, P' are conjugate points of our transformation.

It will suffice to prove that any three of the circles, say those through D, subtend equal angles at P, P'. And since all is invariant under an inversion (or a linear transformation in z) we may suppose D to be at infinity, in which case P, P' are foci of a conic inscribed in the triangle ABC (§ 3).

Now the angle θ subtended by a circle at P, P' is given by

$$\cot^2 \frac{\theta}{2} = \frac{\varpi_P \varpi_{P'}}{R^2 . PP'^2},$$

where $\varpi_P, \varpi_{P'}$ are the powers of P, P' with respect to the circle and R is its radius; in fact the two sides are invariant under inversion and agree when P' is at infinity.

When the circle becomes a straight line we thence have

$$\cot^2 \frac{\theta}{2} = \frac{4PY . P'Y'}{PP'^2},$$

where $PY, P'Y'$ are the perpendiculars from P, P' on the line.

The result now follows at once—the product $PY . P'Y'$ is the same for all tangents to a conic of which P, P' are the foci.

The identification is thus complete.

7. When expressed as geometry on a quadric surface the theorem of § 6 becomes the following:

*If $ABCD$ is a tetrahedron inscribed in a quadric and P is any point on the surface there exists a unique point P' such that a second quadric contains the generators at P, P' of the original and touches the faces of the tetrahedron**.

* Cf. Baker, *Principles of Geometry*, vol. III. p. 42.

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A synthetic proof of this has been given by Dr Baker—as the theorem is analytically equivalent to that of the double six it may be worth while to give an alternative.

Suppose that g is a generator at P and that g' is an arbitrary non-intersecting generator of the original quadric; then

$$g(ABCD) = g'(ABCD),$$

and hence by a theorem of von Staudt the lines g, g' cut the faces of the tetrahedron in projective ranges. It follows that a quadric can be drawn containing g, g' and touching those faces: this meets the original in two further generators h, h' each intersecting g and g'. When g' varies it is quite clear that h and h' do not both remain fixed and we can choose g' so that h, say, is the second generator at P and then h' meets g' in the point P' required.

8. To construct the point P' we have only to translate the method of § 1 into quadric geometry. The result is as follows:

$ABCD$ is a tetrahedron inscribed in a quadric and P is any point of the surface. Lines drawn through P to meet the three pairs of opposite edges cut the quadric again in L, M, N. If we write

$$L = \alpha(P), \quad M = \beta(P), \quad N = \gamma(P),$$

the operators α, β, γ generate an Abelian group

$$1, \alpha, \beta, \gamma, \beta\gamma, \gamma\alpha, \alpha\beta, \alpha\beta\gamma,$$

and

$$(\alpha\beta\gamma)P = P'.$$

9. I propose now to give another proof of the theorem of § 6—it partly depends on the complex variable and it leads to another construction for the point P' when P is known.

Suppose that A', B', C', D' are the centres of the circles BCD, CDA, DAB, ABC and that a, b, c, d are their radii.

It is to be proved that there exists a point whose distances from A', B', C', D' are proportional to a, b, c, d or, what amounts to the same thing, that the inverses of A in the circle $B'C'D'$, of B in the circle $C'D'A'$, etc. are all the same point.

It is clear that A, B, C are the images of D in the sides of the triangle $A'B'C'$ and, since D' is the centre of the circle ABC, D and D' are the foci of a conic inscribed in the triangle mentioned.

I denote the inverse of D in the circle $A'B'C'$ by K. The crux is that K is symmetrically related to the figure as a whole.

To prove this let I, J be the circular points at infinity, let DI cut the sides of $A'B'C'$ in L, M, N and the circle $A'B'C'$ a second

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time in α. It is easy to see that JL, JM, JN pass through A, B, C respectively and that $J\alpha$ passes through K*.

Consider the conics which touch the four lines $B'C'$, $C'A'$, $A'B'$ and DI; among them are the point-pairs A', L; B', M; C', N; a conic having D and D' for foci and one parabola. The last will also touch $J\alpha$ because the two triangles $A'B'C'$, αIJ are inscribed in the same conic (the circle $A'B'C'$), and so their six sides touch a conic. The pairs of tangents from J to the above five conics are

$$JA, JA'; \ JB, JB'; \ JC, JC'; \ JD, JD'; \ JK, JI,$$

and these are accordingly five pairs of an involution. A like result obtains when J is replaced by I and it follows that the five pairs of points

$$(A, A'), (B, B'), (C, C'), (D, D'), (K, \infty)$$

are conjugate pairs of a complex involution.

This property completely defines K and in symmetrical fashion, so that the theorem is proved.

Inversion gives the curious result:

If A, B, C, D, P' are five points in a plane and A', B', C', D' are the inverses of P' in the circles BCD, CDA, DAB, ABC, then the four pairs A, A'; B, B'; C, C'; D, D' belong to a complex involution.

And the conjugate of P' in this involution is the conjugate of P' in the transformation of § 1, namely P.

Regarded as a construction for P (the interchange of P and P' is immaterial) this shows clearly the superiority of that of § 1; so far from putting into evidence the symmetry of the relation between P and P' *qua* A, B, C, D that very symmetry gives rise to new theorems.

10. The properties of the two pentads

$$A\ B\ C\ D\ P$$
$$A'B'C'D'P'$$

(deducible from the uninverted figure) are remarkable:

(i) Two points in the same column such as A and A' are conjugate when the remaining points in either row are the neutral points of § 1.

(ii) All such pairs belong to the same complex involution.

(iii) Two points in either row are inverse for the circle through the other three in the other row.

* I rely on the simple construction for the inverse of a point P in a circle. Suppose that PI, PJ cut the circle again in G, H; then IH and JG meet in the inverse. Similarly for the image of P in a straight line.

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(iv) The relation between the two pentads is in all respects reciprocal—the passage from one to the other is effected by means of (i).

11. The reciprocity can be deduced straightforwardly from the space representation—changing from P, P' to E, E' we have the following theorem:

A, B, C, D, E are five points on a quadric Ω surface; there exists a point E' on the surface such that a second quadric can be drawn to contain the generators of the original at E, E' and to touch the faces of the tetrahedron $ABCD$. In this way arises a second pentad A', B', C', D', E' and the relation between the two is reciprocal.

This follows readily from the extension of a theorem due to Mantel*.

In fact, regarding Ω as the absolute, E, E' are foci of a quadric of revolution inscribed in the tetrahedron $ABCD$; it follows that D', E' are images (in the projective sense) in the plane ABC. The rest is very simple reasoning.

* *Proc. Camb. Phil. Soc.* vol. xxiii. p. 850.

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