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# A GENERAL APPROACH TO THE INTEGRAL FUNCTIONALS OF EPIDEMIC PROCESSES

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**Abstract**

In this paper we consider the integral functionals of the general epidemic model up to its extinction. We develop a new approach to determine the exact Laplace transform of such integrals. In particular, we obtain the Laplace transform of the duration of the epidemic $T$, the final susceptible size $S_T$, the area under the trajectory of the infectives $A_T$, and the area under the trajectory of the susceptibles $B_T$. The method relies on the construction of a family of martingales and allows us to solve simple recursive relations for the involved parameters. The Laplace transforms are then expanded in terms of a special class of polynomials. The analysis is generalized in part to Markovian epidemic processes with arbitrary state-dependent rates.

*Keywords:* General epidemic; Markovian epidemic process; final susceptible size; total area under the trajectory of the infectives and the susceptibles; martingale method; special class of polynomials

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## 1. Introduction

The stochastic epidemic modelling is an important topic in applied probability. An outline of the theory can be found in Daley and Gani (1999) and Andersson and Britton (2000). We also refer the reader to Greenwood and Gordillo (2009) for a nice review.

The most widely studied epidemic process, known as the general Markov epidemic, describes the spread of an infectious disease of the susceptible→infected→removed type in a closed homogeneously mixing population. It is formulated as a bivariate Markovian jump process in continuous time. Let $S_t$, $I_t$, and $R_t$ be the number of susceptibles, infectives, and removed cases at time $t \geq 0$. Initially, $S_0 = n$, $I_0 = m$, and $R_0 = 0$; the population size at any instant is $N \equiv n + m$. The transition probabilities during $(t, t + \mathrm{d}t)$ are defined by

$$\mathbb{P}[S_{t+\mathrm{d}t} = s - 1, I_{t+\mathrm{d}t} = i + 1, R_{t+\mathrm{d}t} = r \mid S_t = s, I_t = i, R_t = r] = \beta si\,\mathrm{d}t + o(\mathrm{d}t), \quad (1)$$

$$\mathbb{P}[S_{t+\mathrm{d}t} = s, I_{t+\mathrm{d}t} = i - 1, R_{t+\mathrm{d}t} = r + 1 \mid S_t = s, I_t = i, R_t = r] = \mu i\,\mathrm{d}t + o(\mathrm{d}t), \quad (2)$$

where $\beta$ and $\mu$ are the positive infection and removal rates. Thus, each infective is infectious during a period of time exponentially distributed of parameter $\mu$. While infected, it can contact any susceptible according to a Poisson process of rate $\beta$. At the end of the infectious period, it becomes removed (dead, isolated, or immune).

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The epidemic terminates at the first instant $T$ when there are no more infectives present ($I_T = 0$). Much attention has been paid to the number $S_T$ of susceptibles who ultimately escape the disease. Another statistic often studied is the total area $A_T$ under the trajectory of the infectives. It represents the cumulated personal time units of infection during the epidemic. As pointed out by Gani and Jerwood (1972), $A_T$ is an important component of the total cost of an epidemic. The distributions of $S_T$ and $A_T$ for the general epidemic were investigated by Ball (1986) and Picard and Lefèvre (1990), amongst many others.

Recently, the general epidemic has been used to build an insurance plan covering an epidemic risk in large companies. We refer the reader to Feng and Garrido (2011) for a study of the deterministic version and to Lefèvre *et al.* (2017) in the case of a Markovian model. In this context, $S_T$ and $A_T$ play an important role, but this is also true for the total area $B_T$ under the trajectory of the susceptibles. Indeed, $B_T$ represents the cumulated personal time units of susceptibility during the epidemic. As the susceptibles are paying premiums as long as they are healthy, the cumulated premiums received by the insurer is measured by the area $B_T$ multiplied by the fixed premium level.

Our purpose in this paper is to develop a new and general approach to obtain the distribution of the integral functionals of the general epidemic up to its extinction. Specifically, consider an integral of the form

$$H_T = \int_0^T h(S_u, I_u)\,\mathrm{d}u,$$

where $h(s, i)$ is any nonnegative function defined on $\{0 \le s \le n, 0 \le i \le N - s\}$. We want to determine the joint generating function and Laplace transform of $(S_T, H_T)$, i.e.

$$\mathbb{E}(x^{S_T}\mathrm{e}^{-\theta H_T}) \quad \text{for } x \text{ real}, \theta \ge 0. \tag{3}$$

In particular, if $\theta = 1$ and $h(s, i) = \theta_1 + \theta_2 i + \theta_3 s$ with $\theta_j \ge 0$, this expectation becomes

$$\mathbb{E}(x^{S_T}\mathrm{e}^{-\theta_1 T - \theta_2 A_T - \theta_3 B_T}),$$

i.e. the joint generating function and Laplace transform of $(S_T, T, A_T, B_T)$. In Lefèvre *et al.* (2017), the equivalence principle in life insurance led the authors to focus essentially on the expected areas $\mathbb{E}(A_T)$ and $\mathbb{E}(B_T)$. The present paper can be seen as a continuation of that work since it deals with the exact distribution of the areas $A_T$ and $B_T$.

The literature on integral functionals of Markov models is widely scattered. A general approach relies on the property that in a finite-state Markov process with one absorbing state, the time to absorption is the stopping time for which a noncurved exponential family holds; see Stefanov (1991) and Stefanov and Pakes (1997). This method was applied by Ball and Stefanov (2001) to compute characteristics of birth–death Markov processes in terms of continued fractions. It can be used for the general Markov epidemic when an absorbing state is added to which an exponential transition is allowed from the final epidemic states. Other approaches, more specific, have also been developed; see, e.g. Puri (1972), Flajolet and Guillemin (2000), Pollett (2003), Gani and Swift (2008), and Khaluf and Dorigo (2016).

By comparison, the new method proposed here is by nature more algebraic. It follows easily from the construction of a family of martingales for the epidemic process. A key advantage of this approach is to provide us with simple recursive procedures to calculate the transform (3) for the general epidemic model and for extended Markovian models (see Section 4). The exponential family method mentioned above gives an in-depth insight into the transforms, but it seems difficult to deduce such explicit results in terms of simple recursions. Moreover, our

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approach allows us to exhibit and exploit a particular polynomial structure which underlies the solutions.

*Preliminaries.* Before beginning, we observe that $H_T$ can be written as

$$H_T = \int_0^T h(S_u, I_u)\, \mathbf{1}_{\{I_u>0\}}\, \mathrm{d}u.$$

Indeed, $I_u > 0$ as long as $u < T$, so that adding the indicator of $\{I_u > 0\}$ in the integral has no impact up to time $T$. The addition of the indicator, however, allows us to let the value of $h(S_u, 0)$ be arbitrary, which will be convenient in Section 2.

Using this trick, we introduce an integral $H_t$, a function of $t$, defined by

$$H_t = \int_0^t h(S_u, I_u)\, \mathbf{1}_{\{I_u>0\}}\, \mathrm{d}u, \qquad t \geq 0. \tag{4}$$

Next, we expand $H_t$ as

$$H_t = \int_0^t h(S_u, I_u)\, \mathbf{1}_{\{S_u\geq 0,\, I_u>0\}}\, \mathrm{d}u = \sum_{0\leq k\leq n,\, 1\leq l\leq N-k} h(k,l) \int_0^t \mathbf{1}_{\{S_u=k,\, I_u=l\}}\, \mathrm{d}u.$$

Defining the integrals

$$B_t(k,l) = \int_0^t \mathbf{1}_{\{S_u=k,\, I_u=l\}}\, \mathrm{d}u, \qquad t \geq 0, \tag{5}$$

we then have

$$H_t = \sum_{0\leq k\leq n,\, 1\leq l\leq N-k} h(k,l) B_t(k,l). \tag{6}$$

We now return to the transform (3). Thanks to (6), it can be written as

$$\mathbb{E}\left(x^{S_T} \exp\left(-\theta \sum_{k,l} h(k,l) B_T(k,l)\right)\right),$$

where the sum is over the set $\{0 \leq k \leq n, 1 \leq l \leq N-k\}$. This is the expression under investigation in what follows.

*Summary.* The paper is organized as follows. In Section 2 we look for the joint distribution of the variables $(S_T, B_T(k,l))$ in the general epidemic model. A family of $n+1$ martingales is constructed (Proposition 1) which allows us to determine the joint Laplace transform of that vector (Propositions 2 and 3). In Section 3 we generalize the previous result to the joint Laplace transform of the variables $(S_T, H_T)$ (Propositions 4 and 5). In Section 4 we show how to adapt our approach to a Markovian right-shift epidemic process with arbitrary state-dependent rates (Propositions 6–8).

## 2. The joint distribution of $(S_T, B_T(k,l))$

Consider $B_t(k,l),\ t \geq 0$, defined by (5) with $0 \leq k \leq n$, $1 \leq l \leq N-k$. Our method relies on the construction of a family of martingales for the epidemic process $\{(S_t, I_t),\ t \geq 0\}$ jointly with $\{B_t(k,l),\ t \geq 0\}$. For that, we introduce an associated stochastic process defined by

$$M_t = a_{k,l}(S_t, I_t;\, \theta)\mathrm{e}^{-\theta B_t(k,l)}, \qquad t \geq 0, \tag{7}$$

where $a_{k,l}(s,i;\,\theta)$ is some function defined on $\{0 \leq s \leq n, 0 \leq i \leq N;\ \theta \geq 0\}$. The first step is to choose the function $a_{k,l}(s,i;\,\theta)$ such that the process $\{M_t,\ t \geq 0\}$ constitutes a martingale.

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**Lemma 1.** *The process* $\{M_t,\ t \geq 0\}$ *is a martingale if the* $a_{k,l}(s, i;\ \theta)$ *satisfy the relations*

$$L[a_{k,l}(s, i;\ \theta)] = \theta\delta_{s,k}\delta_{i,l}a_{k,l}(s, i;\ \theta), \qquad s \geq 0,\ i \geq 1, \tag{8}$$

*where* $\delta_{\cdot,\cdot}$ *is Kronecker's delta and* $L[\cdot]$ *is an operator defined by*

$$\begin{aligned} L[a_{k,l}(s, i;\ \theta)] = {} & \beta si[a_{k,l}(s-1, i+1;\ \theta) - a_{k,l}(s, i;\ \theta)] \\ & + \mu i[a_{k,l}(s, i-1;\ \theta) - a_{k,l}(s, i;\ \theta)] \end{aligned} \tag{9}$$

*with* $a_{k,l}(s, 0;\ \theta)$ *arbitrary and* $a_{k,l}(-1, i;\ \theta) = 0$*, say.*

*Relations (8) and (9) define uniquely the* $a_{k,l}(s, i;\ \theta)$ *once the* $a_{k,l}(s, 0;\ \theta)$ *have been fixed. Specifically, for* $s = 0$*,*

$$a_{k,l}(0, i;\ \theta) = a_{k,l}(0, 0;\ \theta)\left[1 - \mathbf{1}_{\{k=0,\, i\geq l\}}\,\frac{\theta}{\theta + \mu l}\right], \qquad i \geq 1, \tag{10}$$

*and, for* $s \geq 1$*,*

$$a_{k,l}(s, i;\ \theta) = \frac{\beta si a_{k,l}(s-1, i+1;\ \theta) + \mu i a_{k,l}(s, i-1;\ \theta)}{\beta si + \mu i + \theta\delta_{s,k}\delta_{i,l}}, \qquad i \geq 1. \tag{11}$$

*Setting* $\sigma = s + i$*, it then suffices to let* $\sigma$ *vary from* 1 *to* $N$ *and, for fixed* $\sigma$*, to let vary* $s$ *from* 0 *to* $\min(\sigma, n)$*. Furthermore, each* $a_{k,l}(s, i;\ \theta)$ *is a linear combination of the* $a_{k,l}(s', 0;\ \theta)$ *for* $0 \leq s' \leq s$*.*

*Proof.* For clarity, we start by considering the more general stochastic process

$$\{M_t^{\mathrm{g}} = a_{k,l}(S_t, I_t;\ \theta)\mathrm{e}^{-\theta H_t},\ t \geq 0\}$$

with $H_t$ defined by (4). Following the method of Picard (1980), we see that $\{M_t^{\mathrm{g}},\ t \geq 0\}$ forms a martingale if the $a_{k,l}(s, i;\ \theta)$ satisfy the relations

$$\begin{aligned} & [\beta si + \mu i + \theta h(s, i)\,\mathbf{1}_{\{i>0\}}]a_{k,l}(s, i;\ \theta) \\ & \quad = \beta si a_{k,l}(s-1, i+1;\ \theta) + \mu i a_{k,l}(s, i-1;\ \theta), \qquad s, i \geq 0, \end{aligned} \tag{12}$$

with $a_{k,l}(-1, i;\ \theta) = a_{k,l}(s, -1;\ \theta) = 0$, say. Due to the presence of the indicator on the left-hand side, (12) is satisfied for $s \geq 0$, $i = 0$, whatever the value of $h(s, 0)$. Thus, (12) may need to hold only for $s \geq 0$, $i \geq 1$ (as indicated in (8)). Note that without the indicator, (12) for $i = 0$ should yield $0 = 0$, which requires $h(s, 0) = 0$.

We now return to $M_t$, i.e. when $B_t(k, l)$ (with fixed $k, l$) is substituted for $H_t$. From (6), we see that it suffices to choose the function $h(s, i) = \delta_{s,k}\delta_{i,l}$. In that special case, (12) can be rewritten using (8) and (9) as announced.

We first discuss the case where $s = 0$. For $k \geq 1$, (8) and (9) yield $L[a_{k,l}(0, i;\ \theta)] = 0 = \mu i[a_{k,l}(0, i-1;\ \theta) - a_{k,l}(0, i;\ \theta)]$, so that

$$a_{k,l}(0, i;\ \theta) = a_{k,l}(0, i-1;\ \theta) = \cdots = a_{k,l}(0, 0;\ \theta), \qquad i \geq 1.$$

For $k = 0$, $L[a_{0,l}(0, i;\ \theta)] = \theta\delta_{i,l}a_{0,l}(0, i;\ \theta) = \mu i[a_{0,l}(0, i-1;\ \theta) - a_{0,l}(0, i;\ \theta)]$, so that

$$a_{0,l}(0, i;\ \theta) = \frac{a_{0,l}(0, i-1;\ \theta)\mu i}{\theta\delta_{i,l} + \mu i}, \qquad i \geq 1.$$

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Thus, it follows that, for $1 \le i < l$, $a_{0,l}(0, i;\, \theta) = a_{0,l}(0, 0;\, \theta)$ and, for $i \ge l$, $a_{0,l}(0, i;\, \theta) = a_{0,l}(0, 0;\, \theta)\mu l/(\theta + \mu l)$. A compact form of all the results obtained when $s = 0$ is given by (10). Now, when $s \ge 1$, (8) and (9) directly yield (11). Finally, we observe that (10) and (11) provide us with, recursively, the $a_{k,l}(s, i;\, \theta)$ by following the procedure indicated in the statement. □

The choice of the $a_{k,l}(s, 0;\, \theta)$ is a key step for the recursive calculation of the $a_{k,l}(s, i;\, \theta)$. First, consider $\theta = 0$. Then (8) reduces to $L[a_{k,l}(s, i;\, 0)] = 0$, $s \ge 0$, $i \ge 1$. Thus, $k$ and $l$ may play any role and it is possible to exhibit a solution that does not depend on $k$ and $l$. This solution is denoted by $a(s, i)$, i.e.

$$a_{k,l}(s, i;\, 0) \equiv a(s, i) \quad \text{(independent of } (k, l)\text{)}, \qquad s, i \ge 0. \tag{13}$$

The $a(s, i)$ will be made explicit later (see Proposition 1). Of course, they are obtained from the $a(s, 0)$. Now, when $\theta > 0$, we use the same $a(s, 0)$ to set

$$a_{k,l}(s, 0;\, \theta) \equiv a(s, 0) \quad \text{(independent of } \theta\text{)}, \qquad s \ge 0. \tag{14}$$

This choice allows us to determine all the $a_{k,l}(s, i;\, \theta)$. It naturally yields (13) when $\theta = 0$. In what follows, denote

$$d_{k,l}(s, i) = \left[\frac{\mathrm{d}}{\mathrm{d}\theta} a_{k,l}(s, i;\, \theta)\right]_{\theta=0}, \qquad s, i \ge 0, \tag{15}$$

**Lemma 2.** *If $s < k$ or $s+i < k+l$, the $a_{k,l}(s, i;\, \theta)$ are independent of $\theta$, so that $a_{k,l}(s, i;\, \theta) = a(s, i)$ in that case. If $s \ge k$ and $s + i \ge k + l$, the $a_{k,l}(s, i;\, \theta)$ depend on $\theta$. More precisely, $a_{k,l}(s, i;\, \theta)$ can be expressed as*

$$a_{k,l}(s, i;\, \theta) = a(s, i) + d_{k,l}(s, i)\frac{\theta(\beta kl + \mu l)}{\theta + \beta kl + \mu l}, \qquad s \ge 0,\ i \ge 1, \tag{16}$$

*where the $d_{k,l}(s, i)$ satisfy*

$$L[d_{k,l}(s, i)] = \delta_{s,k}\delta_{i,l}a(s, i), \qquad s \ge 0,\ i \ge 1, \tag{17}$$

*with $d_{k,l}(s, 0) = 0$ and $d_{k,l}(-1, i) = 0$, say.*

*Proof.* From (11), $a_{k,l}(s, i;\, \theta)$ is a linear function of $a_{k,l}(s-1, i+1;\, \theta)$ and $a_{k,l}(s, i-1;\, \theta)$ whose coefficients are independent of $\theta$ except if $s = k$ and $i = l$. So, we now apply the recursion described in Lemma 1 by varying $\sigma = s + i$ from 1 to $N$ and, for fixed $\sigma$, $s$ from 0 to $\min(\sigma, n)$. As the $a_{k,l}(s, 0;\, \theta)$ are independent of $\theta$, we see that this property will also hold if $s < k$ or $s + i < k + l$, as announced.

Now, when $(s, i) = (k, l)$, a $1/(\theta + \beta kl + \mu l)$ term will appear. All the next $a_{k,l}(s, i;\, \theta)$ will then depend on $\theta$ (except when $s < k$), and linearly with $1/(\theta + \beta kl + \mu l)$. We can thus express $a_{k,l}(s, i;\, \theta)$ under the form

$$a_{k,l}(s, i;\, \theta) = \alpha_{k,l}(s, i) + \frac{\alpha^*_{k,l}(s, i)}{\theta + \beta kl + \mu l}, \tag{18}$$

where $\alpha_{k,l}(s, i)$ and $\alpha^*_{k;l}(s, i)$ are independent of $\theta$. For $\theta = 0$, from (18), using (13),

$$a(s, i) = \alpha_{k,l}(s, i) + \frac{\alpha^*_{k,l}(s, i)}{\beta kl + \mu l},$$

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while differentiating (18) with $\theta = 0$ yields, using (15),

$$d_{k,l}(s,i) = \frac{-\alpha^*_{k;l}(s,i)}{(\beta kl + \mu l)^2}.$$

By substitution in (18), $a_{k,l}(s,i;\,\theta)$ then becomes

$$\begin{aligned} a_{k,l}(s,i;\,\theta) &= a(s,i) - \frac{\alpha^*_{k,l}(s,i)}{\beta kl + \mu l} + \frac{\alpha^*_{k,l}(s,i)}{\theta + \beta kl + \mu l} \\ &= a(s,i) + d_{k,l}(s,i)(\beta kl + \mu l)^2 \left( \frac{1}{\beta kl + \mu l} - \frac{1}{\theta + \beta kl + \mu l} \right), \end{aligned}$$

which yields (16). Relation (17) is immediate from (8) and using (13)–(15). □

The next step is to make a clever choice for the $a(s,0)$. In Lefèvre *et al.* (2015), all the $a(s,0)$ were taken equal to 1. To avoid confusion, we denote the associated $d_{k,l}(s,i)$ as $b_{k,l}(s,i)$ in that case. From (8) and (9), we then see that $a(s,i) = 1$ for all $s, i \geq 0$, so that (17) yields the following result.

**Lemma 3.** *The $b_{k,l}(s,i)$ satisfy*

$$L[b_{k,l}(s,i)] = \delta_{s,k}\delta_{i,l}, \qquad s \geq 0,\ i \geq 1, \tag{19}$$

*with $b_{k,l}(s,0) = 0$ and $b_{k,l}(-1,i) = 0$, say.*

Using (19), Lefèvre *et al.* (2015) obtained an explicit formula for the $b_{k,l}(s,i)$. This result is not recalled here for reasons of brevity. The distribution of $B_T(k,l)$ was also obtained in that paper. It will be rederived as a special case in (33) below.

In order to obtain the joint distribution of $(S_T, B_T(k,l))$, we choose this time for the $a(s,0)$ distinct values that come from explicitly known values for the $a(s,i)$. In this way, we will be able to derive $n+1$ different families of solutions to (8) and (9). An index $[r]$, $0 \leq r \leq n$, is added in the notation to distinguish these $n+1$ solutions; hence, the notation $a_{[r]}(s,i)$ and $a_{k,l,[r]}(s,i;\,\theta)$.

**Proposition 1.** *When $\theta = 0$, (8) and (9) are satisfied by the $n+1$ families $a_{[r]}(s,i)$, $0 \leq r \leq n$, given by*

$$a_{[r]}(s,i) = s_{[r]} \left( \frac{\mu}{\mu + \beta r} \right)^{s+i-r}, \qquad s,i \geq 0, \tag{20}$$

*where $s_{[0]} = 1$ and $s_{[r]} = s(s-1)\cdots(s-r+1)$, $r \geq 1$. The associated $d_{k,l,[r]}(s,i)$ defined through (17) can then be expressed as*

$$d_{k,l,[r]}(s,i) = a_{[r]}(k,l) b_{k,l}(s,i), \qquad s,i \geq 0, \tag{21}$$

*where the $b_{k,l}(s,i)$ are solutions of (19) (and explicitly known), and $d_{k,l,[r]}(s,0) = 0$.*

*When $\theta > 0$, (8) and (9) are satisfied by the $n+1$ families $a_{k,l,[r]}(s,i;\,\theta)$, $0 \leq r \leq n$, given by*

$$a_{k,l,[r]}(s,i;\,\theta) = a_{[r]}(s,i) + d_{k,l,[r]}(s,i) \frac{\theta(\beta kl + \mu l)}{\theta + \beta kl + \mu l}, \qquad s,i \geq 0. \tag{22}$$

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*Proof.* If $\theta = 0$, from (13) and using (8) and (9), the $a(s, i)$ satisfy

$$\beta si[a(s-1, i+1) - a(s, i)] + \mu i[a(s, i-1) - a(s, i)] = 0, \qquad s \geq 0,\ i \geq 1. \tag{23}$$

We try an expression of the form

$$a(s, i) = c_s \lambda^{s+i}, \qquad s, i \geq 0. \tag{24}$$

Then (23) reduces to

$$\left(\beta s + \mu - \frac{\mu}{\lambda}\right) c_s = \beta s c_{s-1}, \qquad s \geq 0, \tag{25}$$

where $c_{-1} = 0$, say. Choose now $\lambda \equiv \lambda_r = \mu/(\mu + \beta r)$, where $r$ is any nonnegative integer. Note that each $a(s, i)$ is so denoted by $a_{[r]}(s, i)$. From (25), we then obtain

$$(s - r)c_s = s c_{s-1}, \qquad s \geq 0. \tag{26}$$

Moreover, take $c_0 = c_1 = \cdots = c_{r-1} = 0$ and $c_r = r!\,\lambda_r^{-r}$. From (26), we obtain

$$c_s = \frac{s!}{(s-r)!}\lambda_r^{-r}, \qquad s \geq r.$$

Finally, inserting into (24) the expressions derived for $\lambda_r$ and $c_s$ leads us to (20) for $a_{[r]}(s, i)$.

Now, from (17), the corresponding $d_{k,l,[r]}(s, i)$ satisfy

$$L[d_{k,l,[r]}(s, i)] = \delta_{s,k}\delta_{i,l} a_{[r]}(k, l), \qquad s \geq 0,\ i \geq 1, \tag{27}$$

with the border conditions $d_{k,l,[r]}(s, 0) = 0$ from (15). On the one hand, if $k \geq r$, $a_{[r]}(k, l) \neq 0$ (since $k_{[r]} > 0$), so that (27) can be written as

$$L\left[\frac{d_{k,l,[r]}(s, i)}{a_{[r]}(k, l)}\right] = \delta_{s,k}\delta_{i,l}, \qquad s \geq 0,\ i \geq 1.$$

Comparing with (19), we obtain $d_{k,l,[r]}(s, i)/a_{[r]}(k, l) = b_{k,l}(s, i)$. On the other hand, if $k < r$, $a_{[r]}(k, l) = 0$ and (27) becomes

$$L[d_{k,l,[r]}(s, i)] = 0, \qquad s \geq 0,\ i \geq 1.$$

This yields $d_{k,l,[r]}(s, i) = 0$ (equal to $a_{[r]}(k, l)$). In conclusion, $d_{k,l,[r]}(s, i)$ is provided by (21) in all cases.

If $\theta > 0$, (22) follows directly from (16). □

We are ready to obtain the joint distribution of $(S_T,\ B_T(k, l))$. This will be carried out in terms of a family of Abel–Gontcharoff polynomials $\{G_r(x \mid U),\ r \geq 0\}$ of degree $r$ in $x$, that are constructed from a particular sequence of reals $U = \{u_j,\ j \geq 0\}$. For a presentation of these polynomials, we refer the reader to Lefèvre and Picard (1990), (2015). An advantage of these polynomials is that they are computable by recursion and they enjoy simple useful properties. In particular, any polynomial $p(x)$ of degree $n$ admits an Abelian-type expansion, namely,

$$p(x) = \sum_{r=0}^{n} p^{(r)}(u_r) G_r(x \mid U), \tag{28}$$

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where $p^{(r)}(\cdot)$ is the $r$th derivative of $p$. This formula is used below when

$$u_j \equiv \frac{\mu}{\mu + \beta j}, \qquad j \geq 0. \tag{29}$$

Recall that initially there are $S_0 = n$ susceptibles and $I_0 = m$ infectives.

**Proposition 2.** *The $n+1$ following relations hold:*

$$\mathbb{E}(S_{T,[r]}(u_r)^{S_T - r} \mathrm{e}^{-\theta B_T(k,l)}) = a_{k,l,[r]}(n, m;\, \theta), \qquad 0 \leq r \leq n, \tag{30}$$

*where the $u_r$ are defined by (29) and the $a_{k,l,[r]}(n, m; \theta)$ are provided by Proposition 1. Therefore, the joint generating function and Laplace transform of $(S_T, B_T(k, l))$ is given by*

$$\mathbb{E}(x^{S_T} \mathrm{e}^{-\theta B_T(k,l)}) = \sum_{r=0}^{n} a_{k,l,[r]}(n, m;\, \theta) G_r(x \mid U), \qquad x \text{ real}, \ \theta \geq 0. \tag{31}$$

*Proof.* Consider the family of $n+1$ martingales, indexed by $[r]$, that are built from (7) and using Lemma 2 and Proposition 1. They are denoted by $\{M_{t,[r]},\ t \geq 0\}$, $0 \leq r \leq n$. The epidemic is finished at time $T$ when there are no more infectives ($I_T = 0$). The considered martingales are bounded since $S_t$ is bounded by $n$ and $I_t$ by $n + m$. Thus, we may apply the optional stopping theorem to each martingale stopped at $T$. This yields the $n+1$ identities $\mathbb{E}(M_{T,[r]}) = M_{0,[r]}$, i.e. explicitly,

$$\mathbb{E}(a_{k,l,[r]}(S_T, 0;\, \theta) \mathrm{e}^{-\theta B_T(k,l)}) = a_{k,l,[r]}(n, m;\, \theta), \qquad 0 \leq r \leq n. \tag{32}$$

We now determine the $a_{k,l,[r]}(s, 0;\, \theta)$. From (22), $a_{k,l,[r]}(s, 0; \theta) = a_{[r]}(s, 0)$ and, from (20), $a_{[r]}(s, 0) = s_{[r]}(u_r)^{s-r}$ using the notation of (29). By substitution into (32), we then obtain the $n$ relations (30).

Now, from the Abelian expansion (28) for the polynomial $x^{S_T}$, we obtain

$$\mathbb{E}(x^{S_T} \mathrm{e}^{-\theta B_T(k,l)}) = \sum_{r=0}^{n} \mathbb{E}(S_{T,[r]}(u_r)^{S_T - r} \mathrm{e}^{-\theta B_T(k,l)}) G_r(x \mid U).$$

Inserting the $n+1$ relations (30), we deduce the desired formula (31). □

**Proposition 3.** *The Laplace transform of $B_T(k, l)$ is given by*

$$\mathbb{E}(\mathrm{e}^{-\theta B_T(k,l)}) = 1 + b_{k,l}(n, m) \frac{\theta(\beta kl + \mu l)}{\theta + \beta kl + \mu l}, \tag{33}$$

*where $b_{k,l}(n, m)$ is obtained through Lemma 3. In other words, $B_T(k, l)$ is distributed as the product $Y_{k,l} W_{k,l}$, where $Y_{k,l}$ is an exponential random variable with parameter $\beta kl + \mu l$, and $W_{k,l}$ is a Bernoulli random variable independent of $Y_{k,l}$ for which $\mathbb{P}(W_{k,l} = 1) = -b_{k,l}(n, m)(\beta kl + \mu l)$.*

*The probability mass function of $S_T$ is provided by the system of $n+1$ linear relations*

$$\sum_{s=r}^{n} \mathbb{P}[S_T = s] s_{[r]}(u_r)^s = n_{[r]}(u_r)^{n+m}, \qquad 0 \leq r \leq n, \tag{34}$$

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*and its generating function is given by*

$$\mathbb{E}(x^{S_T}) = \sum_{r=0}^{n} n_{[r]}(u_r)^{n+m-r} G_r(x \mid U). \tag{35}$$

*As for the integral $H_T$,*

$$\mathbb{E}(x^{S_T} H_T) = - \sum_{0\le k\le n,\ 1\le l\le N-k} h(k,l) b_{k,l}(n,m) \left[ \sum_{r=0}^{n} k_{[r]}(u_r)^{k+l-r} G_r(x \mid U) \right] \tag{36}$$

*and, in particular,*

$$\mathbb{E}(H_T) = - \sum_{0\le k\le n,\ 1\le l\le N-k} h(k,l) b_{k,l}(n,m). \tag{37}$$

*Proof.* For (33), it suffices to consider $r = 0$ in (30). Indeed, from (29), we have $u_0 = 1$, and from (21) and (22),

$$a_{k,l,[0]}(n,m;\,\theta) = a_{[0]}(n,m) + \frac{d_{k,l,[0]}(n,m)\theta(\beta kl + \mu l)}{\theta + \beta kl + \mu l} = 1 + \frac{b_{k,l}(n,m)\theta(\beta kl + \mu l)}{\theta + \beta kl + \mu l};$$

hence, (33). This result can also be deduced from (31) with $x = 1$ using $u_0 = 1$ and the (known) property that $G_r(u_0 \mid U) = \delta_{r,0}$. From (33), it is easily seen that $B_T(k,l)$ is distributed as the random variable defined in the statement above.

For (34), we take $\theta = 0$ in (30). Indeed, from (20) and (22), and in the notation of (29),

$$a_{k,l,[r]}(n,m;0) = a_{[r]}(n,m) = n_{[r]}(u_r)^{n+m-r}, \qquad 0 \le r \le n;$$

hence, (34). Note that the $r = 0$ case just says that $\mathbb{P}[S_T = 0] + \cdots + \mathbb{P}[S_T = n] = 1$. In the same way, (31) with $\theta = 0$ leads to (35).

For the integral $H_T$, we have, from (6),

$$\mathbb{E}(x^{S_T} H_T) = \sum_{0\le k\le n,\ 1\le l\le N-k} h(k,l)\mathbb{E}(x^{S_T} B_T(k,l)). \tag{38}$$

On the other hand, differentiating (31) with $\theta = 0$ and using (15), we obtain

$$\mathbb{E}(x^{S_T} B_T(k,l)) = - \sum_{r=0}^{n} d_{k,l,[r]}(n,m) G_r(x \mid U), \tag{39}$$

in which, from (20), (21), and (29),

$$d_{k,l,[r]}(n,m) = k_{[r]}(u_r)^{k+l-r} b_{k,l}(n,m), \qquad 0 \le r \le n. \tag{40}$$

Inserting (39) and (40) into (38), we then obtain (36). Finally, (37) follows by taking $x = 1$ in (36), since $u_0 = 1$ and $G_r(1 \mid U) = \delta_{r,0}$. □

We mention that the Laplace transform of $B_T(k,l)$ was already derived by Lefèvre *et al.* (2015). Moreover, the generating function of $S_T$ was first obtained by Picard and Lefèvre (1990).

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## 3. The joint distribution of $(S_T, H_T)$

Consider $H_t$, $t \geq 0$, defined by (6). Now we will build a family of martingales for the epidemic process jointly with $\{H_t,\ t \geq 0\}$. Let $\theta H$ denote the set of parameters $\{\theta h(k,l), 0 \leq k \leq n, 1 \leq l \leq N-k\}$. We define an associated stochastic process of the form

$$M_t = a(S_t, I_t;\ \theta H) \exp\left(-\theta \sum_{k,l} h(k,l) B_t(k,l)\right), \qquad t \geq 0, \tag{41}$$

where $a(s,i;\ \theta H)$ is some function on $\{0 \leq s \leq n, 0 \leq i \leq N-s;\ \theta \geq 0\}$, and the sum is over the set $\{0 \leq k \leq n, 1 \leq l \leq N-k\}$. We also mimic the notation of (15) and write, for $j \geq 1$,

$$d^{(j)}(s,i) = \left[\frac{\mathrm{d}^j}{\mathrm{d}\theta^j} a(s,i;\ \theta H)\right]_{\theta=0}, \qquad s,i \geq 0. \tag{42}$$

Generalizing the previous approach, we can extend Propositions 2 and 3 as follows.

**Proposition 4.** *Let $U = \{u_j,\ j \geq 0\}$ be defined by (29). The following $n+1$ relations hold:*

$$\mathbb{E}\left(S_{T,[r]}(u_r)^{S_T - r} \exp\left(-\theta \sum_{k,l} h(k,l) B_T(k,l)\right)\right) = a_{[r]}(n,m;\ \theta H), \qquad 0 \leq r \leq n, \tag{43}$$

*where the $a_{[r]}(n,m;\ \theta H)$ are obtained through (50) and (51) below. Thus, the joint generating function and Laplace transform of $(S_T, H_T)$ is given, for $x$ real, $\theta \geq 0$, by*

$$\mathbb{E}\left(x^{S_T} \exp\left(-\theta \sum_{k,l} h(k,l) B_T(k,l)\right)\right) = \sum_{r=0}^{n} a_{[r]}(n,m;\ \theta H) G_r(x \mid U), \tag{44}$$

*In particular, the Laplace transform of $H_T$ is*

$$\mathbb{E}(\mathrm{e}^{-\theta H_T}) = a_{[0]}(n,m;\ \theta H), \qquad \theta \geq 0; \tag{45}$$

*hence, its $j$th moments are*

$$\mathbb{E}(H_T^j) = (-1)^j d_{[0]}^{(j)}(n,m), \qquad j \geq 1. \tag{46}$$

*Proof.* First, we determine a function $a(s,i;\ \theta H)$ such that $\{M_t,\ t \geq 0\}$ forms a martingale. Arguing as for Lemma 1, from (41) we see that this process is a martingale if the $a(s,i;\ \theta H)$ satisfy

$$\begin{aligned} L[a(s,i;\ \theta H)] &= \theta \left[\sum_{k,l} h(k,l) \delta_{s,k} \delta_{i,l}\right] a(s,i;\ \theta H) \\ &= \theta h(s,i) a(s,i;\ \theta H), \qquad s \geq 0,\ i \geq 1, \end{aligned} \tag{47}$$

where $L[\cdot]$ is given by (9) and the border conditions are the same as before. As for Lemma 2, the $a(s,0;\ \theta H)$ are taken to be independent of $\theta$. So, (14) is replaced by

$$a(s,0;\theta H) \equiv a(s,0), \qquad s \geq 0. \tag{48}$$

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Moreover, as in Proposition 1, when $\theta = 0$ we choose for $a(s, i;\ 0)$ the $n+1$ families $a_{[r]}(s, i)$, $0 \le r \le n$, given by (20). Thus, applying the stopping time theorem to the associated martingales $\{M_{t,[r]},\ t \ge 0\}$, we obtain, instead of (32),

$$\mathbb{E}\left(a_{[r]}(S_T, 0;\ \theta H)\exp\left(-\theta \sum_{k,l} h(k,l)B_T(k,l)\right)\right) = a_{[r]}(n, m;\ \theta H), \qquad 0 \le r \le n. \quad (49)$$

Since $a_{[r]}(s, 0;\ \theta H) = a_{[r]}(s, 0)$ (see (20)), (49) then become (43). The expansion (44) follows from (28) and (43). Setting $x = 1$ (equal to $u_0$) in (44) yields (45). This yields (46) by successive differentiations. □

It remains to evaluate the $a_{[r]}(s, i;\ \theta H)$. A simple method is to use Taylor's expansion

$$a_{[r]}(s, i;\ \theta H) = a_{[r]}(s, i) + \sum_{j \ge 1} d^{(j)}_{[r]}(s, i)\frac{\theta^j}{j!}, \qquad s, i \ge 0. \quad (50)$$

This requires us, of course, to compute the coefficients $d^{(j)}_{[r]}(s, i),\ j \ge 1$. As shown in (46), the first coefficients $d^{(j)}_{[0]}(n, m)$ allow us to also find the first moments of $H_T$.

**Proposition 5.** *For each $r = 0, \ldots, n$, starting with*

$$d^{(0)}_{[r]}(s, i) \equiv a_{[r]}(s, i), \qquad s, i \ge 0,$$

*the $d^{(j)}_{[r]}(s, i),\ j \ge 1$, are obtained recursively from*

$$d^{(j)}_{[r]}(s, i) = j \sum_{0 \le k \le n,\ 1 \le l \le N-k} h(k, l)d^{(j-1)}_{[r]}(k, l)b_{k,l}(s, i), \qquad s, i \ge 0, \quad (51)$$

*where the $b_{k,l}(s, i)$ correspond to the solutions of (19).*

*Proof.* For $j \ge 1$, we obtain, from (42) and (47),

$$\begin{aligned} L[d^{(j)}_{[r]}(s, i)] &= \left\{\frac{\mathrm{d}^j}{\mathrm{d}\theta^j} L[a_{[r]}(s, i;\ \theta H)]\right\}_{\theta=0} \\ &= \left\{\frac{\mathrm{d}^j}{\mathrm{d}\theta^j}\left[\theta \sum_{k,l} h(k, l)\delta_{s,k}\delta_{i,l}a_{[r]}(s, i;\ \theta H)\right]\right\}_{\theta=0} \\ &= \left\{\frac{\mathrm{d}^j}{\mathrm{d}\theta^j}\left[\theta \sum_{k,l} h(k, l)\delta_{s,k}\delta_{i,l}a_{[r]}(k, l;\ \theta H)\right]\right\}_{\theta=0} \\ &= j \sum_{k,l} h(k, l)\delta_{s,k}\delta_{i,l}\left\{\frac{\mathrm{d}^{j-1}}{\mathrm{d}\theta^{j-1}} a_{[r]}(k, l;\ \theta H)\right\}_{\theta=0} \\ &= j \sum_{k,l} h(k, l)\delta_{s,k}\delta_{i,l}d^{(j-1)}_{[r]}(k, l), \qquad s \ge 0,\ i \ge 1. \end{aligned} \quad (52)$$

The border conditions are $d^{(j)}_{[r]}(s, 0) = 0$ since $a_{[r]}(s, 0;\ \theta H)$ is independent of $\theta$ by (48).

We now show that the solution to (52) is provided by the recursion (51). Denote the right-hand side of (51) by $z^{(j)}_{[r]}(s, i)$, say. First, we note that $z^{(j)}_{[r]}(s, 0) = 0$ since $b_{k,l}(s, 0) = 0$.

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Now, using (19) satisfied by $b_{k,l}(s,i)$, we see that

$$\begin{aligned} L[z_{[r]}^{(j)}(s,i)] &= L\left[ j \sum_{k,l} h(k,l) d_{[r]}^{(j-1)}(k,l) b_{k,l}(s,i) \right] \\ &= j \sum_{k,l} h(k,l) d_{[r]}^{(j-1)}(k,l) L[b_{k,l}(s,i)] \\ &= j \sum_{k,l} h(k,l) d_{[r]}^{(j-1)}(k,l) \delta_{s,k} \delta_{i,l}, \qquad s \geq 0,\ i \geq 1. \end{aligned} \tag{53}$$

From (52), the right-hand side of (53) is equal to $L[d_{[r]}^{(j)}(s,i)]$. Thus, we find that $d_{[r]}^{(j)}(s,i) = z_{[r]}^{(j)}(s,i)$ for all $s, i \geq 0$, i.e. (51) holds. □

Note that (46) for the $j$th moments of $H_T$ reduces to (37) when $j = 1$ since in (51), $d_{[0]}^{(0)}(k,l) = a_{[0]}(k,l) = 1$ from (20).

*An explicit formula.* The previous strategy can be further developed using a partial fractions decomposition of $a_{[r]}(s,i;\,\theta H)$ with respect to the variables $\theta(k,l) \equiv \theta h(k,l)$. This leads us to an explicit (but rather complex) expression for the $a_{[r]}(s,i;\,\theta H)$. Due to a lack of space, we simply state here the structure of the formula without proof. Consider the algorithm for the $a(s,i;\,\theta)$ described in Lemma 1 (just after (11)). The rank of each pair $(k,l)$ successively treated is denoted by $\rho(k,l)$; so, $\rho(k,l)$ goes from 1 to $N \equiv (n+1)(n+m/2) - 1$. We have proved that $a_{[r]}(s,i;\,\theta H)$ can be expanded as

$$a_{[r]}(s,i;\,\theta H) = a_{[r]}(s,0) + \sum_{\nu=1}^{\tilde{N}} \left( \sum_{\rho_\nu} C^{(\nu)}_{(k_1,l_1),\ldots,(k_\nu,l_\nu)}(s,i;\,r) \prod_{u=1}^{\nu} \frac{\theta h(k_u,l_u)(\beta k_u + \mu)}{\theta h(k_u,l_u) + \beta k_u + \mu} \right),$$

where $\rho_\nu$ is the set $\{1 \leq \rho(k_1,l_1) < \cdots < \rho(k_\nu,l_\nu) \leq \tilde{N}\}$, and the $C^{(\nu)}_{(k_1,l_1),\ldots,(k_\nu,l_\nu)}(s,i;\,r)$ are obtained by a recursion on $\nu$ (that looks like (51), roughly).

## 4. Markovian epidemic processes

The method applied to the general Markov epidemic also works for the class of right-shift processes introduced by Severo (1969). These models are Markov processes with a multidimensional state-space on which the infinitesimal transition movement is by a shifting of one unit from one coordinate to some other to its right. A key property of such processes is that the Kolmogorov forward equations can be represented as a triangular system. Applications to epidemic models are numerous; see, e.g. Kryscio and Saunders (1976) and Billard and Zhao (1993).

For right-shift processes, the calculations of integrals made before can be generalized to some extent. However, we focus on a particular extension of the general epidemic model. On the one hand, we assume that a health control organism can detect (and remove) any infective present through a Poisson process. On the other hand, all the Markovian transition rates are allowed to be state-dependent and of general form. Specifically, let $S_t$, $I_t$, $R_t$ be defined as before, and let $Y_t$ denote the number of detected cases at time $t \geq 0$ with $Y_0 = 0$. Recall that

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$S_t + I_t + R_t + Y_t = n + m$. Instead of (1) and (2), we have here

$$\begin{aligned}
&\mathbb{P}[S_{t+\mathrm{d}t} = s-1, I_{t+\mathrm{d}t} = i+1, R_{t+\mathrm{d}t} = r, Y_{t+\mathrm{d}t} = y \mid S_t = s, I_t = i, R_t = r, Y_t = y] \\
&\qquad = \beta(s,i,y)\,\mathrm{d}t + o(\mathrm{d}t), \qquad (54)\\
&\mathbb{P}[S_{t+\mathrm{d}t} = s, I_{t+\mathrm{d}t} = i-1, R_{t+\mathrm{d}t} = r+1, Y_{t+\mathrm{d}t} = y \mid S_t = s, I_t = i, R_t = r, Y_t = y] \\
&\qquad = \mu(s,i,y)\,\mathrm{d}t + o(\mathrm{d}t), \qquad (55)\\
&\mathbb{P}[S_{t+\mathrm{d}t} = s, I_{t+\mathrm{d}t} = i-1, R_{t+\mathrm{d}t} = r, Y_{t+\mathrm{d}t} = y+1 \mid S_t = s, I_t = i, R_t = r, Y_t = y] \\
&\qquad = \alpha(s,i,y)\,\mathrm{d}t + o(\mathrm{d}t), \qquad (56)
\end{aligned}$$

where the infection, removal, and detection rates are positive except that

$$\beta(s,0,y) = \beta(0,i,y) = \mu(s,0,y) = \alpha(s,0,y) = 0.$$

For special cases of state-dependent rates, see, e.g. O'Neill (1997), Ball *et al.* (2007), and Lefèvre *et al.* (2017).

Our purpose is to show how to adapt the analysis made in Section 2 to the model in (54)–(56). So, instead of (5), we introduce the integrals

$$B_t(k,l,w) = \int_0^t \mathbf{1}_{\{S_u=k,\, I_u=l,\, Y_u=w\}}\,\mathrm{d}u, \qquad t \geq 0,$$

where $0 \leq k \leq n$, $1 \leq l \leq N-k$, and $0 \leq w \leq N-k-l$. As in (4) and (6), we also consider an integral of the form

$$\begin{aligned}
H_t &= \int_0^t h(S_u, I_u, Y_u)\,\mathbf{1}_{\{I_u>0\}}\,\mathrm{d}u \\
&= \sum_{0\leq k\leq n,\, 1\leq l\leq N-k,\, 0\leq w\leq N-k-l} h(k,l,w)B_t(k,l,w), \qquad t \geq 0. \qquad (57)
\end{aligned}$$

Now, instead of (7), we introduce the stochastic process

$$M_t = a_{k,l,w}(S_t, I_t, Y_t;\, \theta)\mathrm{e}^{-\theta B_t(k,l,w)}, \qquad t \geq 0. \qquad (58)$$

Lemmas 1–3 can then be easily generalized as follows.

**Lemma 4.** *The process* $\{M_t,\ t \geq 0\}$ *is a martingale if the* $a_{k,l,w}(s,i,y;\,\theta)$ *satisfy*

$$L[a_{k,l,w}(s,i,y;\,\theta)] = \theta\delta_{s,k}\delta_{i,l}\delta_{y,w}a_{k,l,w}(s,i,y;\,\theta), \qquad s \geq 0,\ i \geq 1,\ y \geq 0, \qquad (59)$$

*where* $L[\cdot]$ *is given by*

$$\begin{aligned}
L[\cdot] = {}& \beta(s,i,y)[a_{k,l,w}(s-1,i+1,y;\,\theta) - a_{k,l,w}(s,i,y;\,\theta)] \\
&+ \mu(s,i,y)[a_{k,l,w}(s,i-1,y;\,\theta) - a_{k,l,w}(s,i,y;\,\theta)] \\
&+ \alpha(s,i,y)[a_{k,l,w}(s,i-1,y+1;\,\theta) - a_{k,l,w}(s,i,y;\,\theta)] \qquad (60)
\end{aligned}$$

*with* $a_{k,l,w}(s,0,y;\,\theta)$ *arbitrary and* $a_{k,l,w}(-1,i,y;\,\theta) = 0$, *say. The* $a_{k,l,w}(s,i,y;\,\theta)$ *are defined uniquely by (59) and (60) once the* $a_{k,l,w}(s,0,y;\,\theta)$ *have been fixed. Equations (10) and (11) can be directly adapted, as well as the recursive method of calculation.*

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Following (13) and (14), we choose

$$a_{k,l,w}(s,i,y;\,0) \equiv a(s,i,y) \quad \text{(independent of } (k,l,w)\text{)}, \qquad s,i,y \geq 0,$$
$$a_{k,l,w}(s,0,y;\,\theta) \equiv a_{k,l,w}(s,0,y;\,0) = a(s,0,y) \quad \text{(independent of } \theta\text{)}, \qquad s,y \geq 0,$$

and as in (15), we set

$$d_{k,l,w}(s,i,y) = \left[\frac{\mathrm{d}}{\mathrm{d}\theta} a_{k,l,w}(s,i,y;\,\theta)\right]_{\theta=0}, \qquad s,i,y \geq 0. \tag{61}$$

**Lemma 5.** *The $a_{k,l,w}(s,i,y;\,\theta)$ are independent of $\theta$ if $s < k$, or $y > w$, or $s+i+y < k+l+w$. In a general way, they can be expressed as*

$$a_{k,l,w}(s,i,y;\,\theta) = a(s,i,y) + d_{k,l,w}(s,i,y)\frac{\theta[\beta(k,l,w)+\mu(k,l,w)+\alpha(k,l,w)]}{\theta+\beta(k,l,w)+\mu(k,l,w)+\alpha(k,l,w)}, \tag{62}$$

*where the $d_{k,l}(s,i)$ satisfy*

$$L[d_{k,l,w}(s,i,y)] = \delta_{s,k}\delta_{i,l}\delta_{y,w}a(s,i,y), \qquad s,y \geq 0,\ i \geq 1,$$

*with $d_{k,l,w}(s,0,y) = 0$ and $d_{k,l,w}(-1,i,y) = 0$, say.*

First, we take all the $a(s,0,y)$ equal to 1. This implies that $a(s,i,y) = 1$ for $s,i,y \geq 0$. As in Lemma 3, the $d_{k,l,w}(s,i,y)$ obtained in this case are denoted by $b_{k,l,w}(s,i,y)$.

**Lemma 6.** *The $b_{k,l,w}(s,i,y)$ satisfy*

$$L[b_{k,l,w}(s,i,y)] = \delta_{s,k}\delta_{i,l}\delta_{y,w}, \qquad s,y \geq 0,\ i \geq 1, \tag{63}$$

*with $b_{k,l,w}(s,0,y) = 0$ and $b_{k,l,w}(-1,i,y) = 0$, say.*

Then we are in a position to generalize, in part, Proposition 3 with, in particular, the distribution of $B_T(k,l,w)$.

**Proposition 6.** *The Laplace transform of $B_T(k,l,w)$ is given by*

$$\mathbb{E}(\mathrm{e}^{-\theta B_T(k,l,w)}) = 1 + b_{k,l,w}(n,m,0)\frac{\theta[\beta(k,l,w)+\mu(k,l,w)+\alpha(k,l,w)]}{\theta+\beta(k,l,w)+\mu(k,l,w)+\alpha(k,l,w)}, \tag{64}$$

*i.e. $B_T(k,l,w)$ has the same distributional representation as (33). For the expectation $\mathbb{E}(H_T)$, (37) is replaced by*

$$\mathbb{E}(H_T) = -\sum_{0\leq k\leq n,\ 1\leq l\leq N-k,\ 0\leq w\leq N-k-l} h(k,l,w)b_{k,l,w}(n,m,0). \tag{65}$$

*Proof.* From the martingale (58), we obtain

$$\mathbb{E}(a_{k,l,w}(S_T,0,Y_T;\,\theta)\mathrm{e}^{-\theta B_T(k,l,w)}) = a_{k,l,w}(n,m,0;\,\theta). \tag{66}$$

As $a(s,0,y) = 1$, this yields

$$\mathbb{E}(\mathrm{e}^{-\theta B_T(k,l,w)}) = a_{k,l,w}(n,m,0;\,\theta).$$

From (62), $a_{k,l,w}(n,m,0;\,\theta)$ is given by the right-hand side of (64); hence, the result (64).

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For $\mathbb{E}(H_T)$, the expansion (57) implies that

$$\mathbb{E}(H_T) = \sum_{0\le k\le n,\, 1\le l\le N-k,\, 0\le w\le N-k-l} h(k,l,w)\mathbb{E}(B_T(k,l,w)).$$

By differentiation of (64) and using (61), we see that

$$\mathbb{E}(B_T(k,l,w)) = -b_{k,l,w}(n,m,0),$$

which leads us to (65). □

For instance, using (65), we can compute the expectations $\mathbb{E}(T)$, $\mathbb{E}(A_T)$, $\mathbb{E}(B_T)$, and $\mathbb{E}(\int_0^T Y_u\,\mathrm{d}u)$ in terms of the $b_{k,l,w}(n,m,0)$.

Second, suppose that for $\theta = 0$, it is possible to determine explicitly a particular set of solutions to (59) and (60), denoted by $\hat{a}(s,i,y)$. In Section 2 this was verified with the $a_{[r]}(s,i)$ given by (20). As for (21), we easily see that the associated $d_{k,l,w}(s,i,y)$ can be expressed by

$$d_{k,l,w}(s,i,y) = \hat{a}(k,l,w)b_{k,l,w}(s,i,y), \qquad s,i,y \ge 0. \tag{67}$$

Using (62), we are then able to obtain the corresponding $a_{k,l,w}(s,i,y;\,\theta)$.

It remains to find such $\hat{a}(s,i,y)$. To do so, we choose any admissible state $(s_0,0,y_0)$, and we consider the solutions to (59) and (60) with $\theta = 0$, denoted by $f_{s_0,0,y_0}(s,i,y)$, that satisfy the conditions

$$f_{s_0,0,y_0}(s,0,y) = \delta_{s,s_0}\delta_{y,y_0}, \qquad s,y \ge 0. \tag{68}$$

Note that any function $g(s,i,y) = \sum_{s_0,y_0} g_{s_0,0,y_0} f_{s_0,0,y_0}(s,i,y)$ with given coefficients $g_{s_0,0,y_0}$ will be a solution to (59) and (60) with $\theta = 0$ that takes the value $g_{s_0,0,y_0}$ when $(s,i,y) = (s_0,0,y_0)$. This allows us to generate all the possible $\hat{a}(s,i,y)$.

**Proposition 7.** *The $f_{s_0,0,y_0}(s,i,y)$ are given by*

$$\begin{aligned} f_{s_0,0,y_0}(s,i,y) = {} & -\mu(s_0,1,y_0)b_{s_0,1,y_0}(s,i,y) - \alpha(s_0,1,y_0-1)b_{s_0,1,y_0-1}(s,i,y) \\ & + \delta_{s,s_0}\delta_{i,0}\delta_{y,y_0}, \qquad s,i,y \ge 0. \end{aligned} \tag{69}$$

*Proof.* We try to construct $f_{s_0,0,y_0}(s,i,y)$ under the form

$$\begin{aligned} f_{s_0,0,y_0}(s,i,y) = {} & c_1 b_{s_0,1,y_0}(s,i,y) + c_2 b_{s_0,1,y_0-1}(s,i,y) \\ & + \delta_{s,s_0}\delta_{i,0}\delta_{y,y_0}, \qquad s,i,y \ge 0, \end{aligned} \tag{70}$$

where $c_1$ and $c_2$ are two constants to be determined. The condition (68) is automatically satisfied since the $b_{k,l,w}(s,0,y)$ are null (see Lemma 6). Thus, it is enough to make sure that $L(f) = 0$ when the right-hand side of (70) is substituted for $f$. From (63), this relation becomes

$$c_1\delta_{s,s_0}\delta_{i,1}\delta_{y,y_0} + c_2\delta_{s,s_0}\delta_{i,1}\delta_{y,y_0-1} + L(\delta_{s,s_0}\delta_{i,0}\delta_{y,y_0}) = 0. \tag{71}$$

Taking $(s,i,y) = (s_0,1,y_0)$ in (71) yields

$$c_1 + 0 + [L(\delta_{s,s_0}\delta_{i,0}\delta_{y,y_0})]_{(s,i,y)=(s_0,1,y_0)} = 0,$$

which leads to $c_1 = -\mu(s_0,1,y_0)$. Similarly, (71) for $(s,i,y) = (s_0,1,y_0-1)$ yields

$$0 + c_2 + [L(\delta_{s,s_0}\delta_{i,0}\delta_{y,y_0})]_{(s,i,y)=(s_0,1,y_0-1)} = 0,$$

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so that $c_2 = -\alpha(s_0, 1, y_0 - 1)$. Finally, it is easily checked that

$$[L(\delta_{s,s_0}\delta_{i,0}\delta_{y,y_0})]_{(s,i,y)\neq(s_0,1,y_0),(s_0,1,y_0-1)} = 0.$$

Thus, with these values of $c_1$ and $c_2$, (71) holds as desired, and (70) yields the announced (69), completing the proof. □

Thanks to this result, we can obtain, for instance, the generating function of $(S_T, Y_T)$ jointly with the integral $B_T(k, l, w)$.

**Proposition 8.** *We have*

$$\begin{aligned}\mathbb{E}(x^{S_T} z^{Y_T} B_T(k, l, w)) &= b_{k,l,w}(n, m, 0) \\ &\quad \times \sum_{0\le s_0\le n,\, 0\le y_0\le N-s_0} x^{s_0} z^{y_0} [\mu(s_0, 1, y_0) b_{s_0,1,y_0}(k, l, w) \\ &\qquad + \alpha(s_0, 1, y_0 - 1) b_{s_0,1,y_0-1}(k, l, w)], \qquad x, z \text{ real.} \quad (72)\end{aligned}$$

*Proof.* We start from (66) which also holds here. By differentiation and setting $\theta = 0$, we obtain

$$\mathbb{E}(\hat{a}(S_T, 0, Y_T) B_T(k, l, w)) = -d_{k,l,w}(n, m, 0). \quad (73)$$

As indicated before, we may choose, e.g.

$$\hat{a}(s, i, y) = \sum_{0\le s_0\le n,\, 0\le y_0\le N-s_0} x^{s_0} z^{y_0} f_{s_0,0,y_0}(s, i, y). \quad (74)$$

Note that, for $i = 0$, (74) reduces to

$$\hat{a}(s, 0, y) = \sum_{0\le s_0\le n,\, 0\le y_0\le N-s_0} x^{s_0} z^{y_0} f_{s_0,0,y_0}(s, 0, y) = x^s z^y, \quad (75)$$

by virtue of (68). Inserting (75) into (73) thus yields

$$\mathbb{E}(x^{S_T} z^{Y_T} B_T(k, l, w)) = -d_{k,l,w}(n, m, 0). \quad (76)$$

Moreover, from (67), we have

$$d_{k,l,w}(n, m, 0) = b_{k,l,w}(n, m, 0) \sum_{0\le s_0\le n,\, 0\le y_0\le N-s_0} x^{s_0} z^{y_0} f_{s_0,0,y_0}(k, l, w). \quad (77)$$

Recall that $l \ge 1$ by construction. Thus, (69) for $f_{s_0,0,y_0}(k, l, w)$ yields

$$f_{s_0,0,y_0}(k, l, w) = -\mu(s_0, 1, y_0) b_{s_0,1,y_0}(k, l, w) - \alpha(s_0, 1, y_0 - 1) b_{s_0,1,y_0-1}(k, l, w). \quad (78)$$

Combining (76)–(78), we then deduce (72). □

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