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AN "*i*" FOR AN *i*: SINGULAR TERMS, UNIQUENESS, AND REFERENCE

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Abstract. There is an interesting logical/semantic issue with some mathematical languages and theories. In the language of (pure) complex analysis, the two square roots of -1 are indiscernible: anything true of one of them is true of the other. So how does the singular term '*i*' manage to pick out a unique object? This is perhaps the most prominent example of the phenomenon, but there are some others. The issue is related to matters concerning the use of definite descriptions and singular pronouns, such as donkey anaphora and the problem of indistinguishable participants. Taking a cue from some work in linguistics and the philosophy of language, I suggest that *i* functions like a parameter in natural deduction systems. This may require some rethinking of the role of singular terms, at least in mathematical languages.

§1. Speaking of "*i*": reference to indiscernible objects. Some critics of my ante rem structuralism (*Philosophy of Mathematics: Structure and Ontology* (1997)) took me to task over structures that have indiscernible places. This initiated a small, but lively discussion on the topic of indiscernibility.[1] A structure is said to be *rigid* if its only automorphism is the trivial one based on the identity mapping. It is an easy theorem that isomorphic structures are equivalent: Let f be an automorphism on a given structure M and let $\Phi(x_1, \ldots, x_n)$ be any formula in the language of the structure. Then for any objects $a_1, \ldots, a_n$ in the domain of M, M satisfies $\Phi(a_1, \ldots, a_n)$ if and only if M satisfies $\Phi(fa_1, \ldots, fa_n)$. If f is nontrivial, then there is an object a such that $fa{=}a$. In this case, a and fa will be indiscernible, at least concerning the language of the structure. Anything that can be said about a, using a formula in this language (possibly with parameters) will be true of fa.

One common example is complex analysis. Start with the language of fields, and consider the algebraic closure of the reals, which is unique up to isomorphism (in its second-order formulation). The complex numbers are the intended model. The function that takes a complex number $a{+}bi$ to its conjugate $a{-}bi$ is an automorphism. It follows that for any formula $\Phi(x)$, with x free, $\Phi(a{+}bi)$ if and only if $\Phi(a{-}bi)$. In particular, $\Phi(i)$ if and only if $\Phi(-i)$. It follows that i and $-i$ are indiscernible; they have the same properties, at least among those that can be expressed in the language. Another oft-cited example is Euclidean space, where things are even worse. Any two points in Euclidean space can be connected with a rigid translation, which is an automorphism. So, it seems, *all* of the points

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[1] The early critics include Burgess (1999, pp. 287–288), Hellman (2001, pp. 192–196), and Keränen (2001, 2006). More recent participants, on both sides, include Ladyman (2005), Button (2006), Ketland (2006), MacBride (2005, sec. 3, 2006b), and Leitgeb & Ladyman (2008). My own contributions are Shapiro (2006a, 2006b, 2008). Section 3 of the latter includes a brief treatment of some of the topics broached in the present paper.

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in Euclidean space are indiscernible, at least with respect to properties that can be expressed in the language of geometry. Leitgeb & Ladyman (2008) point out that since some (simple) graphs have no relations, *any* bijection on them is an isomorphism. So with those graphs, every point is indiscernible from every other. The simplest of these simple graphs—those that are finite and have no edges—are isomorphic to the finite cardinal structures introduced in my chapter on epistemology in Shapiro (1997, chap. 4).

Some ill-chosen remarks in my book at least suggest a principle of the identity of indiscernibles, which, in light of these examples, would indeed be trouble. There are two distinct complex square roots of -1; my view had better not entail that there is just one. There are uncountably many points in Euclidean space; my view had better not entail that there is just one. One might think that my adoption of the finite cardinal structures would countersuggest that, in the end, I don't accept the identity of indiscernibles, but I do not propose to engage in self-exegesis here.

Much of the discussion of this issue is metaphysical. Burgess (1999, p. 288) points out that although the two complex roots of -1 are distinct, on my view "there seems to be *nothing* to distinguish them." This seems to invoke something in the neighborhood of the Principle of Sufficient Reason. If something is so, then there must be something that makes it so, or at least something that explains why it is so. Keränen (2001) articulates a general metaphysical thesis that anyone who puts forward a theory of a type of object must provide an account of how those objects are to be "individuated." According to Keränen, for each object a in the purview of a proposed theory, we have to be told "the fact of the matter that makes a the object it is, distinct from any other object" of the theory, by "providing a *unique* characterization thereof."

Some authors entered the discussion, on my behalf, by suggesting metaphysical principles that are weaker than Keränen's individuation requirement but still meet Burgess's request that the theorist find something that distinguishes distinct objects. The weakest of these is a requirement that for any a, b, if $a \neq b$ then there is an irreflexive relation R such that Rab (Ladyman, 2005). Complex analysis and Euclidean geometry easily pass this test, although the finite cardinal structures and some simple graphs still fail it, unless nonidentity counts as an irreflexive relation (in which case the proposed criterion is not helpful, and begs any question there may be here). Ketland (2006) and Leitgeb & Ladyman (2008) seem to agree with me that there is no individuation or distinguishing requirement at all, or at least none that requires the structuralist to provide anything beyond an ordinary mathematical axiomatization of common structures.

I wish to put aside these metaphysical matters here, at least as far as possible. I ask the reader to assume that the ante rem structuralist has won this metaphysical war. There are some related and, I think, more interesting issues concerning the semantics and pragmatics of mathematical languages, and perhaps languages generally. These issues also bear on logic, and they go well beyond local disputes concerning ante rem structuralism. How do we manage to talk about, and thus, in some sense, refer to indiscernible objects?

The informal language of complex analysis has a term "*i*" which is supposed to denote one of the square roots of -1. At least grammatically, "*i*" is a constant, a proper name. And, of course, the role of a constant is to denote a single object—at least in a sufficiently regimented language. But which of the square roots does "*i*" pick out? Is it not as if the mathematical community has managed to single out one of the roots, in order to baptize it with the name "*i*." It seems that they cannot do so, as the two roots are indiscernbile.

Some time ago, I found myself visiting a friend, a computer engineer. He was explaining the basics of complex analysis to his young son. My friend mentioned various properties

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of the complex unit which, of course, he called "*i*." I interrupted, and asked him, "What is '*i*'?" He answered, with the intonation of a question, "The square root of minus 1?" I then asked him, "Which square root?," reminding him that there are two such. He thought about this for a few moments, and responded, "that's cute," and went back to talking to his son.

> Frege (1884) seems to have noted our problem:
>
> > We speak of 'the number 1', where the definite article serves to class it as an object (§57).
> > If, however, we wished to use [a] concept for defining an object falling under it [by a definite description], it would, of course, be necessary first to show two distinct things:
> >
> > > that some object falls under the concept;
> > > that only one object falls under it (§74n).
> >
> > Nothing prevents us from using the concept 'square root of −1'; but we are not entitled to put the definite article in front of it without more ado and take the expression 'the square root of −1' as having a sense. (§97)

Presumably, the "more ado" here is to prove existence and uniqueness.

Complex analysis is the most salient example of the logic-semantic phenomenon in question, but there are some others, at least if we go with a straightforward reading of various mathematical languages (see Brandom, 1996) . Consider, for example, the integers, but with addition being the only operation. It is, of course, an abelian group, whose elements are:

$$\ldots -3, -2, -1, 0, 1, 2, 3, \ldots$$

In the relevant language, the operation that takes any integer a to its inverse $-a$ is an automorphism. So anything in the relevant language that holds of an element a also holds of $-a$. In this structure, then, 1 is indiscernible from -1, but, of course, 1 is distinct from -1. Another example is the Klein group. It has four elements, which are usually called e, a, b, and c, and there is one operation, given by the following table:

e	a	b	c
a	e	c	b
b	c	e	a
c	b	a	e

It is easy to verify that for any permutation f on this domain, if $fe = e$, then f is an automorphism. The three nonidentity elements are thus indiscernible, in the language of groups, and yet there are three such elements and not just one. So which one is a?

In a given category, an object O is called *terminal* if for each object A, there is exactly one arrow from A to O. If a category has a terminal object, it is common to introduce a singular term 1 to denote "the" terminal object of that category, and to define an *element* of an object A to be an arrow from 1 to A. But in typical categories, including for example, the much studied set theory ETCS, if there is one terminal object, there are many. So which one of them is 1? And which is the object that determines what the elements of each object are, by serving as domain of the relevant arrows?

Actually, *all* of the terminology in category theory has the feature in question. Indeed, that is often touted as one of the features of the enterprise. For example, a product of two

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objects A, B is an object O together with an arrow from O to A and an arrow from O to B, that has a certain universal property. The custom is to let "$A \times B$" stand for "the" product of A and B. Grammatically, at least, "$\times$" is a function symbol, so that "$A \times B$" is a (structured) singular term. But in typical categories (including ETCS), if there is one product, there are many. So for a given pair A, B, which object is $A \times B$?

To foreshadow where we will be going, there is a related phenomenon concerning the use parameters or free variables, which act like singular terms in context. Suppose that in the course of a demonstration, a geometer says "let *ABCD* be any parallelogram, with the line *AB* congruent and parallel to the line *CD*." It follows that the pair of points A,B, (and the line segment *AB*) are indiscernible from the pair C,D (and the segment *CD*). Anything one can say about one of the pairs (and one of the segments) will be true of the other pair (and the other segment). So which one is *AB* and which one is *CD*?

The talk of automorphism highlights the fact that the indiscernibility is relative to whatever expressive resources are available. To return to the standard example, suppose that we just add a constant i to the official language of complex analysis, with the obvious axiom $i^2 = -1$. Then the structure becomes rigid: there are no nontrivial automorphisms. The reason is that isomorphisms must preserve all of the structure in the language, and, in particular, it must preserve the denotations of the constants. If f is an isomorphism between M_1 and M_2, in the language of arithmetic, for example, then if "0" denotes a in M_1, then "0" denotes fa in M_2. Similarly, let N be any model of complex analysis, in the envisioned language, and let f be an automorphism. If "i" denotes a in N, then "i" denotes fa in N. That is, $a = fa$.

It seems to me that, in the relevant sense, the two square roots of -1 are still indiscernible. Let N' be a model of the theory that is just like N, except that in N, "i" denotes $-a$ (and thus "$-i$" denotes a). Technically, N is not isomorphic to N', for the above reason. However, it seems to me that the two models are equivalent, in an intuitive sense. Both have the same domain and agree on the operations. In particular, in each model, the same two objects are the roots of -1. The only difference between them is that *N calls* one of them "i": and N' *calls* the other one "i". And, at least intuitively, that is not a significant difference.

To develop this point a bit, let us go up a level, so to speak, and think of the semantic relations themselves in formal, or structural terms. Begin with a simple graph that has two nodes and no edges. It is completely homogeneous, as noted above. Now add two new objects, a, b, and a relation R to the structure. The new item a bears R to one of the nodes in the original graph structure and the other new item b bears R to the other node. This is the structure of some very simple semantic relations on the simple graph: think of a and b as names, and R as the reference relation. This mathematical-cum-semantic structure is not rigid. If we modify it by switching the "referents" of a and b, we get an automorphism. And, intuitively, we have not really changed the structure with this switch. It is still the same simple graph with the same two new objects, the same relation R, and the same structural semantic relations.

We can do the same with our more standard mathematical example. Consider a structure M that includes the places and relations of our model N, of complex analysis. In addition, M has items representing the primitive *terms* of the language of complex analysis ("0," "i," "+," "$\cdot$"), and a relation R representing reference. So, for example, Rix would be an atomic formula in the envisioned object language, saying that 'i' refers to, or denotes, x. The theory would include the axioms of complex analysis (restricted to the domain of N) and the Tarskian satisfaction clauses between the "terms" and the relevant items constructed from N. So, for example, our structure would satisfy $\forall x \forall y((Rix \& Riy) \rightarrow x = y)$ and Ria (recalling that a is one of the square roots of -1 in N).

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This mathematical-cum-semantic structure is not rigid. The function that takes $x+ay$ to $x-ay$ (within N), and takes each term $(0,1,i,+,\cdot)$, to itself, and adjusts the relation R accordingly, is an automorphism. We still have a model of complex analysis, as above, and all of the Tarskian satisfaction clauses are still satisfied (see Leitgeb, 2007, pp. 133–134). The problem, here, is to say something about the semantics and logic of the languages of mathematics, so construed.

The issues here are related to those in Black's (1952) celebrated discussion of the identity of indiscernibles. The paper is in the form of a dialogue. One character, A, takes the identity of indiscernibles to be "obviously true," while his opponent, B, takes it to be "obviously false." The latter gives a thought experiment meant to refute the principle:

> Isn't it logically possible that the universe should have contained nothing but two exactly similar spheres? We might suppose that each was made of chemically pure iron, had a diameter of one mile, that they had the same temperature, color, and so on, and that nothing else existed. Then every quality and relational characteristic of the one would also be a property of the other. (p. 156)

Black's two spheres are analogous to the two square roots of -1. Of course, no one is claiming that there is a possible world which consists of just these two complex numbers. But the rest of the analogy holds, fairly well.

The main thrust of Black (1952) is metaphysical. Along the way, however, the characters raise the sort of logico-semantic issues of concern here. The defender of the indiscernibility of identicals, A, first denies that B has described a coherent possibility, and then continues, "But supposing that you *have* described a possible world, I still don't see that you have refuted the principle. Consider one of the spheres, a." At this point, B interrupts, protesting: "How can I, since there is no way of telling them apart? Which one do you want me to consider?" That is, B refuses to let his interlocutor use a variable, parameter, or singular term for one of the spheres. Character A responds: "This is very foolish. I mean either of the two spheres, leaving you to decide which one you wished to consider." In our case, it strikes me as eminently reasonable to say, "let i designate one of the square roots of -1. I don't care which." The present problem is to make sense of this locution.

> Brandom (1996, p. 298) puts our issues in similar terms:
>
> Now if we ask a mathematician '*Which* square root of -1 is i?', she will say 'It doesn't matter: pick one'. And from a *mathematical* point of view this is exactly right. But from the *semantic* point of view, we have the right to ask how this trick is done—how is it that I *can* 'pick one' if I can't tell them apart? What must I do in order to be *picking* one, and picking *one*? For we really *cannot* tell them apart—and . . . not just because of some lamentable incapacity of ours.

The next exchange in Black's dialogue puts some detail to the differing presuppositions. Character A, the proponent of the (obvious truth of) the identity of indiscernibles, continues, "If I were to say to you 'Take any book off the shelf" it would be foolish on your part to reply 'Which?"' B retorts: "It's a poor analogy. I know how to take a book off a shelf, but I don't know how to identify one of the two spheres supposed to be alone in space . . ." It seems that, for the purposes of this argument, B claims that one cannot use a singular term to designate an object without first "identifying" it, or at least knowing how to identify one

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of the objects in question. That, I take it, is the matter at hand here. The character *A* takes the bait: "Can't you imagine that one sphere has been designated as '*a*'?" The dialogue continues:

> *B*. I can imagine only what is logically possible. Now it is logically possible that somebody should enter the universe I have described, see one of the spheres on his left hand and proceed to call it '*a*' . . .
>
> *A*. Very well, now let me try to finish what I began to say about *a*. . . [ellipsis in original]
>
> *B*. I still can't let you, because you, in your present situation, have no right to talk about *a*. All I have conceded is that if something were to happen to introduce a change in my universe, so that an observer entered and could see the two spheres, one of them could then have a name. But this would be a different supposition from the one I wanted to consider. My spheres don't yet have names . . . You might just as well ask me to consider the first daisy in my lawn that would be picked by a child, if a child were to come along and do the picking. This doesn't now distinguish any daisy from the others. You are just pretending to use a name.
>
> *A*. And I think you are just pretending not to understand me.

An exchange or two later, *B* broaches matters mathematical, as an aside: "You remind me of the mathematicians who thought that talking about an Axiom of Choice would really allow them to choose a member of a collection when they had no criterion of choice." I do not think that the mathematicians in question, Zermelo and Cantor, say, were as confused as Black's character *B* seems to think they were, although the axiom of choice is perhaps ill named. *B* is certainly correct that there is no *choosing* going on, despite the name of the axiom, and the informal language sometimes used in mathematics. What *is* going on is on our agenda here.

§2. What? me worry? It is worth a brief look at how the present issues play themselves out for various philosophical conceptions of (various branches of) mathematics. I do not claim that every philosophical view has an issue concerning reference and indiscernibility, but some prominent ones do.

At places, the characters in Black (1952) each adopt verificationist or positivist principles. For example, *A* argues that if there is no way to verify that an object *a* is distinct from an object *b*, then $a = b$. There is just one object there, and not two.[2] The opponent, *B*, will not allow the use of a singular term, or a definite description, until its referent is verified—or at least not until one has shown how to pick out a referent. I do not know what the mathematical analogue of these verificationist principles would be. Perhaps one might claim that if the language of a mathematical theory has two singular terms *a*, *b*, and it is

[2] Something like this is not uncommon, even in post-positivist philosophy. See, for example, Quine (1986, chap. 5) and Kraut (1980). One burden of Shapiro (2008) is to show that such a principle has no place in mathematics.

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not possible to prove that $a \neq b$, then $a = b$. I see no reason to adopt such a principle, and do not see it as conforming to mathematical practice, so we can set that aside.[3]

There is an important distinction in how various (formalized) mathematical theories are construed. Say that a given theory is *algebraic* if it is intended to be about any structure that satisfies its axioms. To take some simple examples, a group is whatever satisfies the axioms of group theory; and a ring is whatever satisfies the axioms of ring theory. There is no such thing at "the" group or "the" ring. Much of category theory is to be understood in this algebraic perspective. Indeed, a category itself is defined to be any system that satisfies the Eilenberg–MacLane axioms.

In contrast, to follow a suggestion of Hellman (2003), say that a theory is *assertory* if it is intended to be about a fixed subject matter, a fixed structure or system of mathematical objects. Theorems of assertory theories are meant to express propositions with fixed truth values. Theorems of algebraic theories are schematic, applying to any system of objects that meets certain given conditions.

The much-discussed Frege–Hilbert exchange, concerning Hilbert's celebrated Hilbert's (1899) *Grundlagen der Geometrie*, turns on this distinction.[4] In present terms, Frege insisted that geometry is assertory—it is about *space*. At the time, at least, Hilbert's perspective was that geometry, and perhaps all of mathematics, is algebraic:

> . . . it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points, I think of some system of things, e.g., the system love, law, chimney-sweep . . . and then assume all my axioms as relations between these things, then my propositions, e.g., Pythagoras' theorem, are also valid for these things . . . [A]ny theory can always be applied to infinitely many systems of basic elements. One only needs to apply a reversible one-one transformation and lay it down that the axioms shall be correspondingly the same for the transformed things . . . [This] . . . can never be a defect in a theory, and it is in any case unavoidable.

See Shapiro (2005).

When it comes to algebraic theories, there is no problem concerning reference to indiscernible objects. To put it roughly, the reason is that in such cases, there is no reference. It is perhaps natural to think of the aforementioned Klein group in algebraic terms. From that perspective, there is no such thing as *the* Klein group (despite the fact that "the Klein group" is a common locution). Rather, there are a lot of Klein groups. So there is no need to worry about what the "referents" of "a," "b," and "c" (and "e" for that matter) are. Those are just schematic letters that are to represent the elements of a given Klein group.

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[3] An intuitionist, following the Brouwer–Holmogorov–Heyting reading of the logical connectives, would hold that if it is impossible to prove that $a \neq b$, then $\neg(a \neq b)$, which is $\neg\neg(a = b)$. But this is not the identity of indiscernibles. At most, it is a principle of the nondistinctness of indiscernibles. Moreover, the antecedent of the conditional is rather strong. To conclude $\neg\neg(a = b)$, it does not suffice to show that any formula of some given language that holds of a also holds of b. For the intuitionist to conclude that $\neg\neg(a = b)$, she must reduce to absurdity the assumption that a cannot be distinguished from b.

[4] The correspondence between Frege and Hilbert is published in Frege (1976) and translated in Frege (1980).

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One might go on to claim that in any given Klein group, the four objects can be properly individuated.

One can think of complex analysis in algebraic terms. Accordingly, there is no such thing as *the* complex numbers. The theory is about any and all algebraic closures of the real numbers. Suppose that the language of this algebraic theory contains a constant "*i*." This is just to insist that in any algebraic closure we consider, one of the square roots of -1 is to be distinguished.[5] Similarly, in any category with a terminal object, one such terminal objects is to be distinguished as the referent of "1." This will settle what the elements of objects in that category are.

There are some perspectives that avoid the present problem altogether, and, as a side benefit, they allow the philosopher to maintain the identity of indiscernibles, if she is so inclined. One such perspective is to hold, apparently following Hilbert, that all of mathematics is algebraic.[6] So, in effect, there is no reference to worry about. A less radical view would be to hold that any legitimate *assertory* theory has to be about an rigid structure. This philosophy allows that there can be an assertory theory of the natural numbers (under both addition and multiplication), the real numbers (under addition and multiplication), and the iterative hierarchy (under membership), but there can be no assertory theory of the complex numbers, nor of Euclidean space. An even less radical view would be to hold that no legitimate assertory theory can have a constant that purports to refer to one of several indiscernible objects. This would be just to deny the complex analyst the use of "*i*."

One view in this neighborhood would hold that there is really only one assertory theory, say Zermelo–Fraenkel set theory, whose subject matter is the iterative hierarchy (which, as it happens, is rigid). This domain supplies ontology for all of the rest of mathematics, construed as algebraic. So on this view, real analysis, for example, is the study of all collections of sets that have relations on them constituting a Dedekind-continuous, real closed field; complex analysis is the study of all algebraically closed fields over one of those, etc. It is not too much of a distortion to read Benacerraf (1965) as suggesting something along these lines; see also Linnebo (2008).

Of course, there are other ways to maintain the identity of indiscernibles, if one is so inclined. With a little bookkeeping, that the mathematician is free to ignore, the symmetries can be broken by thinking of each putative nonrigid structure as embedded in a rigid one. For example, one can think of the integers under addition alone as a substructure of the integers under addition and multiplication. There is no problem of distinguishing 1 and -1 in the richer background. To study the integers under addition alone, we just ignore the extra structure. Similarly, one can think of Euclidean space as the study of pairs or triples of real numbers, that is, $\mathbb{R}^2$ or $\mathbb{R}^3$. To do synthetic geometry, we would just focus on what is invariant under different coordinate systems. Concerning the main example, one can think of complex numbers as pairs of reals, following a now common mathematical technique. In that case, i is the pair $<0,1>$ and $-i$ is $<0,-1>$. Those are easily distinguished from each other,[7] in the language of $\mathbb{R}^2$.

[5] A number of prominent logicians have pressed this interpretation on me.

[6] See Awodey (2004). Shapiro (2005) argues that it is difficult to think of metamathematics in algebraic terms.

[7] I asked a few mathematicians about the referent of '*i*'. Some recognized the issue under study here, but others suggested that there is no serious problem, since they think of complex numbers as pairs of real numbers. There is an interesting complication, however. Nowadays, it is more

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I suspect that there would not be a problem for Frege concerning our issue.[8] As noted above, he demanded *two* things before one could use a definite article in a properly rigorous mathematical treatment. One is to show "that some object falls under" the concept in question, and the other is "that only one object falls under it" (Frege, 1884, §74n). It would not do, for Frege, to simply declare that the complex numbers are the algebraic closure of the reals, or even to say that we are interested in *an* algebraic closure of the reals. In doing this, we would fail the *first* requirement, of showing that some object falls under the concept "square root of -1." How do we know that there are any algebraic closures of the real numbers? And how do we specify a unique one? Presumably, Frege would have given an explicit definition of the complex numbers, perhaps as pairs of reals (which, in turn, would be defined in terms of certain courses-of-values). This explicit definition would break the symmetry between the two square roots of -1.

For what it is worth, my ante rem structuralism proposes to combine the algebraic and assertory approaches to at least some theories, including arithmetic, real analysis, complex analysis, various geometries, and set theory. I note different orientations toward mathematical theories. From one perspective, which I dubbed "places-are-offices," they are algebraic, applying to whatever systems satisfy them. However, if the axioms of a branch of mathematics are satisfiable and categorical, then they characterize a single structure, and the axioms are true *of it*. I call this the "places-are-objects" perspective. The idea is that places in a structure are bona fide objects, and we can have quantifiers ranging over them. The structure itself is a chunk of reality, and the theory is assertory about that structure. So the same axioms are algebraic from one perspective, and assertory from another. Since there are satisfiable and categorical, but nonrigid structures, the present issue comes up.

In fact, the issue arises for any philosophy that holds that there are assertory mathematical theories of nonrigid structures, especially if some such theory has a singular term that (apparently) denotes one of a batch of indiscernible objects. It does not matter, at least at first, what the metaphysical nature of mathematical objects may be. Our philosopher may be a traditional platonist, or she may hold that mathematical objects are somehow mental constructions, or social constructs,[9] or perhaps she is a quietist about mathematical ontology, or whatever. All that matters, for now, is that the languages be understood literally, as assertory, and that some numerically distinct objects are indistinguishable.

Without much loss of generality, we might as well keep on with our standard example: our philosopher holds that the complex numbers exist and that the square roots of -1 are indiscernible. So our problem arises. He must either come up somehow with a referent for 'i', which would be to break the symmetry, or else he must describe the logico-semantic role of that term.

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common to use polar coordinates. So i is the pair $<1,\pi/2>$ and $-i$ is $<1,3\pi/2>$. So now it is a matter of distinguishing $\pi/2$ from $3\pi/2$ *as angles*. That, in turn, depends on distinguishing "clockwise" from "counterclockwise," or "up" from "down," or "positive" from "negative" (among angles). But all of those are symmetric. So, it seems, the move to polar coordinates reintroduces the symmetry we tried to break.

[8] One must be speculative here, since Frege only gave the barest hints at how *real* analysis was to fit into his logicist program (see, e.g., Simons, 1987). Thanks to Michael Dummett for some key insights along the foregoing lines.

[9] If a philosopher thinks of mathematical objects as "constructed" somehow one at a time, then the symmetries can be broken. For example, one can let "i" denote the square root of -1 that was constructed first.

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Suppose, finally, that our philosopher is a nominalist, who denies the existence of mathematical objects altogether. For this philosopher, mathematics has no distinct ontology. Then the devil is in the details of the view. I do not see an issue for a fictionalist. One can surely tell a coherent *story* about objects that are indiscernible as far as the details provided by the story go. Consider, for example, the following short story: "One day, two people met, fell in love, and lived together, happily ever after." Whatever its literary merits, this is surely a coherent piece of fiction. It is reminiscent of Black's story about the two iron spheres. In both cases, there is nothing in the story to distinguish the characters. Anything in the language of the story that holds of one also holds of the other. And, of course, we have nothing to go on besides the details of the story, plus common knowledge of human psychology, naming conventions, and the like. Consider this variation on our story: "One day, Chris met Kelly. They fell in love, and lived together, happily ever after." One might claim that, here, Chris is distinguished from Kelly because he or she is the one who is *named* "Chris" in that story. But, as with complex analysis, this does not seem like a distinction that matters. To be sure, there are interesting issues concerning the semantics and, perhaps, the ontology and metaphysics of fiction, but I do not propose to enter that realm here.

Reconstructive nominalists provide translations of mathematical languages into vocabulary that does not commit the mathematician to the existence of mathematical objects. Typically, singular terms and bound variables in mathematical languages are rendered as bound variables in the scope of modal operators. Instead of speaking of what exists, the reconstructive nominalist speaks of what might exist (as in Hellman, 1989), what can be constructed (ala Chihara, 1990), or what follows from axioms, or whatever. Whether an issue analogous to the present one arises depends on the details of the translation, and I propose to avoid that as well.[10]

In the sequel, then, I continue to assume that the languages of mathematics are to be understood at, or at least near, face value, and that there are at least prima facie indiscernible objects in some of the theories. What is the role and function of singular terms (or linguistic items that look and function like singular terms), and how do such terms get introduced into conversations, and into the language generally? At this point, I propose an excursion into matters linguistic.

§3. Definites in natural language. The situation with i, certain categories, and perhaps the elements of the Klein group (if we take the theory to be assertory), seem to be unique to mathematics. I know of no nonmathematical (prima facie) *proper names*, for which one cannot somehow specify the referent, at least in principle—once ambiguities and contextually sensitive matters are resolved. But natural languages have other devices for referring to individual objects: definite descriptions, singular demonstratives ("this," "that"), and singular pronouns ("he," "she," "it"), at least in some of their uses. A good theory of those may shed some light on our issues. On the other hand, theories in linguistics are fraught with controversy, and a bad theory of definites just might serve to further confuse our issues. But let us at least try.

[10] Chihara's (1990) modal constructivism is an interesting case here. Accordingly, a singular term, such as a numeral, represents the possibility of constructing an open sentence with certain semantic features. So one can wonder which open sentence would correspond to "i," as opposed to the open sentence that corresponds with "$-i$." I presume that Chihara would liken complex numbers to pairs of reals, as above.

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Russell's (1905) "On denoting" provides a once prominent analysis of definite descriptions. Let P and Q be predicates. According to Russell, the proper reading of a sentence in the form "The P is Q" is:

There is one and only one x such that x is P, and x is Q.

In symbols:

$$\exists x\,(\forall y\,(Py \equiv x = y)\,\&\,Qx)\,.$$

So it would seem that, for Russell, the uniqueness of definite descriptions is registered at the level of logic form: it *follows* from "The P is Q" that something is P and that at most one thing is P.

Almost half a century later, Strawson (1950) launched a sustained critique of Russell's account of definite descriptions. One of the chief complaints is that the Russell account is not true to how definite descriptions function in ordinary discourse. Suppose, for example, that two people, Fred and Wilma, each believe that their town has a mayor when, in fact, it doesn't (or suppose, instead, that it has two co-mayors). Wilma says that the mayor of their town is wise, and Fred disputes this, since the town is so badly run. He says, emphatically, "No, it is not true that the mayor is wise. He is an idiot." According to Russell's analysis, Fred has spoken a truth with his first utterance (although it is not so clear what to make of his second sentence, with the pronoun). But upon learning that his town has no mayor at all (or that it has two), Fred certainly would not feel vindicated.

In more contemporary terms, Strawson argues that existence and uniqueness are *presuppositions* of the felicitous use of definite descriptions, but existence and uniqueness do not *follow* from a sentence that contains a definite description. Someone who says "The P is Q" does not thereby *say* that there is one and only one P. Geach (1950) draws a similar conclusion.

In his polemical reply to Strawson, Russell (1957) reveals his orientation to the enterprise at hand. He writes that there is

> a fundamental divergence between myself and many philosophers with whom Mr. Strawson appears to be in general agreement. They are persuaded that common speech is good enough, not only for daily life, but also for philosophy. I, on the contrary, am persuaded that common speech is full of vagueness and inaccuracy, and that any attempt to be precise and accurate requires modification of common speech both as regards vocabulary and as regards syntax . . . For technical purposes, technical languages differing from those of daily life are indispensable . . . In philosophy, it is syntax, even more than vocabulary that needs to be corrected . . . My theory of descriptions was never intended as an analysis of the state of mind of those who utter sentences containing descriptions . . . I was concerned to find a more accurate and analysed thought to replace the somewhat confused thoughts which most people at most times have in their heads. (Russell, 1957, pp. 387–388)

The "data" that support Strawson's (and Geach's) arguments are the untutored reactions of native speakers (perhaps when they are being careful). For Russell's revisionist project, these considerations are (at least largely) irrelevant.

To be sure, we have come some distance from the practice of and debate over Oxford-style ordinary language philosophy. I suggest, however, that a linguistic analysis of definites in ordinary language can shed some light on our problem. The languages of mathematics

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are, or at least seem to be, *languages*. In the informal language of complex analysis, "*i*" at least appears to be a singular term, and the locution "the square root of −1" at least appears to be a definite description. We might learn something by looking at ways similar matters are handled in ordinary language.

a. Some data. When it comes to definite descriptions, and other definites, it is manifest that existence and uniqueness must be registered *somehow*. The question is how, and where. Much of the following is culled from Roberts (2003).[11]

Let us start with a straightforward example:

(1) The President of the United States is eager to cooperate with his political opponents.

A Russell-style analysis would fit here as well as anywhere. It is common knowledge that there is one and only one President at a given time (or at least at almost all given times). So there is little cost to saying that the existence and uniqueness conditions are entailed by this sentence. In many cases, however, such common knowledge fails us, and the Russell analysis is more strained:

(2) I found a box in my attic the other day. I opened the lid and pushed the button I found inside. You won't believe what happened.

Presumably, it is common knowledge that many or perhaps typical boxes have lids, and most of those have only one lid. But it is not common knowledge that boxes have buttons inside, let alone unique buttons. This example illustrates one of Strawson's points. The speaker is not *saying* that the box had one (and only one) button inside. Given the context, a listener would realize that the speaker found one and only button inside. Existence and uniqueness are *presuppositions* of the utterance, not consequences of it.

The next example is similar:

(3) Teacher, giving directions: On the next page, you will find a puzzle. Find the clown in the puzzle.

Here, too, the sentence presupposes that there is one and only one puzzle on the next page, and also that there is one and only one (picture or drawing of a) clown in the puzzle. The children would realize this, and act accordingly.

(4) Every room has a copy of the bible in it. In this room, it was hidden behind the TV.

The first sentence here does not entail, or even implicate, that there is only one bible in each room. Yet the use of the singular pronoun "it" in the second sentence is felicitous. If the context indicates that it is a hotel, then perhaps it is common knowledge that there is usually no more than one bible in each room. Or perhaps it is a presupposition that the room in question has only one, or else the speaker had a particular bible in mind.

(5) A wine glass broke last night. It/The glass was very expensive.

The first sentence does not entail, or implicate, that one wine glass broke. The second sentence is not only felicitous, but can even be true even if more than one glass broke, and perhaps even if more than one *expensive* glass broke. An ordinary listener would assume that the speaker had a particular glass in mind.

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[11] I am thoroughly indebted to Roberts for much fruitful conversation, and for her articles on this and related topics. Of course, any errors and misrepresentations that follow are my own.

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The next example is also clearly felicitous:

(6) Every car in this lot has a statue on the dashboard.

It is common knowledge that each car has one and only one dashboard (or at least typical cars do). But what is the referent of the term "the dashboard" in this sentence? There are, presumably, lots of dashboards in the lot, one for each car. It seems that this sentence is elliptical for something like the following more awkward:

For each car in this lot, there is a statue on the dashboard of that car.

In a straightforward formalization, the definite description here applies to a predicate with a variable that is bound from the outside: "dashboard of x." The definite description is thus unsaturated, in Frege's sense, and denotes or represents a *function*. Compare (6) to the following:

(6′) Every unicycle in this lot has a silver spoke in the wheel.

(6″) #Every car in this lot has a puncture in the tire.

The sentence (6′) is felicitous, but (6″) is not—unless there are some contextual clues indicating which tire we are talking about.

Our next example is similar, but it involves a pronoun, not a definite description:

(7) Every man who has a wife sits next to her.

What is the referent of "her"? At the event in question, there were, or at least could have been, lots of women. The sentence says that every man who has a wife sits next to his wife. So "her" does duty for the value of the relevant function. It is, of course, common knowledge, nowadays, that (almost) every man with a wife has only one such. I do not know if (7) would be felicitous in a polygamous society. One would not think that cultural conventions are part of semantics.

The next examples illustrate the use of definite descriptions and singular pronouns when *existence* fails:

(8) If a strange man and a curious woman live here, the strange man will scare my cat, and the curious woman will make friends with it.

(8′) If a strange man and a curious woman live here, he will scare my cat, and she will make friends with it.

It does not follow from the antecedent of these conditionals that there is only one strange man and only one curious woman living here. Nor does it follow from (8) and (8′) that there is even one strange man or curious woman living here. So neither existence nor uniqueness is implicated or presupposed. But the sentences are felicitous, and may even be true. A Russellian could handle (8) by embedding the existence and uniqueness conditions in the consequent of the conditional (i.e., giving the quantifiers narrow scope).

Consider, next, the much discussed phenomenon sometimes called "donkey anaphora":

(9) Every farmer that owns a donkey beats it.

This is felicitous, but what is the referent of "it"? There surely is more than one donkey among the farmers. We can think of the pronoun as indicating a function, along the lines of some of our previous examples, but this works only if each farmer in the range of the opening quantifier has at most one donkey. Otherwise, it is more like (6″) above, which is not felicitous. It seems to me that (9) is still felicitous even if there are farmers with two or

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more donkeys, and the sentence is true (albeit brutal) if each such farmer beats all of his poor donkeys. It may even be true if each farmer who owns donkeys beats at least one of them. The truth conditions of donkey sentences is fraught with controversy, of course, but clearly they are felicitous utterances. So it seems that we *can* use a singular pronoun here, even when uniqueness fails. This case is at least roughly analogous to our present concern with the complex unit and the like.

Consider, next, the so-called problem of indistinguishable participants. Here are two instances:

(10) If a bishop meets another bishop, he blesses him.

(11) If a man lives with another man, he shares the housework with him.

Given the symmetries, and absent any other conversational cues, there is no way to distinguish the two bishops and no way to distinguish the two men living together. So which one is "he" and which one is "him." In a sense, of course, it does not matter. If the first sentence is true, then any two (or more?) meeting bishops will bless each other, and if the second sentence is true, then any two (or more?) men who live together will share the housework with each other.[12] But we still need some guidance on how to parse the two sentences.

Our final examples are probably the closest to mathematical matter at hand. The next one is sometimes called the "sage plant" sentence:

(12) Everyone who bought a sage plant here bought at least eight others along with it.

(13) Remember that chess set that came with an extra pawn? I could have used an extra king, but I never needed the extra pawn.

In (12), we are given no way to indicate the referent of "it." Which of the sage plants was *the* one that the customer in question purchased, along with the eight (or more) *others*? In some circumstances, there will be a unique sage plant that can be thought of as the referent of "it." Suppose, for example, that each customer who bought sage plants first took one of them up to the checkout counter, and then ran back and got at least eight more. But the sentence is true, if, instead, each customer who bought sage plants bought at least nine of them, all at once. As for (13), the chess set in question presumably came with nine pawns of one color. Again, in some circumstances, one of those pawns could have been distinguished as *the* extra one. Suppose, for example, that in the original box, a pawn was taped to the bottom, with the rest of them inside the box. In that case, the taped one is the extra pawn. But the sentence is true even if all of the pieces were thrown together inside the box, with none of the nine pawns of that color distinguished in any relevant way. The following also seems to be felicitous:

(13)′ Remember that chess set that came with nine white pawns? I could have used an extra king, but I never needed the extra pawn.

b. Introducing discourse referents. According to Roberts, the existence and uniqueness of definites is registered at the level of pragmatics (see also Roberts, 2002, 2004). Her

[12] Thanks to Jason Stanley, who put me onto this fascinating issue in linguistics and the philosophy of language. For the general problem of indistinguishable participants, see, for example, Heim (1990), Roberts (2003), and Elbourne (2003, 2005, §1.3.2 and chap. 4). Here is an example with (presumably) a bit less symmetry: "When a mafia hit-man meets another mafia hit-man, he kills him."

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account makes heavy use of the notion of conversational record. This is a sort of running database that keeps track of assumptions, presuppositions, comparison classes, quantifier ranges, and other items implicitly agreed to in the course of a conversation. The record is continually updated, in that items are put on it and removed from it in the course of a conversation. It is a theoretical posit.

For present purposes, it will suffice to sketch one of the earliest articulations of the notion, due to Lewis (1979). He calls it "conversational score," in analogy with the scoreboard of a baseball game in progress (runs, people on base, balls, strikes, outs). He delimits some features of the database:

> . . . the components of a conversational score at a given stage are abstract entities. They may not be numbers, but they are other set-theoretic constructs: sets of presupposed propositions, boundaries between permissible and impermissible courses of action, or the like.
>
> What play is correct depends on the score. Sentences depend for their truth value, or for their acceptability in other respects, on the components of the conversational score at the stage of the conversation when they are uttered . . . [T]he constituents of an uttered sentence—subsentences, names, predicates, etc.—may depend on the score for their intension or extension.
>
> Score evolves in a more-or-less rule-governed way. There are rules that specify the kinematics of score:
>
> > If at time t the conversational score is s, and if between time t and time t', the course of the conversation is c, then at time t' . . . the score is some member of the class S of possible scores, where S is determined in some way by s and c.
>
> The conversationalists may conform to directives, or may simply desire, that they strive to steer components of the conversational score in certain directions . . .
>
> To the extent that conversational score is determined, given the history of the conversation and the rules that specify its kinematics, these rules can be regarded as constitutive rules akin to definitions. (Lewis, 1979, p. 345)

Lewis points out that unlike most games, conversations tend to be cooperative. This is a crucial insight for present purposes. One ramification of this is that, as Lewis (1979, p. 347) puts it, "rules of accommodation . . . figure prominently among the rules governing the kinematics of conversational score." The idea is that the conversational record tends to evolve in such a way that, other things equal, whatever is said will be construed as to count as correct—felicitous and true—if this is possible. The record will be updated to make this so. Suppose, for example, that someone utters a sentence "Our banjo player no longer owns a Porsche," which has the presupposition that the banjo player used to own a Porsche. Unless someone objects, the presupposition goes on the record. To be sure, cooperation is "not inevitable, but only a tendency," as Lewis puts it, and presuppositions can be retracted later.

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Lewis (1979) goes on to use this notion of conversational score to illuminate six diverse features of the semantics and pragmatics of natural language. The notion has evolved since then, and it has been put to work in numerous ways, under various guises.[13] It seems clear, for example, that for a given conversation, the assumption that there is a single score is an idealization. It is better to postulate a separate score for each participant in the conversation. A given conversationalist's score keeps track of what he or she takes to be the assumptions, presuppositions, comparison classes, quantifier ranges, and other items implicitly agreed to in the course of the conversation (see Stalnaker, 1999, and much of the subsequent literature). The original, simple picture will do here, at least for a start.

Roberts adopts a view that the conversational record contains *discourse referents* which are used to track individual objects or persons. Discourse referents, in this pragmatics-based use, are sometimes called "tags." There are at least three ways that discourse referents are created and placed on the conversational record. First, it can happen explicitly, when an object or person is mentioned. Suppose, for example, that someone says, "I saw our friend Jack yesterday." If everyone knows who I am talking about, then there is a discourse referent for Jack. Otherwise, I may be asked to clarify. If I go on to say, "He was upset about the state of the economy," the pronoun will be anaphoric to that discourse referent. Another device for explicitly introducing discourse referents is indefinite descriptions. We will encounter several examples below.

A second way for a discourse referent to come about is via common knowledge, as with our above example:

(1) The President of the United States is eager to cooperate with his political opponents.

In such cases, we can think of the discourse referent as being created simultaneously with the utterance of "The President of the United States," since the speakers all know (or at least assume) that there is exactly one such, or else we can posit that once a title is established generally, a discourse referent corresponding to it is always available, and does not need to be created each time.

A third way for is for a discourse referent to be introduced through what Lewis calls "accommodation." That occurs in the following example:

(2) I found a box in my attic the other day. I opened the lid and pushed the button I found inside. You won't believe what happened.

When a speaker utters (2), the listeners assume that she is saying something correct. The indefinite "a box" creates one discourse referent, and "the lid" is anaphoric to that, via accommodation and/or common knowledge concerning boxes (i.e., that they typically have lids). The listeners also realize, at least implicitly, that the speaker cannot be communicating anything unless there was a button in the box, and so they supply a discourse referent for that as well.

The next example is similar:

(3) Teacher, giving directions: On the next page, you will find a puzzle. Find the clown in the puzzle.

When the students hear this, they assume that the teacher is giving coherent directions. This requires that there be a clown in the puzzle (or at least that the teacher thinks that there is

[13] Advertisement: My (broadly) contextualist account of vagueness (Shapiro, 2006c) makes heavy use of conversational score. On my account of the kinematics of conversation, the conversational record tracks which borderline cases have been called.

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one). So a discourse referent is created and put on the record. Notice, incidentally, that the discourse referent for the definite description "the puzzle" at the end was introduced explicitly, with the earlier indefinite "a puzzle."

On Roberts view, the existence and uniqueness of definites come from discourse referents, and not from anything in the logical form of the sentences or propositions in question. The idea is that each felicitous use of a definite description, singular pronoun, or singular demonstrative requires that there be a unique discourse referent. In line with what Lewis says about conversational score, discourse referents are abstract, theoretical entities. They are *not* the referents of the singular terms in question, in the usual sense of "referent." Each discourse referent comes with a description, such as "being named 'Jack'," "President of the United States," "box in my attic," and "puzzle." Truth conditions for utterances in conversation will, in general, turn on whether there are objects that satisfy these descriptions. Exactly how this works is, of course, a key item on the agenda.

For present purposes, the crucial feature is that, depending on what presuppositions and the like are on the record, a discourse referent need not correspond to any object or person *in the world*. And a discourse referent need not correspond to a *unique* such object either. Consider, for example, the above example:

(8) If a strange man and a curious woman live here, the strange man will scare my cat, and the curious woman will make friends with it.

(8)′ If a strange man and a curious woman live here, he will scare my cat, and she will make friends with it.

When a speaker utters the antecedent of one of these sentences, two discourse referents are created, and go on the record (explicitly), due to the indefinite descriptions in the common antecedent: "a strange man" and "a curious woman." The definite descriptions and pronouns in the consequents are anaphoric to the discourse referents so created.

So let us turn to the examples that are closest to our present concern, with the complex unit:

(9) Every farmer that owns a donkey beats it.

(10) If a bishop meets another bishop, he blesses him.

(12) Everyone who bought a sage plant here bought at least eight others along with it.

(13) Remember that chess set that came with an extra pawn? I could have used an extra king, but I never needed the extra pawn.

In each case, the relevant discourse referents are introduced explicitly. In (9), it comes from the indefinite "a donkey"; the pronoun "it" is anaphoric to that discourse referent. In (10), there are two discourse referents, one from "a bishop" and the other from "another bishop," and the two pronouns go with those, "he" with the first, and "him" with the second. With (12), we get a discourse referent from the phrase "a sage plant"; the pronoun "it" goes with that. And with (13), the discourse referent is introduced when the speaker utters the indefinite "an extra pawn" in that context. The definite description "the extra pawn" goes with that.

Finally, consider

(13)′ Remember that chess set that came with nine white pawns? I could have used an extra king, but I never needed the extra pawn.

On the surface, it does not look like there is a discourse referent for the definite description "the extra pawn." To get an idea of what is going on, consider the simpler:

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(14) Last night, I dropped nine quarters, and could only find eight of them. The missing quarter must have slid under my dresser.

Here, too, we have no explicit discourse referent for the definite description, "the missing quarter." But the sentence is felicitous. The reason is that a listener will do the simple subtraction (or assume that the speaker did), and supply a discourse referent, via accommodation.[14] Our example (13)′ is similar. The common (or accommodated) knowledge here is that a normal chess set contains only eight white pawns, and, of course, the facts about subtraction.

Following Roberts, say that a definite is *semantically unique* if there is a unique object (in the relevant domain) that satisfies the associated description.[15] In these terms, the key point here is that semantic uniqueness is not always required for the felicitous use of a definite. Roberts shows, however, that there are some situations where the felicitous and true use of a definite description does require, or at least presuppose, that it is semantically unique. One such is the use of titles, as in our first example above:

(1) The President of the United States is eager to cooperate with his political opponents.

As Roberts (2003, pp. 314–315) puts it,

> A title is created, whether consciously or through continued usage, with a view to use by the public, including people with all levels of information about the kinds of entities in question . . . [So] in order to insure that a title in the form a definite description will always be felicitous its designers must believe that its denotation will be semantically unique at any given time . . . Hearers know this about titles, and this leads to an epistemic version of semantic uniqueness. That is, the hearer knows (at least tacitly) the felicity conditions for definite descriptions . . . , knows what a title is, and assumes that the title in question, in the form of a definite description, was appropriately designed to assure its felicity in any context of use. Having such knowledge leads to the assumption that the title's denotation at any given time is believed to be unique in the world in bearing the descriptive content of the definite.

Our main example in this paper, of course, is the complex unit i. That particle does come with something that at least looks like a title, in the form of a definite description, namely, "the square root of -1." So it is an exception to Roberts's principle (as there are two

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[14] Consider the following:

> #Last night, I dropped nine quarters, and could only find eight of them. It must have slid under my dresser.

This one is not felicitous (although most listeners will know what is meant). According to Roberts, this has to do with an important difference between definite descriptions and pronouns. For pronouns, the discourse referent must be salient.

[15] This definition could be made more complicated to accommodate definites that occur in the antecedents of conditionals, within the scope of a modal operators and propositional attitude verbs, or the like. In a worlds-based semantic framework, we could speak of a unique object in the indicated possible world (or impossible world, if such there be). The simple definition will do for present purposes.

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complex square roots of −1). As we shall see, however, it is not an exception to the *reason* Roberts gives for the principle.[16]

If I can coin a phrase, say that a use of a definite description or a singular pronoun is *pragmatically unique* if either it is semantically unique or some aspect of the context (broadly construed), the circumstances of evaluation, or the context of assessment, indicates a unique object in the relevant domain as the referent. It seems to me that the felicity of some utterances does require pragmatic uniqueness. Consider, for example,

(5) A wine glass broke last night. It/The glass was very expensive.

As above, this example does not require *semantic* uniqueness. Many glasses may have broken the night before, even many expensive glasses. But the felicity of the utterance does seem to demand pragmatic uniqueness. The speaker would not have said this if he were just commenting on a bunch of glasses that broke, no one glass in particular.

Other uses of definite descriptions and singular pronouns do not even require pragmatic uniqueness. Such is the case with at least some donkey sentences (9), some cases of indistinguishable participants (10)–(11), and, especially, the sage plant (12) and chess set (13) examples. Consider, for example, the last of these:

(13) Remember that chess set that came with an extra pawn? I could have used an extra king, but I never needed the extra pawn.

As noted above, some utterances of (13) would have "the extra pawn" be at least pragmatically unique—if we construe "context" broadly enough. One pawn may have been taped to the bottom of the box when it was purchased, or it may have been distinguished as "extra" in some other relevant way. The crucial observation here, however, is that these definites can be felicitous, and true, even if they are not even pragmatically unique. And the bare use of this sentence does not carry any presupposition of semantic or pragmatic uniqueness, one way or the other.[17]

To be sure, I have not done much justice to the deep and subtle treatment in Roberts (2002, 2003, 2004), nor to the wealth of data cited there, and by critics of views like this. Moreover, many of the details are not crucial for present purposes. What is needed here is only that something along theses lines is plausible, at least for the languages of mathematics. I take the supporting data, above, to show that existence and uniqueness (in the world, so to speak) are not always needed for the felicitous use of definite descriptions and singular pronouns. The hope is that the resolution of this issue concerning natural language can be mobilized for our problem concerning some languages of mathematics.

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16 Roberts points out that there are other situations which presuppose semantic uniqueness. One such is when a definite description appears as part of giving directions, as in (3) above. Other cases, such as (2), (4) involve accommodation and conversational implicature. It all turns on what can and should be assumed of listeners and readers, in order for communication to be effective. Considering such interesting material here would take us too far afield.

17 Pragmatic uniqueness can be added to the record later in the course of the conversation. Suppose the speaker goes on to say, "It turns out that I could not use it (the extra pawn) anyway, as it was twice the size of the others, and it looked too much like a bishop." That would *add* pragmatic uniqueness to the conversational record. Or suppose that someone else says, "I hope you never used it/the extra pawn, as I think it was cursed. I heard that a group of witches have been sneaking cursed pawns into sets." That would presuppose that at least the second speaker believed that "the extra pawn" is pragmatically unique.

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c. Speaking of i. Recall that in the language of (pure) complex analysis, "*i*" is, at least grammatically, a proper name, and a permanent fixture in the language. The above examples supporting Roberts's analysis involved definite descriptions and/or pronouns. With the exception of the title in (1), all of those items are temporary, used only used during a particular conversation, and, in some cases, only when certain assumptions are active (of which more later).

The actual history of complex analysis, and the evolution of its language, is a complex affair (no pun intended). I gather that, originally, terms like "the square root of -1" and "$\sqrt{-1}$" were introduced only as computational aids in solving equations with real variables. The terms were not thought to have any meaning at all, and certainly no referents. The use of the word "imaginary" is probably derived from this orientation. This is not the place to recount this history, nor is it particularly relevant to the matters under study here. Eventually, complex analysis became a field of study in its own right, with its own subject matter, and the terms in question started to get used as something like genuine singular terms. Or so I assume here.

The following is a sort of rational reconstruction of how a term for the complex unit could have come about, in a Roberts-style framework. Some members of the mathematical community came to realize that there is an algebraic closure of the real numbers, or perhaps they simply assume that there is, or can be, such a structure. The phrase "an algebraic closure of the real numbers" introduces a discourse referent. The community can thus subsequently use a phrase "the algebraic closure of the real numbers"—along the lines of (13) above. Recall that the existence of a discourse referent does not imply uniqueness of a referent in the world. So there is no assumption that there is but one algebraic closure of the reals (although all such closures are isomorphic to each other). The locution "the complex numbers" can be introduced as a title for the algebraic closure of the real numbers.[18]

The mathematicians then notice that in the complex numbers (or in any algebraic closure of the real numbers), there is square root of -1:

$$\exists x \left(x^2 = -1\right).$$

This introduces another discourse referent, and allows the felicitous use of the expression, "the square root of -1," again along similar lines as those of (13) above.

At this point, the members of our community realize that they are going to continue to discuss these and other square roots of negative numbers, and so they introduce a singular term "*i*" for this purpose. This introduction is something like a baptism, except that in this case, there is not a single object or person "before us" getting named. Or one can think of "*i*" as simply synonymous with "the square root of -1."

To make the connection with the Roberts account tighter, we can say that the (postulated) "conversation," which was begun way back when, is still going on, and so the discourse referent is still available to current speakers, as the source of "the square root of -1," and thus "*i*." Or maybe the use of the term became something like common knowledge, at least among the mathematicians and the relevant informed members of the linguistic, lay community. So the expression is what we called a "title" above.

Alternately, one could think of conversations concerning the complex numbers as starting anew each time, with a new discourse referent introduced on each occasion. This would

[18] Since the theory is satisfiable and categorical, there is only one ante rem structure that is an algebraic closure of the reals. And so, on my view, "the complex numbers" is semantically unique.

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be like thinking of "i" as like a pronoun that changes its "referent" in different contexts. In standard Kaplan-style terms, perhaps "i" has a constant character, but its content changes from context to context. Sometimes "i" denotes one of the square roots of -1 (but don't ask which) and sometimes it denotes the other. I do not think much turns on this, but I think it is better to think of "i" more along the lines of any other numeral or proper name, such as "3" or "π," which does not change its referent from conversation to conversation (so long as one is discussing the same collection of numbers). The reason is that the "referents" of the different uses of "i," on different occasions, are thought to line up with each other.

I do not think that very much turns on this last matter, however. It *is* important, however, that the use of related expressions line up with each other within a given conversation. The phrase "the square root of x" indicates a function, and so does the radical expression "$\sqrt{x}$." So we need a unique value for each argument x on which the function is defined. Recall that each positive *real* number r has two square roots. Of course, the custom is to use the expression, "the square root of r" and "$\sqrt{r}$," to stand for the nonnegative root. In the complex numbers, each negative real number $-r$ also has two square roots, and the custom is to use "the square root of $-r$," and the corresponding radical, to stand for the *positive* multiple of "i." So, for example, *the* square root[19] of -4 is $2i$.

Russell (1957, p. 385), for example, uses the locution: "The square root of minus one is half the square root of minus 4," to make a point against Strawson. Clearly, he intended this sentence to be true, and this requires that the two definite descriptions be coordinated with each other.[20]

As noted above, Roberts argues that titles have to be semantically unique. That is, the felicitous introduction and use of a title requires that there be one and only one object (or person) that satisfies its semantic content. There should be one and only one President of the United States (at least most of the time), one and only one Prime Minister of the United Kingdom, etc. The phrase "the square root of -1" seems to be a title, as it is a definite description with a permanent and uniform use, but, once again, it is not semantically unique in the complex numbers.

This does not represent a problem for Roberts's overall account. The reason she gives for why titles have to be semantically unique is that they are intended for use in all sorts of contexts, in all sorts of conversations, come what may. If there were two objects or people that satisfied the semantic content of a title, then there could be a conversation in which both objects or people were in the intended domain, and each already had its own discourse referent on the conversational record. This would violate the guiding principle

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[19] Notice, incidentally, that the two square roots of $-2i$ are $(1-i)$ and $(-1+\mathrm{i})$. As far as I know, no one is tempted to call one of them "the" square root of $-2i$. However, it is common to dub the square root whose angle is smallest, when expressed in polar coordinates, the "principle square root" of the number. In this case, it is $-1+i$. (See note 7 above.)

[20] There is an interesting irony here. On Russell's own analysis of definite descriptions, his sentence comes to this:

> there is one and only x such that $x^2=-1$; there is one and only one y such that $y^2=-4$; and $y=2x$.

This sentence is *false*. Perhaps Russell was thinking of a context in which the symmetry is broken, and in which there is a convention concerning the definite description. Russell's comment, however, was made in a spirited defense of his own account of definite descriptions. So presumably he meant for this very sentence to be analyzed accordingly.

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that the felicitous use of a definite requires a *unique* discourse referent. Notice that *this* problem cannot arise in complex analysis. Of course, in discussions concerning the complex numbers, both square roots of -1 are in the intended domain (duh). But there cannot be separate discourse referents for those two numbers, since they are indistinguishable. There is no way to pick them out separately, and independently,[21] then wonder which one is supposed to be *the* square root of -1.

It is common in mathematics to embed one structure into another one, in order to shed light on the properties of the former. Sometimes, they go so far as to "identify" the objects in the original structure with their images under the embedding. We can think of natural numbers as real numbers or complex numbers, or finite, von Neumann ordinals; we can think of the real numbers as Dedekind cuts, etc. (see Shapiro, 2006a). As noted above, it is the current custom to identify the complex numbers with the elements of another, rigid structure, usually pairs of reals, or the complex plane. The foregoing account of definite descriptions, and the (modified) clause concerning titles, would "predict" (or postdict) that whenever this is done, it is necessary to stipulate which square root of -1 is to be the referent of "*i*." And this, of course, is what we see. The unit i is the pair $(0,1)$ (as opposed to $(0,-1)$), or in polar coordinates, $(1,\pi/2)$. It is not left indeterminate which square root is the referent of i.

Brandom (1996, §6) proposes a similar, perhaps identical resolution of the general problem concerning terms like "*i*" (but without the direct reference to the foregoing linguistic theory). He calls items like "*i*," "merely distinguishing terms," to distinguish them from ordinary proper names, which do pick out unique objects (or people). He writes:

> Frege's practice . . . would seem to show that what matters for him is that we understand the proper use of the expressions we introduce: what commitments their use entails, and how we can become entitled to those commitments. We can be entitled to use merely distinguishing terms . . . provided we are careful never to make inferences that depend upon the . . . specifiablity of what is labeled—that is that our use of the [terms] respects the [automorphisms] that precluded such specifiablity . . . [T]here is nothing mysterious about the rules governing [these terms].

Brandom suggests that we have a choice. We can think of ordinary singular terms and what he calls "merely distinguishing terms" as two separate syntactic/semantic categories, or else we can think of all singular terms as merely distinguishing terms, and note that some such terms have an extra feature, the property that Roberts calls "semantic uniqueness." Nothing of significance turns on this choice, perhaps, but I propose to take the former route, and to sketch an account of how "merely distinguishing terms" are introduced, and how they function.

d. Taking parameters seriously. Introduction. Systems of natural deduction are, I believe, good models of ordinary deductive reasoning in ordinary discourse, at least for the most part, and they provide the key for developing the foregoing account.[22] Let us focus

[21] Of course, once the title is established, it is possible to "refer" to the other square root of -1. One can just use the phrase "the other square root of -1," as another definite description, dependent on the first. Or one can just call it "$-i$." The second discourse referent is somehow dependent on the first. Again, there is no possibility of confusion.

[22] I understand that thinking about natural deduction was part of the motivation behind Roberts (2003).

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on one of the roles of singular terms in natural deduction—and in the reasoning modeled by natural deduction systems. Consider the rule of existential elimination. Suppose that in doing a derivation, someone reaches a conclusion in the form:

$$(\exists x)\, \Phi\, (x)\,,$$

resting on some premises or assumptions. In typical systems, the reasoner proceeds by making a new *assumption* $\Phi(b)$, where b is a singular term that does not occur in the formula $(\exists x)\Phi(x)$, or in any undischarged premise supporting that formula. Then she proceeds to deduce a formula Ψ, in which the term b does not occur, and which does not rest on any premises or undischarged assumptions that contain b, other than the introduced assumption $\Phi(b)$. She is then entitled to discharge that assumption, and have Ψ rest on whatever $(\exists x)\Phi(x)$ rests on.

In some natural deductive systems, the introduced term b is a free variable. One might just assume $\Phi(x)$. It is more common, however, to require that only closed formulas (sentences) appear in derivations. In such cases, b has to be a constant. In either case, b functions, grammatically, as a singular term in the formula in which it occurs.

The rule of existential elimination corresponds to a common pattern in ordinary reasoning. In actual reasoning, in mathematical or natural language, it would be strange—to say the least—to use an existing *proper name* in the role of b. Suppose, for example, that someone is doing a deduction in the language of arithmetic, and gets to a conclusion of the form $(\exists x)\Phi(x)$. Then he notices that the numeral '39' does not occur in any undischarged premise of the deduction. So he assumes that $\Phi(39)$. This fits the letter of the rule for invoking existential elimination, at least as the rule is formulated in most logic books. But that assumption would be misleading at best, and potentially confusing. The term '39' already has a role in the language, namely that of denoting the number 39, and it may well be that our formula Φ does not hold of 39. Suppose, for example, that it follows from $\Phi(n)$ that n is prime. In that case, I would not want to assume $\Phi(39)$—even if I am planning on discharging that assumption. And besides, there is a danger of forgetting that "39" was used just for the purpose of existential elimination, and the reasoner might draw some unwarranted conclusion, thinking of that numeral as a name for the number 39. Similarly, suppose I am thinking about people I know, and get to an intermediate conclusion that someone has lied to me in the past week. It would then be weird to assume that my mother lied to me, to start the process of existential elimination.

What is usually done, in mathematics at least, is to introduce an *unused* constant for this purpose—and that is the advice given to students assigned to do exercises in natural deduction. Of course this presumes that there is always a stock of such unused constants available. Suppose that in informal reasoning, one gets to a conclusion of the form $(\exists x)\Phi(x)$. The standard move is to say something like this: let b be a Φ. At some point, it is customary to remind the reader that "b is arbitrary," or that b is an arbitrary Φ—usually at or near the point where we think of the assumption of $\Phi(b)$ as discharged. My mother is hardly an arbitrary person, among those I know, and 39 is anything but an arbitrary natural number. As Frege and others pointed out, *no* natural number is "arbitrary." In sum, to avoid confusion, no name that (already) denotes a particular number should be used in existential elimination.

It seems to me that part of the problem here is to figure out what the locution "arbitrary" means here. Our introduced b is not, or need not be, an *arbitrary object*, in the sense of Fine (1985) (of which more later). The "arbitrariness" concerns the inferential role of the

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linguistic item, not the object supposedly denoted by the item. In the reasoning in question, one is not to assume anything about b other than $\Phi(b)$.

To model how the inference in question is deployed in real life, it might be better to follow some authors and introduce a new category of singular term for use in existential elimination (and universal introduction). Call them *parameters*. In some ways, parameters function as constants; in others they function as variables. In the case of existential elimination, we have it that some object in the domain satisfies Φ. The semantic role of the term b is to denote *one* such object. So in that sense, the parameter is like a constant. But it is crucial that we do not specify *which* such object, even if we could. The rules of engagement require the reasoner to avoid saying anything about b that does not hold of *any* object that satisfies Φ. In that sense, b functions more like a variable, ranging over the Φ's. Free variables, when they are allowed to occur in deductions, often have the same mixed role.

It seems to me that we are departing a bit from ordinary, nonmathematical reasoning here. Outside of mathematics, there is no stock of generic names available for purposes like this, although I presume it would not be too much of a stretch to add them, and that ordinary speakers would know what is going on if we did. In some reasoning type situations, we do invoke words that are ordinarily used as names in the role of parameter:

(15) Consider an arbitrary philosophy philosopher; call her "Joan." Suppose Joan has just published a new article, . . .

(16) We know that somebody has been blackmailing the Senator. Call her "Joan." We know that Joan is good at spoofing email addresses . . .

These locutions would be effective only if no one else named "Joan" is salient to the conversation in question. So it functions like a parameter.

I suggest that it not particularly natural (no pun intended) to think of existential elimination as beginning with an *assumption*, to be discharged later. Following ordinary mathematical practice, think of the move as two steps, one a stipulation, and one a (trivial) inference. Recall that in the envisioned derivation, we have reached a formula $(\exists x)\Phi(x)$. The reasoner stipulates, "let b be a Φ." Then he *infers* $\Phi(b)$, and writes that sentence on the next line. So the proposed rule is this:

> from a sentence in the form,
>
> $$(\exists x)\ \Phi\ (x)\ ,$$
>
> one can infer
>
> $$\Phi(b),$$
>
> provided that the term b does not occur previously in the deduction. The inferred formula $\Phi(b)$ rests on whatever premises and assumptions the existential formula $(\exists x)\Phi(x)$ rests on.

This is a version of a rule once called existential instantiation.[23]

[23] One can relax the condition a little, requiring only that b does not occur in any premise on which $(\exists x)\Phi(x)$ depends, but that subtlety need not concern us here. Incidentally, Fine (1985) also sees existential instantiation as the more natural inference, and devotes much of the book to showing what the semantics of this would be, using his account of arbitrary objects.

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It seems to me that existential instantiation fits mathematical practice slightly better than the now more usual existential elimination does.[24] As noted, in an actual, informal deduction, one typically starts the procedure by saying something like "let b be a Φ," rather than "assume $\Phi(b)$." Suppose that the reasoner goes on to deduce a sentence Ψ, which does not contain an instance of b. Typically, the reasoner does not repeat this sentence on a new line, and explicitly note the discharge of the assumption $\Phi(b)$. Instead, she just goes on from there, perhaps reminding the reader that b is arbitrary.

It is not much of a stretch to assimilate the reasoning in question to the foregoing account of definites, due to Roberts (2003). Indeed, I submit that parameters *are* definites—much like pronouns—and they are subject to the same rules. That is, in the informal language being regimented, the felicitous use of a parameter requires a (unique) discourse referent to be available.

In a typical instance of existential instantiation, the reasoner has inferred a formula in the form $\exists x \Phi(x)$. An English rendering of this is something like, "There is a Φ." This is an indefinite, and so introduces a discourse referent. This sanctions the introduction of our new parameter b, as it would a singular pronoun or definite description.

Of course, this pleasant conclusion may just turn on how I chose to render the existential quantification into natural language.[25] Instead of saying "There is a Φ," I could have said "There are Φ's." The plural phrase does not, by itself, generate a *singular* discourse referent. Moreover, the plural expression would not license either a definite description or a singular pronoun. For example, neither of the following is felicitous:

(17) Some men came in the room. #He was quite angry.

(17)′ Some men came in the room. #The man who came in was quite angry.

Recall that, in the informal gloss, the use a parameter first requires a *stipulation*, followed by a trivial inference. The stipulation would be something like: "Let b be a Φ" or "let b be one of the Φ's." Following a suggestion made above, with (15) and (16), we can modify the infelicitous (17) and (17)′:

Some men came in the room. Call one of them "Jack." Now, Jack was quite angry.

Some men came in the room. Let Jack be one of them. Now, Jack was quite angry.

These *are* felicitous. What I suggest, then, is that locutions like "a Φ," "one of the Φ's," or even "one of them," create the indicated discourse referents, and the parameter so introduced is anaphoric to those discourse referents.

To be sure, in some of the locutions in question, such as "let b be a Φ" or "let Jack be one of them," the creation of the discourse referent comes *after* the parameter first appears, but it comes immediately after, and, in general, the hearer or reader has no trouble making the connection. It is something like accommodation. Notice that in some cases, pronouns

[24] The proposed rule of existential instantiation corresponds to the way the existential quantifier is handled in the methods of truth trees and semantic tableau. If, in developing an argument, a reasoner reaches a sentence in the form $(\exists x)\Phi(x)$, she is to pick an unused constant (or free variable) b and write $\Phi(b)$, and then operate on that sentence. The idea, I suppose, is that b should stand for an object (or any object) that satisfies Φ (see, e.g., Smullyan, 1995). Quine's (1982, §30) *Methods of Logic* invokes something similar as well. But, it seems, neither truth trees, semantic tableau, nor Quine's procedures are intended to be models of actual deductive reasoning. They are more like abstract tests for validity. The claim here is that the rule of existential instantiation, as formulated here, does in some way reflect actual reasoning with existential statements.

[25] Thanks to Steve Yablo here.

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occur before their antecedents, as in "Although he is normally prompt, our speaker this afternoon will be delayed."

Returning to the main case at hand, I submit that we think of "*i*" as a parameter, introduced into the language of complex analysis. Again, we note that in the algebraic closure of the reals, there is at least one square root of -1:

$$\exists x \left(x^2 = -1\right) .$$

So we let i be one such square root, and go on from there. The difference between "*i*" and more typical parameters is that it is a permanent fixture in the language of complex analysis.[26] One might note, in line with existential elimination, that there is nothing to be said about i, in the language of fields, that does not hold of every square root if -1. This is as it should be, since the two roots are indiscernible.

As an aside, I suggest that at least typical instances of the rule of *universal introduction* can also be understood as also deploying parameters. The reasoner makes a stipulation, "let c be any object." At some point later, she deduces a formula in the form $\Phi(c)$. Then she concludes $(\forall x)\Phi(x)$, perhaps reminding the reader that c is arbitrary. To maintain the connection with Roberts (2003), perhaps we can say that locutions like "any object" generate discourse referents.

e. Semantics of parameters. The syntax of parameters is straightforward: they are singular terms, like variables and constants. The deductive role of parameters is also quite straightforward. In the style of natural deduction, we start with rules for the introduction of formulas containing parameters. One introduction rule is that of existential instantiation: if one has deduced a formula in the form $(\exists x)\Phi(x)$, then one may produce any *previously unused* parameter b, declare "let b be a Φ" and infer $\Phi(b)$. Call this b an *existential parameter*. It is tied to its *introducing formula*, $(\exists x)\Phi(x)$. In the addendum to this section, we'll raise an issue concerning how durable existential parameters are. Another introduction rule for parameters is this: at any point, one may declare "let c be any object," where c is a unused parameter. Call c a *universal parameter*. Here there is no the introducing formula to keep track of.

Existential parameters do not need a special elimination rule. As singular terms, they are available for use in existential introduction: from $\Psi(b)$ infer $(\exists x)\Psi(x)$. Universal parameters are also available for that, but they can be also be used in universal introduction: from $\chi(c)$, infer $(\forall x)\chi(x)$, provided that c does not occur in an active assumption.

What of semantics? An inferentialist about meaning, or at least the meaning of logical terminology, has a relatively easy path here. On such views, the above rules give the meaning of the parameters. The inferentialist must then show that adding parameters to the languages, with the foregoing rules, is a conservative extension of the language without parameters. That is, he must show that the new rules do not allow us to deduce invalid arguments in the previous language—the language without parameters. The usual soundness proofs are easily modified to do just that.

---

[26] John Burgess once suggested a similar resolution of the issue concerning "*i*" to me, also modeled on natural deduction. Pettigrew (2008) coins the phrase "dedicated parameter" for linguistic items originally introduced via something like existential elimination, but kept in the language for future use. He argues that virtually all of the singular terms of mathematics, such as "N" and "0," should be understood this way, thus supporting an Aristotelian philosophy of mathematics, a sort of eliminative structuralism. The project advanced in this paper thus gives aid and comfort to one of the main opponents of ante rem structuralism.

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The model-theorist, of course, does semantics in terms of recursive satisfaction conditions on formulas. One option for the model-theorist is to wax instrumentalist, and hold that formulas with parameters have no meaning. They are just aids in reasoning, perhaps analogous to how terms like "$\sqrt{-1}$" were once thought to be mere aids in computation. The model-theorist would then follow the proof-theorist and show that the expanded system is conservative over the one without parameters. Again, the usual soundness proofs do the trick.

This cop-out is not in the spirit of the present paper, which implicitly assumes that terms like "i," and the other such linguistic items, are meaningful and central parts of mathematical languages. What is needed, for the model-theorist, is an account of the *satisfaction conditions* of formulas that contain parameters and, consequently, truth conditions for closed formulas that contain parameters. I'll explore some options.

Suppose that b is an existential parameter, which was introduced in a deduction via the formula $(\exists x)\Phi(x)$. Let $\Psi(x)$ be any formula whose only free variable is x. One might think that $\Psi(b)$ is true, in a given interpretation, if and only Ψ is satisfied by every Φ. That is, one might think that $\Psi(b)$ is equivalent to

$$(*) \qquad (\forall x)\,(\Phi\,(x)\ \rightarrow\ \Psi\,(x))\,,$$

perhaps with $(\exists x)\Phi(x)$ as a presupposition.

It does seem that (*) gives something like the *assertion condition* for $\Psi(b)$. That is, one is entitled to assert $\Psi(b)$ only if she shows that Ψ holds of every Φ. Indeed, that is the point. In our target example, in (pure) complex analysis, one can assert something about i only if that something holds of both square roots of -1. However, I do not think that (*) gives the *content* or meaning of $\Psi(b)$ in general. To see this, suppose that a reasoner introduces $\Psi(b)$ as an *assumption* into a derivation (formally or informally). What is the content of that assumption? What, exactly, is being assumed? Not (*). That is, the reasoner assumes that Ψ holds *of* b, not of every Φ.

To see this, suppose that a new nonlogical predicate B is added to the language of complex analysis, and that someone assumes Bi. What is being assumed here? Surely not that B holds of *every* square root of -1—of both i and $-i$. For example, it would not follow from that assumption alone that B holds of at least two complex numbers, would it? Unhelpfully, the intuitive content of the assumption is that B holds of i, not of both square roots. But, of course, we have never singled out one of the square roots of -1 to be i. We can't. That's the issue that got this whole project started.

One way to proceed is to introduce a choice function f into the metatheory. Suppose that M is an interpretation whose domain is d. As usual, we assume that d is not empty. Let $c \subseteq d$. If c is not empty, then $fc \in c$. If c is the empty set, then let fc be fd.

We use this choice function to assign a *quasi-referent* to each existential parameter, as it is introduced into a derivation. Suppose, then, that a reasoner has derived a sentence in the form $(\exists x)\Phi(x)$, and wishes to introduce a parameter b (along the lines of the locution, "let b be a Φ"). Let c be the set of all $n \in d$ such that n satisfies Φ in M. Then let the quasi-referent of b be fc. In other words, we use our choice function to deliver a "referent" for the parameter among the Φ's, if there is such a referent to be had. Otherwise, we give the parameter a dummy "referent."[27]

[27] We need the "otherwise" clause, since it may happen that there are no Φ's. In the derivation, the sentence $(\exists x)\Phi(x)$ may depend on a false assumption. That case will occupy us briefly in the addendum to this section. In correspondence, Jeffrey Ketland suggested that (existential)

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To be clear, I am not suggesting that the quasi-referent gives the referent of the parameter, in any intuitive sense of "referent." The choice function is just a technical device to establish satisfaction conditions. Moreover, in general, it is not possible to specify a *particular* choice function. We cannot do so in the case of complex analysis, for example. In effect, our name "f" above is a *parameter* in the metalanguage. We know (or assume) that there is a choice function, and we let f be one such. That is, the present proposal (as well as others) invoke the very item in the metatheory for which we are trying to give a semantics. I do not see this circularity as problematic, and it is virtually unavoidable.

An equivalent route would be to think of existential parameters along the lines of a Hilbert-style ε-operator. An existential parameter b introduced from a sentence $(\exists x)\Phi(x)$ would be $(\varepsilon x)\Phi(x)$. In a Hilbert-style system, one does not need quantifiers to be among the primitive lexicon: $(\exists x)\Phi(x)$ is just an abbreviation of $\Phi((\varepsilon x)\Phi(x))$. In the present system, one can infer $\Phi(b)$ from $(\exists x)\Phi(x)$ and vice versa.

I'd prefer instead to think of parameters more along the lines of free variables. If a parameter b occurs in a sentence $\Psi(b)$, then the sentence has no truth-value by itself, even under an interpretation—just as the open formula $\Psi(x)$ does not have a truth-value by itself. The analogy would be to a pronoun used without an antecedent, even implicitly: If someone says, out of the blue, "She is cold," the proper response would be "Who are you talking about?"

The proposal is to define satisfaction as a relationship between an interpretations, a formula, an assignment to the variables, and an assignment to the parameters. In contemporary linguistics and philosophy of language, the word "context" has several, overlapping functions. In one sense, the term applies to relevant items picked up from the context of utterance, the circumstances of evaluation, the context of assessment, and the like (depending on one's view). Bach (1999) suggests that things like speaker intentions count as "wide context," in this sense. We have invoked such in discussing the data in section 3 above (e.g., (2), (3), (5)), and in our talk of pragmatic uniqueness. In a second role, "context" supplies what must be added to certain utterances, or sentences, to get a truth-value. The present proposal is that assignments to parameters are contexts in the second function, and—to repeat the point of this paper—not necessarily the first. That is, in the cases of interest, there is nothing in the context of utterance, circumstances of evaluation, context of assessment, etc., that gives a referent to the parameter. Yet, on the present proposal, a sentence with parameters gets a truth value only relative to an assignment to the parameters—only relative to getting a referent for the parameter.

For existential parameters, it is natural to supervaluate. Again, suppose that b is an existential parameter introduced from the formula $(\exists x)\Phi(x)$, and fix an interpretation for the language. The intended range of b, of course, are the Φ's (if there are any) in the domain of the interpretation. Let $\Psi(x)$ be any formula in which only x occurs free. We can say that

---

parameters be called "Skolem terms," since their role is similar to that of the introduced constants in the Henkin proof of Gödel's completeness theorem (of which the Löwenheim–Skolem theorem is corollary). The idea is to extend a given theory to one such that for each formula $\Phi(x)$ with only x free, there is a constant c such that $(\exists x)\Phi(x) \rightarrow \Phi(c)$ is a theorem. The constant c is to act as an arbitrary "witness" for Φ. One shows that any model of the original theory can be extended to a model of the expanded system (with the new constants), thus showing that the latter is a conservative extension of the former (see van Dalen, 2004, 136ff; Hodges, 1997; or just about any standard first text in mathematical logic). In the present context, this corresponds to a proof of the soundness of the indicated rule of existential instantiation, and the conservativeness of the system.

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$\Psi(b)$ is super-true if Ψ holds of all of the Φ's (and there are such Φ's), that $\Psi(b)$ is super-false if Ψ is false of all of the Φ's (and there are such), and that $\Psi(b)$ is indeterminate otherwise. Of course, super-truth is not compositional, and the satisfaction relation is not defined in terms of it.

Notice that the present proposal also has the circularity of above suggestion concerning quasi-referents, via a choice function. Even if, say, the Φ's are indiscernible, we still presuppose that it is acceptable to speak of an assignment that maps the parameter onto one of Φ's. And in the metalanguage, we can introduce a (meta-)parameter for one such assignment, when elaborating the model-theoretic semantics. The character *B* in the Black (1952) dialogue (section 1 above) would surely balk at this, saying that we have no right to speak of an "assignment" to the parameter, since the objects so assigned are indistinguishable. The burden of this paper is that this is simply mistaken.

f. Addendum: an issue concerning intuitionistic logic, assumptions, and parameters. In some contexts, an unrestricted rule of existential instantiation is actually a bit stronger than the ordinary rule of existential elimination. The difference illustrates an interesting issue concerning the role of parameters, and other definites in natural and mathematical language, and it bears on complications that arise when definites appear in the antecedents of conditionals, under modal subordination, and the like.

All along, I have been presupposing that the background logic is classical. For now, however, assume that the ordinary *intuitionistic* introduction and elimination rules hold for the connectives, and for the universal quantifier. Now add the above rule of existential instantiation: from $(\exists x)\Phi(x)$, (stipulate, "let b be a Φ," and) infer $\Phi(b)$, where b is a new parameter. Then we can deduce the following argument:

$$\textbf{(EMP)} \qquad (E \rightarrow (\exists x)\, Kx) \models (\exists x)\,(E \rightarrow Kx)\,.$$

The so-called "Empire State Building Paradox" is in the form (EMP). The premise is that if the Empire State Building falls down, then someone will be killed. The conclusion is that there is someone who will be killed if the Empire State Building Falls. So under the premise, which is eminently plausible, there is a person whose continued life depends on the Empire State Building staying up. Nevertheless, the argument (EMP) is classically valid. Here is a deduction in our envisioned system:

(1)	$E \rightarrow (\exists x)Kx$	Premise
(2)	E	Assumption
(3)	$(\exists x)Kx$	from (1) and (2), by $\rightarrow$E
(4)	Kb	from (3), by Existential Instantiation
(5)	$E \rightarrow Kb$	from 2–4, by $\rightarrow$I (discharging (2))
(6)	$(\exists x)(E \rightarrow Kx)$	from (5), by $\exists$I

Notice that an attempt to derive the conclusion this way using the more common existential elimination rule will fail. Line (4) would be an *assumption* of *Kb*, not a derived *conclusion*. So one could not discharge the assumption from line (2) at line (5), since there is another assumption still active, namely *E*. More importantly, when you do the existential elimination, at step (6), the conclusion would still depend on what the existential statement at line (3) depends on, namely the assumption of *E* at line (2). A typical derivation of (EMP) in a classical system begins with an instance of excluded middle, $E \vee \neg E$, and proceeds by cases.

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It easy to see that (EMP) is not intuitionistically valid. Consider a Kripke structure with three nodes, m, n, o, where n and o are accessible from m, but not vice versa, and neither n nor o is accessible from the other:

$$\begin{array}{ccc} (E, Ka)n & & o(E, Kb) \\ & \bigvee & \\ & m & \end{array}$$

There are two objects, a,b in the domain of all three nodes. The sentence E holds at n and at o, but not at the base m. The extension of K is empty at m, while at n, Ka holds and Kb does not, and at o, Kb holds and Ka does not. In the example, n and o represent two futures in which the Empire State building falls. In one of them a is killed (and not b), and in the other b is killed (and not a). It is straightforward to check that $E \rightarrow (\exists x)Kx$ holds at m, since at each future in which the Empire State Building falls, someone is killed. But $(\exists x)(E \rightarrow Kx)$ does not hold at m. In this model, there is no individual who is killed in *every* future in which the Empire State Building falls.[28]

So the present rule of existential instantiation is not conservative over a standard intuitionistic system. As indicated in the previous subsection, of course, the rule is classically valid, and so if we are thoroughgoing classical logicians and mathematicians, we can rest content with leaving it in the deductive system. But there is something awkward, at best, in natural language analogues of the above derivation. One would go something like this:

> We have that if the Empire State Building Falls, then someone will be killed. So assume that the Empire State Building does fall. Then someone will be killed. Call her "*b*". So, given the assumption, *b* is killed. So *if* the Empire State Building falls, then *b* will be killed. So there is indeed someone (namely *b*) who will be killed if the Empire State Building falls.

I think it must be admitted that this is not the most straightforward way to reason. To follow the foregoing Roberts-style analysis, the intermediate conclusion that someone will be killed produced a discourse referent. This made the use of the parameter *b* legitimate. But this discourse referent, and the legitimacy of the parameter, depends on the *assumption* that the Empire State Building falls. Once that is discharged—in effect, taken off the conversational record—the discourse referent should no longer be available. So, one might think, we are no longer entitled to use the parameter.

We can illustrate the situation with one of the examples above:

> (8) If a strange man and a curious woman live here, the strange man will scare my cat, and the curious woman will make friends with it.

[28] Branching is not required to refute the argument. Consider a Kripke structure that has only two nodes, m and n, with n accessible from m, but not vice versa. The domain at n is $\{a\}$, and the domain at m is $\{a,b\}$. The sentence E holds at n, but not at m; and the extension of K is empty at m, and is $\{b\}$ at n. So at the base m, in the one future in which the Empire State Building falls, someone is killed, but there is no one that exists at m who will be killed if the Empire State Building falls. Adding the present, unrestricted rule of existential instantiation to an intuitionistic system is equivalent to adding an axiom scheme: $(\exists x)((\exists y)\Phi(y) \rightarrow \Phi(x))$: there is something that has Φ if anything does. The rule is valid in linear Kripke structures which have constant domain. Fine (1985, §21) contains a thorough study of existential instantiation in the context of natural deduction.

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As noted above, this is felicitous. Consider the following:

> (8)" If a strange man and a curious woman live here, he will scare my cat, and she will make friends with it. But, as it happens, nobody strange or curious lives here. #So the strange man will not scare my cat and the curious woman will not make friends with it. #Instead, he will pet the cat and she will ignore it.

The last two sentences here are clearly not felicitous (absent some substantial context setting). Indeed, they do not even make sense. As above, the pronouns in the first sentence are anaphoric to the discourse referents introduced with the phrases "a strange man" and "a curious woman." But those phrases occur in the antecedent of the conditional. When that antecedent is denied, we are no longer entitled to the discourse referent. The pronouns here occur in what is sometimes called a *scope island*; they can be used only under the given antecedent.

To follow this insight here, one might restrict the rule of existential instantiation. The rule would be that from a formula $(\exists x)\Phi(x)$, one can deduce $\Phi(b)$, where the parameter b does not occur previously in the derivation. The line $\Phi(b)$ rests on the same assumptions that $(\exists x)\Phi(x)$ rests upon, and *the parameter may not occur after any of those assumptions are discharged*.

Much of the above motivation for introducing parameters into the language of mathematics makes sense in an intuitionistic setting, as well as in other nonclassical situations, such as those involving intentional objects. In cases like that, it seems, the proper rule for existential instantiation should be restricted, along the present lines. But, as noted, the unrestricted rule is classically valid, even if its use is awkward in English.

Suppose that a parameter b is introduced from a line $(\exists x)Bx$. In sketching one of the model-theoretic options in the previous subsection, we assigned a "quasi-referent" to b using a choice function on the domain. In effect, we think of b as "denoting" some object that has B, if something does, and we let b "denote" any object otherwise (in line with the use of Skolem terms, see note 27). In effect, we invoke an "if, then, else" condition to assign a quasi-referent to the parameter. As far as I know, the kinematics of conversational English do not have anything analogous to the "else" clause. Moreover, the clause invokes excluded middle, in the metatheory. We assume that either there is an object that satisfies the formula or there is not.

§4. Other views. I know we have been at this a long time. Before knocking off, I'd like to close with a brief overview of other ways to resolve the present issues concerning indiscernible objects and the relevant locutions in mathematical languages.

One contender is Fine's (1985) account of arbitrary objects. On the above view, phrases like "arbitrary" refer to the reference relation. It is arbitrary which object gets "picked out," to speak loosely. In contrast, according to Fine, arbitrary objects occupy a special ontological category. They are a certain type of *object*. The theory was developed to handle phrases like, "let n be an even natural number," which, of course, play a considerable role here. On Fine's view, such an "n" designates a particular object, an arbitrary even natural number. So, in a sense, this n is distinct from each of the particular even natural numbers, such as 4, 12, and 248, that we are more familiar with. Fine's book contains a rich and subtle semantic account of how arbitrary objects can figure in mathematical reasoning, or at least in a certain rational reconstruction thereof.

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The application of Fine's theory to the present cases is straightforward—once one has digested the details of the treatment. In the case of complex analysis, the reconstruction would go like this: At some point in history, the mathematical community elected to have the term "*i*" designate an arbitrary square root of −1, in the complex numbers. So, on this view, *i* is not an individual number, on a par with, say, 24, *e*, and π. Just as the above "*n*" stands for an arbitrary even natural number, "*i*" stands for an arbitrary (complex) square root of −1. The use of the definite article, "the," as in "the square root of −1" is a (slight) abuse of language, since there is more than one arbitrary square root of −1 (or perhaps the foregoing Roberts account can be invoked here). Indeed, $-i$ is another, arbitrary square root of −1. In Fine's sense, $-i$ depends on i.

It should be noted that Fine's account is not as ontologically extravagant as it may appear at first sight. As Fine sees things, arbitrary objects are not metaphysically "primitive," or sui generis. Rather, they are set-theoretic constructions on the more usual mathematical objects. Nevertheless, the formal semantics and model theory get quite complicated, with terms designating arbitrary objects interacting with ordinary singular terms, and the like. And, of course, the theory is not beyond criticism (see Tennant, 1983).

Martino (2001) is concerned with mathematical languages whose intended domains are infinite, or perhaps uncountable. It would seem that we cannot refer to, say, each real number, as there are too many of them. Yet, Martino notes, we use expressions like "let *a* be a real number" ubiquitously. He argues that use of such locutions presupposes a "thesis of arbitrary reference," that it is ideally possible to refer "individually to any object of the universe." The reference here is not possible for flesh and blood humans, of course. So Martino speaks of ideal mathematicians, in analogy with the intuitionistic talk of ideal construction.

In cases like the complex numbers and the Euclidean plane, I would think that even an ideal mathematician cannot pick out, and thus refer to, individual objects. The two square roots of −1 and the points in Euclidean space are too homogeneous to be picked out individually, even ideally. But that may depend on what we mean by "ideal." Martino's envisioned idealization goes well the usual matter of waiving the limitations on attention span, lifetime, and the like. And one might wonder what the abilities of idealized creatures like this have to do with our own linguistic practices. How does the activities of ideal beings bear on our own linguistic practices? Martino's thought, it seems, is that the legitimacy of our practices depends somehow on what the postulated ideal mathematicians can do.

In any case, the burden of the present account is to make it plausible that the locutions in question do not presuppose even ideal reference to individual objects. They do not presuppose reference at all, in the sense of somehow picking out individual objects, one at a time.

A third option is modeled after epistemicist accounts in vagueness (see, Williamson, 1994; Sorenson, 2001).[29] Suppose that, in the course of a demonstration, someone says "let *n* be an even natural number." On Fine's account, the introduced term *n* denotes an arbitrary even natural number (or perhaps the independent arbitrary even natural number). On the rival account under study here, *n* denotes a *particular* even natural number, such as 6, or 22, or 267352564570098146094000678. We have no way of knowing *which* even number "*n*" denotes—thus the connection with epistemicism—but the view is that *n* denotes one such number nonetheless. Thus, the term "*n*" acts just like any other singular term, and so the rules for reasoning with this term are straightforward.

[29] I am indebted to conversations with Ofra Magidor here. See Breckenridge and Magidor (2012).

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To put the view metaphorically, whenever someone says something like, "let n be an even natural number," the world somehow serves up one such number, and the reference relation has "n" denote that served up number. Again, the application to the present block of issues is straightforward. At some point, a mathematician said, "let i be a square root of -1." The world then cooperated, and served up one of the two square roots. The introduced name "i" then denotes that number, and it has denoted the same number ever since. Fortunately, it does not matter which number was served up, since we have no way of knowing what number that is.

A fourth option derives from Priest's (2005) Meinongian view of intentional objects. According to Priest, we can directly designate a (nonexistent) intentional object just by thinking about it. For example, suppose that little Jackie Paper comes to hope that a magic dragon is lurking near his home. Then, on Priest's view, there is such a magic dragon (although it may not exist), and Jackie can name it, as he can name any other object, existent or otherwise, that he manages to get in touch with. Suppose he calls the dragon "Puff." Then he can say "I love that rascal Puff," and, if he is sincere, that statement will be true.

In fact, Priest takes mathematical objects to themselves be intentional objects, and he is a Meinongian about those too (Priest, 2005, chap. 7; see also Priest, 2003). So the same philosophical and semantic account applies in mathematics as it does to other intentional objects. On the relevant rational reconstruction, some time long ago, the mathematical community managed to think about—have singular thoughts about—a square root of -1, and they designated "i" to name *that* complex number. The term designates that same complex number to this day, since we intend that our use of the term should line up with theirs. It does not matter if we forgot which of the square roots is i.

An analogous view would be available to a Platonist who adopted a naive Platonic epistemological position that we have some sort of direct access to individual mathematical objects. On such a view, we can somehow intuit the two square roots of -1, individually. We can pick out one of them—with the mind's eye—and designate it as "i".

I do not claim that the foregoing accounts exhaust the possibilities, but they are at least simplified presentations of each of the alternatives that I know of. I think it is fair to note that they will invoke incredulous stares, due to their metaphysical and/or epistemological extravagance. The present account may not be as exciting, but I submit that it is more sober, and more in line with actual practice.

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