Probability in the Engineering and Informational Sciences, **25**, 2011, 219–235.
doi:10.1017/S0269964810000367

ACTIVE REDUNDANCY ALLOCATIONS IN SERIES SYSTEMS

NEERAJ MISRA, AMIT KUMAR MISRA, AND ISHWARI DUTT DHARIYAL
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur
Kanpur, 208 016, India
E-mails: neeraj@iitk.ac.in; amishra@iitk.ac.in; idd@iitk.ac.in

We consider the problem of optimally allocating one/two active spares in series systems. Lifetimes of different configurations are compared with respect to the reversed failure rate order, the failure rate order, the increasing concave order, and the stochastic precedence order.

1. INTRODUCTION

The performance of a system can often be improved by introducing some kind of redundancy into the system. The problem of optimally allocating redundant spares to the components of a system has been vastly studied in the literature (see Boland, El-Neweihi, and Proschan [3], Li and Hu [8], Misra, Dhariyal, and Gupta [9], Romera, Valdés, and Zequeira [11], Shaked and Shanthikumar [13], Singh and Misra [15], and Valdés and Zequeira [17] and references cited therein).

Two commonly used types of redundancy are the active redundancy and the standby redundancy. In active redundancy, available spares are attached in parallel to components of the system. In standby redundancy, spares are attached to components of the system in a manner that a spare starts functioning only after the failure of the component to which it is attached. Two different allocations of spares can be compared through stochastic orders between the lifetimes of corresponding systems. We recall some of the definitions of stochastic orders relevant to the context of the article.

Let X and Y be random variables having the common support $[0, \infty)$, distribution functions $F(\cdot)$ and $G(\cdot)$, the Lebesgue density functions $f(\cdot)$ and $g(\cdot)$, the failure rate functions $r(\cdot)$ and $\mu(\cdot)$, and the reversed failure rate functions $\tilde{r}(\cdot)$ and $\tilde{\mu}(\cdot)$, respectively. Let $\bar{F}(\cdot) = 1 - F(\cdot)$ and $\bar{G}(\cdot) = 1 - G(\cdot)$ be the survival functions of

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X and Y, respectively. The notation $X =_{\text{st}} Y$ will be used to indicate that the random variables X and Y have the same distribution. When we say that a function is increasing (decreasing), it means that the function is nondecreasing (nonincreasing). We also use the following notation: $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = (0, \infty)$. For a set, $A \subseteq \mathbb{R}$, $I_A(\cdot)$ will denote the indicator function of A. For a differentiable function $h(\cdot)$ defined on $\mathbb{R}$, $h'(\cdot)$ will denote its derivative.

DEFINITION 1.1: *X is said to be smaller than Y in the*

(i) usual stochastic order (written as $X \leq_{\text{st}} Y$) if $\bar{F}(t) \leq \bar{G}(t), \forall t \in \mathbb{R}_+$;

(ii) failure rate order (written as $X \leq_{\text{fr}} Y$) if $\bar{G}(t)/\bar{F}(t)$ is increasing in $t \in \mathbb{R}_+$, or equivalently if $r(t) \geq \mu(t), \forall t \in \mathbb{R}_+$;

(iii) reversed failure rate order (written as $X \leq_{\text{rfr}} Y$) if $G(t)/F(t)$ is increasing in $t \in (0, \infty)$, or equivalently if $\tilde{r}(t) \leq \tilde{\mu}(t), \forall t \in (0, \infty)$;

(iv) increasing concave order (written as $X \leq_{\text{icv}} Y$) if $E[\phi(X)] \leq E[\phi(Y)]$ for all increasing concave functions $\phi : \mathbb{R} \to \mathbb{R}$, for which the expectations exist;

(v) stochastic precedence order (written as $X \leq_{\text{sp}} Y$) if $P(Y < X) \leq P(X < Y)$.

For definitions and properties of various stochastic orders, one can refer to Müller and Stoyan [10] and Shaked and Shanthikumar [14]. For a notion of stochastic precedence order, readers can refer to Boland, Singh, and Cukic [4] and Singh and Misra [15].

Consider a series system consisting of components $C_1, C_2, \ldots, C_n$. Suppose that we also have spares R, R_1, and R_2. Let the nonnegative and statistically independent random variables $X_1, X_2, \ldots, X_n, X, Y_1$, and Y_2 denote the lifetimes of $C_1, C_2, \ldots, C_n, R, R_1$, and R_2, respectively. First, consider a model in which a spare R is available for active redundancy that can be allocated either to component C_1 or to component C_2. Then the lifetimes of two systems are given respectively by

$$V_1 = \wedge\{\vee(X_1, X), X_2, X_3, \ldots, X_n\} \quad \text{and} \quad V_2 = \wedge\{X_1, \vee(X_2, X), X_3, \ldots, X_n\}, \quad \textbf{(1.1)}$$

where the symbols $\wedge$ and $\vee$ represent min and max, respectively. Boland et al. [3] proved that if $X_1 \leq_{\text{st}} X_2$, then $V_2 \leq_{\text{st}} V_1$. Singh and Misra [15] established that if $X_1 \leq_{\text{st}} X_2$, then $V_2 \leq_{\text{sp}} V_1$. Boland et al. [3] and Singh and Misra [15] proved similar results for k-out-of-n systems. For two-component systems ($n = 2$), Li and Hu [8] showed that if $X_1 \leq_{\text{icv}} X_2$, then $V_2 \leq_{\text{icv}} V_1$ and that if $X_1 \leq_{\text{icv}} X_2$ and if X and X_1 (or X_2) have convex survival functions, then $V_2 \leq_{\text{sp}} V_1$.

Now, consider a model in which two spares, R_1 and R_2 (possibly identical), are available for active redundancy and, due to some constraints, only one of them can be used in the following manner: either R_1 with C_1, or R_2 with C_2. Then the lifetimes of

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two systems are given by

$$U_1 = \wedge\{\vee(X_1, Y_1), X_2, X_3, \ldots, X_n\} \quad \text{and} \quad U_2 = \wedge\{X_1, \vee(X_2, Y_2), X_3, \ldots, X_n\}, \tag{1.2}$$

respectively. Note that if Y_1 and Y_2 are identically distributed, then this model reduces to the one described in preceding paragraph. For two-component systems ($n = 2$), Valdés and Zequeira [17] proved that if $X_1 \leq_{\text{fr}} X_2, X_1 \leq_{\text{fr}} Y_1, Y_2 \leq_{\text{fr}} Y_1$, and the ratio of failure rate functions of X_2 to X_1 is decreasing, then $U_2 \leq_{\text{fr}} U_1$. They also provided conditions under which $U_2 \leq_{\text{st}} U_1$ holds. Romera et al. [11] proved that if $X_1 \leq_{\text{st}} X_2$ and $\bar{F}_2(t)\bar{G}_1(t) \geq \bar{F}_1(t)\bar{G}_2(t)$ for all $t \geq 0$, then $U_2 \leq_{\text{sp}} U_1$. Recently Valdés, Arango, and Zequeira [16] provided conditions under which $U_2 \leq_{\text{icv}} U_1$ holds. In Section 2 we will provide other conditions under which $U_2 \leq_{\text{fr}} U_1, U_2 \leq_{\text{icv}} U_1, U_2 \leq_{\text{sp}} U_1$, and $U_2 \leq_{\text{rfr}} U_1$ hold, thereby supplementing the results of Valdés and Zequeira [17] and Romera et al. [11].

Now, consider another model in which two spares, R_1 and R_2, are available for active redundancy in one of the following two ways: R_1 with C_1 and R_2 with C_2, or R_1 with C_2 and R_2 with C_1. Then the lifetimes of corresponding systems are given by

$$Z_1 = \wedge\{\vee(X_1, Y_1), \vee(X_2, Y_2), X_3, \ldots, X_n\} \tag{1.3}$$

and

$$Z_2 = \wedge\{\vee(X_1, Y_2), \vee(X_2, Y_1), X_3, \ldots, X_n\}, \tag{1.4}$$

respectively. Boland, El-Neweihi, and Proschan [2] proved that if $X_1 \leq_{\text{st}} X_2$ and $Y_1 \leq_{\text{st}} Y_2$, then $Z_1 \leq_{\text{st}} Z_2$. Romera et al. [11] proved that if $X_1 \leq_{\text{fr}} X_2$ and $Y_2 \leq_{\text{fr}} Y_1$, then $Z_2 \leq_{\text{sp}} Z_1$. Boland et al. [2] and Romera et al. [11] proved similar results for k-out-of-n systems. Recently, Valdés et al. [16] showed that $Z_2 \leq_{\text{sp}} Z_1$ also holds if $X_1 \leq_{\text{rfr}} X_2$ and $Y_2 \leq_{\text{rfr}} Y_1$. They also proved that if $X_i =_{\text{st}} Y_i, i = 1, 2$, and $X_1 \leq_{\text{rfr(fr)}} X_2$ or $X_2 \leq_{\text{rfr(fr)}} X_1$, then $Z_1 \leq_{\text{sp}} Z_2$. For $n = 2$, Valdés and Zequeira [18] proved that if $X_i =_{\text{st}} Y_i, i = 1, 2, X_1 \leq_{\text{fr}} X_2$, and the ratio of failure rate functions of X_2 to X_1 is decreasing, then $Z_1 \leq_{\text{fr}} Z_2$. In Section 3 we provide some additional conditions under which $Z_2 \leq_{\text{sp}} Z_1$ holds. We will also derive conditions under which $Z_1 \leq_{\text{fr}} Z_2, Z_1 \leq_{\text{rfr}} Z_2$, and $Z_1 \leq_{\text{sp}} Z_2$ hold when $X_i =_{\text{st}} Y_i, i = 1, 2$. These results supplement the results of Romera et al. [11], Valdés and Zequeira [18], and Valdés et al. [16].

Let $F_i(\cdot), \bar{F}_i(\cdot), f_i(\cdot), r_i(\cdot)$, and $\tilde{r}_i(\cdot)$ denote the distribution function, the survival function, the Lebesgue density function, the failure rate function, and the reversed failure rate function of $X_i, i = 1, 2, \ldots, n$. Furthermore, let $G_i(\cdot), \bar{G}_i(\cdot), g_i(\cdot), \mu_i(\cdot)$, and $\tilde{\mu}_i(\cdot)$ denote the distribution function, the survival function, the Lebesgue density function, the failure rate function, and the reversed failure rate function of $Y_i, i = 1, 2$. The common support of all the random variables considered in the article is assumed to be $\mathbb{R}_+$. Moreover, all of the random variables considered in the article are assumed to have a probability density with respect to the Lebesgue measure on $\mathbb{R}$.

https://doi.org/10.1017/S0269964810000367 Published online by Cambridge University Press

2. ALLOCATION OF ONE ACTIVE REDUNDANCY

In this section we will make stochastic comparisons between U_1 and U_2, defined by Eq. (1.2).

THEOREM 2.1: *Suppose that* $X_1 \leq_{\text{rfr}} X_2, X_1 \leq_{\text{rfr}} Y_1, Y_2 \leq_{\text{rfr}} Y_1$, *and* $F_1(t)G_1(t) \geq F_2(t)G_2(t)$ *for all* $t \in \mathbb{R}_+$. *Then, for* $n = 2$, $U_2 \leq_{\text{rfr}} U_1$.

PROOF: We will show that

$$h_1(t) = \frac{P(U_1 \leq t)}{P(U_2 \leq t)} = \frac{F_2(t) + F_1(t)G_1(t) - F_1(t)F_2(t)G_1(t)}{F_1(t) + F_2(t)G_2(t) - F_1(t)F_2(t)G_2(t)}$$

is increasing in $t \in (0, \infty)$. For $t \in (0, \infty)$, it can be easily verified that

$$\begin{aligned}
\Delta_1(t) &\equiv [P(U_2 \leq t)]^2 h_1'(t) \\
&= \tilde{r}_2(t)F_1(t)F_2(t)[1 - F_1(t)G_1(t) - G_1(t)G_2(t) + F_1(t)G_1(t)G_2(t)] \\
&\quad - \tilde{r}_1(t)F_1(t)F_2(t)[1 - F_2(t)G_2(t) - G_1(t)G_2(t) + F_2(t)G_1(t)G_2(t)] \\
&\quad + \tilde{\mu}_1(t)F_1(t)\bar{F}_2(t)G_1(t)[F_1(t) + F_2(t)G_2(t) - F_1(t)F_2(t)G_2(t)] \\
&\quad - \tilde{\mu}_2(t)\bar{F}_1(t)F_2(t)G_2(t)[F_2(t) + F_1(t)G_1(t) - F_1(t)F_2(t)G_1(t)] \\
&\geq \tilde{r}_1(t)F_1(t)F_2(t)[G_1(t)G_2(t)(F_1(t) - F_2(t)) - F_1(t)G_1(t) + F_2(t)G_2(t)] \\
&\quad + \tilde{\mu}_1(t)[F_1^2(t)G_1(t)(1 - F_2(t)) - F_2^2(t)G_2(t)(1 - F_1(t))] \\
&\qquad \text{(using } X_1 \leq_{\text{rfr}} X_2 \text{ and } Y_2 \leq_{\text{rfr}} Y_1\text{)} \\
&\geq \tilde{r}_1(t)F_1(t)F_2(t)[G_1(t)G_2(t)(F_1(t) - F_2(t)) - F_1(t)G_1(t) + F_2(t)G_2(t)] \\
&\quad + \tilde{\mu}_1(t)F_1(t)G_1(t)(F_1(t) - F_2(t)) \\
&\qquad \text{(since } F_1(t)G_1(t) \geq F_2(t)G_2(t), \forall t \in \mathbb{R}_+\text{)}.
\end{aligned}$$

Conditions $X_1 \leq_{\text{rfr}} Y_1, X_1 \leq_{\text{rfr}} X_2$, and $Y_2 \leq_{\text{rfr}} Y_1$ imply that $\tilde{\mu}_1(t) \geq \tilde{r}_1(t), F_1(t) \geq F_2(t)$, and $G_2(t) \geq G_1(t), \forall t \in \mathbb{R}_+$. Therefore,

$$\Delta_1(t) \geq \tilde{r}_1(t)F_1(t)G_1(t)[F_1(t) - F_2(t)][1 - F_2(t) + F_2(t)G_2(t)] \geq 0, \quad \forall t \in (0, \infty);$$

that is, $h_1(t)$ is increasing in $t \in (0, \infty)$. ■

Remark 2.1: The condition $F_1(t)G_1(t) \geq F_2(t)G_2(t), \forall t \in \mathbb{R}_+$, is equivalent to the condition $\vee\{X_1, Y_1\} \leq_{\text{st}} \vee\{X_2, Y_2\}$.

COROLLARY 2.1: *Suppose that* $X_1 \leq_{\text{rfr}} X_2, X_1 \leq_{\text{rfr}} Y_1, Y_2 \leq_{\text{rfr}} Y_1$, *and* $\tilde{\mu}_1(t) - \tilde{\mu}_2(t) \leq \tilde{r}_2(t) - \tilde{r}_1(t)$ *for all* $t \in (0, \infty)$. *Then, for* $n = 2$, $U_2 \leq_{\text{rfr}} U_1$.

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PROOF: Consider the function

$$\phi(t) = \ln(F_1(t)G_1(t)) - \ln(F_2(t)G_2(t)), \qquad t \in (0, \infty).$$

Then

$$\phi'(t) = \tilde{\mu}_1(t) - \tilde{\mu}_2(t) - \tilde{r}_2(t) + \tilde{r}_1(t) \leq 0, \quad \forall t \in (0, \infty).$$

Therefore, $\phi(t) \geq \lim_{x\to\infty} \phi(x) = 0, \forall t \in (0, \infty)$; that is, $F_1(t)G_1(t) \geq F_2(t)G_2(t)$, $\forall t \in (0, \infty)$. The result now follows from Theorem 2.1. ■

The following corollary follows from Theorem 2.1 by taking $Y_i =_{\text{st}} X, i = 1, 2$, and on using that $X_1 \leq_{\text{rfr}} X_2$ implies $X_1 \leq_{\text{st}} X_2$.

COROLLARY 2.2: *Suppose that* $X_1 \leq_{\text{rfr}} X_2$, *and* $X_1 \leq_{\text{rfr}} X$. *Then, for* $n = 2, V_2 \leq_{\text{rfr}} V_1$.

The following example illustrates a few situations in which assumptions of Theorem 2.1 hold.

Example 2.1: Let $\bar{F}_i(t) = e^{-\lambda_i t}$ and $\bar{G}_i(t) = e^{-\mu_i t}, t \geq 0, \lambda_i > 0, \mu_i > 0, i = 1, 2$. Using Theorem 3.1 of Dykstra, Kochar, and Rojo [5], it follows that if $\min(\lambda_1, \mu_1) \geq \min(\lambda_2, \mu_2)$ and $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$, then $\vee\{X_1, Y_1\} \leq_{\text{lr}} \vee\{X_2, Y_2\}$ (and hence, in particular, $\vee\{X_1, Y_1\} \leq_{\text{st}} \vee\{X_2, Y_2\}$). Therefore, the conditions of Theorem 2.1 are satisfied if $\lambda_2 \leq \mu_1 \leq \min(\lambda_1, \mu_2)$ and $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$. One such choice of parameters is $\lambda_1 = 1.5, \lambda_2 = 0.5, \mu_1 = 1$, and $\mu_2 = 2$.

For two-component ($n = 2$) systems, Valdés and Zequeira [17, Prop. 1] proved that if $X_1 \leq_{\text{fr}} X_2, X_1 \leq_{\text{fr}} Y_1, Y_2 \leq_{\text{fr}} Y_1$, and $r_2(t)/r_1(t)$ is decreasing on $\mathbb{R}_+$, then $U_2 \leq_{\text{fr}} U_1$. The following theorem, as we will see later (Corollary 2.3), generalizes this result of Valdés and Zequeira [17].

THEOREM 2.2: *Suppose that* $X_1 \leq_{\text{st}} X_2$, $X_1 \leq_{\text{fr}} Y_1$, $Y_2 \leq_{\text{fr}} Y_1$, *and* $r_1(t)F_2(t) \geq r_2(t)F_1(t)$ *for all* $t \in \mathbb{R}_+$. *Then* $U_2 \leq_{\text{fr}} U_1$.

PROOF: We will show that

$$h_2(t) = \frac{P(U_1 > t)}{P(U_2 > t)} = \frac{\bar{F}_2(t)[1 - F_1(t)G_1(t)]}{\bar{F}_1(t)[1 - F_2(t)G_2(t)]}$$

is increasing in $t \in \mathbb{R}_+$. For $t \in \mathbb{R}_+$, it can be easily verified that

$$\begin{aligned}\Delta_2(t) &\equiv \frac{[P(U_2 > t)]^2 h_2'(t)}{\bar{F}_1(t)\bar{F}_2(t)\bar{G}_2(t)} \\ &= \bar{G}_1(t)[r_1(t)F_2(t) - r_2(t)F_1(t) + F_1(t)F_2(t)(\mu_2(t) - \mu_1(t))] \\ &\quad + \bar{F}_2(t)\frac{\bar{G}_1(t)}{\bar{G}_2(t)}(r_1(t) - \mu_1(t)F_1(t)) - \bar{F}_1(t)(r_2(t) - \mu_2(t)F_2(t)).\end{aligned}$$

We have $r_1(t)F_2(t) - r_2(t)F_1(t) + F_1(t)F_2(t)(\mu_2(t) - \mu_1(t)) \geq 0$, $r_1(t) - \mu_1(t)F_1(t) \geq 0, \bar{G}_1(t) \geq \bar{F}_1(t)$, and $\bar{G}_1(t) \geq \bar{G}_2(t)$, for every $t \in \mathbb{R}_+$. Therefore, for all

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$t \in \mathbb{R}_+$,

$$\Delta_2(t) \geq r_1(t)[1 - F_1(t)F_2(t)] - r_2(t)\bar{F}_1(t)[1 + F_1(t)] + \bar{F}_1(t)F_1(t)F_2(t)(\mu_2(t) - \mu_1(t)) - F_1(t)\bar{F}_2(t)\mu_1(t) + \bar{F}_1(t)F_2(t)\mu_2(t).$$

Using $r_1(t)F_2(t) \geq r_2(t)F_1(t)$, for all $t \in \mathbb{R}_+$, we get

$$\begin{aligned}\Delta_2(t)F_1(t) &\geq r_1(t)[F_1(t) - F_2(t)] + \mu_2(t)\bar{F}_1(t)F_1(t)F_2(t)(1 + F_1(t)) \\ &\quad - \mu_1(t)F_1^2(t)[\bar{F}_1(t)F_2(t) + \bar{F}_2(t)] \\ &\geq \mu_1(t)\bar{F}_1(t)[F_1(t) - F_2(t)] \\ &\qquad \text{(since } X_1 \leq_{\text{st}} X_2, X_1 \leq_{\text{fr}} Y_1, \text{ and } Y_2 \leq_{\text{fr}} Y_1\text{)} \\ &\geq 0 \qquad \text{(since } X_1 \leq_{\text{st}} X_2\text{)}.\end{aligned}$$

We conclude that $\Delta_2(t) \geq 0, \forall t \in \mathbb{R}_+$; that is, $h_2(t)$ is increasing in $t \in \mathbb{R}_+$. ■

Remark 2.2:

(i) The set of conditions $X_1 \leq_{\text{st}} X_2$ and $r_1(t)F_2(t) \geq r_2(t)F_1(t), \forall t \in \mathbb{R}_+$, is equivalent to the set of conditions $X_1 \leq_{\text{fr}} X_2$ and $r_1(t)F_2(t) \geq r_2(t)F_1(t), \forall t \in \mathbb{R}_+$.

(ii) The hypotheses of Theorem 2.2 are satisfied if $X_1 \leq_{\text{st}} X_2, X_2 \leq_{\text{fr}} Y_2, Y_2 \leq_{\text{fr}} Y_1$, and $r_1(t)F_2(t) \geq r_2(t)F_1(t), \forall t \in \mathbb{R}_+$.

COROLLARY 2.3: *Suppose that* $X_1 \leq_{\text{st}} X_2, X_1 \leq_{\text{fr}} Y_1, Y_2 \leq_{\text{fr}} Y_1$, *and* $r_2(t)/r_1(t)$ *is decreasing in* $t \in \mathbb{R}_+$. *Then* $U_2 \leq_{\text{fr}} U_1$.

PROOF: The result would follow if we show that the given conditions imply $r_1(t)F_2(t) \geq r_2(t)F_1(t)$ for all $t \in \mathbb{R}_+$. Fix $t \in \mathbb{R}_+$. Then, for some $\xi \in (0, t)$,

$$\frac{F_2(t)}{F_1(t)} = \frac{F_2(t) - F_2(0)}{F_1(t) - F_1(0)} = \frac{f_2(\xi)}{f_1(\xi)} = \frac{r_2(\xi)\bar{F}_2(\xi)}{r_1(\xi)\bar{F}_1(\xi)} \geq \frac{r_2(t)}{r_1(t)}$$

since $X_1 \leq_{\text{st}} X_2, \xi \in (0, t)$ and $r_2(t)/r_1(t)$ is decreasing in $t \in \mathbb{R}_+$. ■

The following corollary is immediate on taking $Y_i =_{\text{st}} X$, $i = 1, 2$.

COROLLARY 2.4: *Suppose that one (or both) of the following conditions holds:*

(a) $X_1 \leq_{\text{st}} X_2, X_1 \leq_{\text{fr}} X$, *and* $r_1(t)F_2(t) \geq r_2(t)F_1(t)$ *for all* $t \in \mathbb{R}_+$;

(b) $X_1 \leq_{\text{st}} X_2, X_1 \leq_{\text{fr}} X$, *and* $r_2(t)/r_1(t)$ *is decreasing in* $t \in \mathbb{R}_+$.

Then $V_2 \leq_{\text{fr}} V_1$.

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Valdés and Zequeira [17, Prop. 2] also proved that if $X_1 \leq_{\text{fr}} X_2, X_1 \leq_{\text{fr}} Y_1, Y_2 \leq_{\text{fr}} Y_1$, and $\alpha(t) = r_2(t)/r_1(t) \leq 1 - m^{m/(1-m)} + m^{1/(1-m)}, \forall t > 0$, where $m = \inf_{t>0} \alpha(t)$, then $U_2 \leq_{\text{fr}} U_1$. The following example gives a practical case in which the conditions of Propositions 1 and 2 of Valdés and Zequeira [17], described earlier, are not satisfied, but the result of Theorem 2.2 is valid.

Example 2.2: Let $r_1(t) = at + b$ and $r_2(t) = ct + d, t \in \mathbb{R}_+$, with $a \geq c > 0, b \geq d > 0$, and $bc - ad > 0$. Suppose that $\mu_1(t)$ and $\mu_2(t)$ are such that $r_1(t) \geq \mu_1(t)$ and $\mu_2(t) \geq \mu_1(t), t \in \mathbb{R}_+$. It can be easily verified that $r_1(t) \geq r_2(t)$ and $\alpha(t)$ is strictly increasing in $t \in \mathbb{R}_+$. Thus, the conditions of Proposition 1 of Valdés and Zequeira [17] are not satisfied. Furthermore, if we take $\inf_{t>0} \alpha(t) = d/b = m = 0.3$ and $\sup_{t>0} \alpha(t) = c/a > 0.58$, then the conditions of Proposition 2 of Valdés and Zequeira [17] are not satisfied (see Example 3 of Valdés and Zequeira [17]). In the sequel, we will show that if $bd > c$ also holds, then the result of Theorem 2.2 is valid. One particular set of a, b, c, and d satisfying these assumptions is $a = 5, b = 10, c = 3$, and $d = 3$. For $t \in \mathbb{R}_+$, consider

$$\begin{aligned}\psi(t) &= r_1(t)F_2(t) - r_2(t)F_1(t) \\ &= (at + b)(1 - e^{-(1/2)ct^2 - dt}) - (ct + d)(1 - e^{-(1/2)at^2 - bt}).\end{aligned}$$

Then

$$\begin{aligned}\psi'(t) &= (at + b)(ct + d)(e^{-(1/2)ct^2 - dt} - e^{-(1/2)at^2 - bt}) + a(1 - e^{-(1/2)ct^2 - dt}) \\ &\quad - c(1 - e^{-(1/2)at^2 - bt}) \\ &\geq [act^2 + (ad + bc)t + bd - c](e^{-(1/2)ct^2 - dt} - e^{-(1/2)at^2 - bt}) \qquad (\text{since } a \geq c) \\ &\geq 0, \quad \forall t \in \mathbb{R}_+ \qquad (\text{since } a \geq c > 0, b \geq d > 0, bd - c > 0).\end{aligned}$$

Thus, $\psi(t)$ is increasing in $t \in \mathbb{R}_+$ and, hence, $\psi(t) \geq \psi(0) = 0, \forall t \in \mathbb{R}_+$.

Remark 2.3: The assumption that $r_2(t)/r_1(t)$ is a decreasing function on $\mathbb{R}_+$ is related to the notion of relative aging. Let F and G be the distribution functions of the random variables X and Y, respectively. According to Kalashnikov and Rachev [6], X is said to be aging faster than Y (written as $X \prec_c Y$) if the random variable $Z = -\ln(\bar{G}(X))$ has an increasing failure rate (IFR) distribution. This stochastic order is based on the notion of relative aging of two life distributions. If the failure rate functions of X and Y are $r(\cdot)$ and $\mu(\cdot)$, respectively, it can be verified that $X \prec_c Y$ if and only if $\mu(\cdot)/r(\cdot)$ is a decreasing function. Sengupta and Deshpande [12] further studied this order and proposed two new notions of relative aging and studied their interrelations. Lai and Xie [7] studied the relative aging properties of two parallel systems.

Recently, Valdés et al. [16, Prop. 1] proved that $U_2 \leq_{\text{icv}} U_1$ holds provided either of the following two conditions hold: (a) $X_1 \leq_{\text{icv}} X_2$ and $Y_2 \leq_{\text{st}} Y_1$ or (b) $X_1 \leq_{\text{icv}}$

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$X_2, X_1 \leq_{\text{st}} Y_1$, and $Y_2 \leq_{\text{st}} X_2$. In the following theorem and subsequent corollary we generalize/supplement the results of Valdés et al. [16].

THEOREM 2.3: *Suppose that any one of the following conditions holds:*

(a) $X_1 \leq_{\text{icv}} X_2$ *and* $\bar{F}_2(t)\bar{G}_1(t) \geq \bar{F}_1(t)\bar{G}_2(t), \forall t \geq 0$;
(b) $Y_1 \leq_{\text{icv}} Y_2$ *and* $\bar{F}_2(t)\bar{G}_1(t) \geq \bar{F}_1(t)\bar{G}_2(t), \forall t \geq 0$;
(c) $X_1 \leq_{\text{icv}} X_2$ *and* $F_2(t)G_2(t) \geq F_1(t)G_1(t), \forall t \geq 0$.

Then $U_2 \leq_{\text{icv}} U_1$.

PROOF: Let $\bar{H}(t) = \prod_{i=3}^{n} \bar{F}_i(t), t \in \mathbb{R}$. The result would follow (see Shaked and Shanthikumar [14, p. 182]) if we show that

$$\Delta_3(x) = \int_0^x P(U_2 \leq t)\, dt - \int_0^x P(U_1 \leq t)\, dt \geq 0, \quad \forall x \in \mathbb{R}_+.$$

(a) For $x \in \mathbb{R}_+$,

$$\Delta_3(x) = \int_0^x [1 - F_1(t)G_1(t)]\bar{F}_2(t)\bar{H}(t)\, dt - \int_0^x [1 - F_2(t)G_2(t)]\bar{F}_1(t)\bar{H}(t)\, dt \tag{2.1}$$

$$= \int_0^x [F_1(t)\bar{F}_2(t)\bar{G}_1(t) - F_2(t)\bar{F}_1(t)\bar{G}_2(t)]\bar{H}(t)\, dt \tag{2.2}$$

$$\geq \int_0^x [\bar{F}_2(t) - \bar{F}_1(t)]\bar{F}_1(t)\bar{G}_2(t)\bar{H}(t)\, dt$$

(since $\bar{F}_2(t)\bar{G}_1(t) \geq \bar{F}_1(t)\bar{G}_2(t), \forall t \geq 0$)

$$\geq 0,$$

where the last inequality follows from the assumption that $X_1 \leq_{\text{icv}} X_2$ and Theorem 7.3 of Barlow and Proschan [1, p. 121].

(b) Using Eq. (2.2), we have, for $x \in \mathbb{R}_+$,

$$\Delta_3(x) = \int_0^x [\bar{F}_2(t)\bar{G}_1(t) - \bar{F}_1(t)\bar{G}_2(t)]\bar{H}(t)\, dt + \int_0^x \bar{F}_1(t)\bar{F}_2(t)\bar{H}(t)[\bar{G}_2(t) - \bar{G}_1(t)]\, dt \geq 0,$$

where the inequality follows from the assumptions $Y_1 \leq_{\text{icv}} Y_2$, $\bar{F}_2(t)\bar{G}_1(t) \geq \bar{F}_1(t)\bar{G}_2(t), \forall t \geq 0$, and Theorem 7.3 of Barlow and Proschan [1, p. 121].

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(c) Using Eq. (2.1), we have, for $x \in \mathbb{R}_+$,

$$\Delta_3(x) \geq \int_0^x [1 - F_2(t)G_2(t)][\bar{F}_2(t) - \bar{F}_1(t)]\bar{H}(t)\,dt.$$

Again using $X_1 \leq_{\text{icv}} X_2$ and Theorem 7.3 of Barlow and Proschan [1, p. 121], we get $\Delta_3(x) \geq 0, \forall x \in \mathbb{R}_+$. ■

As a consequence of Theorem 2.3, we have the following corollary.

COROLLARY 2.5: *Suppose that one (or more) of the following conditions holds:*

(a) $X_1 \leq_{\text{icv}} X_2, X_1 \leq_{\text{st}} Y_1$, *and* $Y_2 \leq_{\text{st}} X_2$;
(b) $Y_1 \leq_{\text{icv}} Y_2, X_1 \leq_{\text{st}} Y_1$, *and* $Y_2 \leq_{\text{st}} X_2$;
(c) $X_1 \leq_{\text{icv}} X_2, X_2 \leq_{\text{st}} Y_1$, *and* $Y_2 \leq_{\text{st}} X_1$;
(d) $X_1 \leq_{\text{icv}} X_2$ *and* $r_2(t) - r_1(t) \leq \mu_2(t) - \mu_1(t), \forall t \geq 0$;
(e) $Y_1 \leq_{\text{icv}} Y_2$ *and* $r_2(t) - r_1(t) \leq \mu_2(t) - \mu_1(t), \forall t \geq 0$;
(f) $X_1 \leq_{\text{icv}} X_2$ *and* $\tilde{r}_2(t) - \tilde{r}_1(t) \leq \tilde{\mu}_1(t) - \tilde{\mu}_2(t), \forall t > 0$.

Then $U_2 \leq_{\text{icv}} U_1$.

PROOF: Parts (a) and (b) follow from Theorem 2.3(a) and 2.3(b) on using the fact that

$$X_1 \leq_{\text{st}} Y_1 \text{ and } Y_2 \leq_{\text{st}} X_2 \Leftrightarrow \bar{F}_1(t) \leq \bar{G}_1(t) \text{ and } \bar{G}_2(t) \leq \bar{F}_2(t), \quad \forall t \geq 0$$
$$\Rightarrow \bar{F}_1(t)\bar{G}_2(t) \leq \bar{F}_2(t)\bar{G}_1(t), \quad \forall t \geq 0.$$

Part (c) follows from Theorem 2.3(c) on using the fact that

$$X_2 \leq_{\text{st}} Y_1 \text{ and } Y_2 \leq_{\text{st}} X_1 \Leftrightarrow F_2(t) \geq G_1(t) \text{ and } G_2(t) \geq F_1(t), \quad \forall t \geq 0$$
$$\Rightarrow F_2(t)G_2(t) \geq F_1(t)G_1(t), \quad \forall t \geq 0.$$

Parts (d), (e), and (f) follow from Theorem 2.3 on using the idea in the proof of Corollary 2.1. ■

The following example illustrates some situations in which the conditions of Proposition 1 of Valdés et al. [16] are not satisfied, but the result of Theorem 2.3 is valid.

Example 2.3: Let $\bar{F}_i(t) = e^{-\lambda_i t}$ and $\bar{G}_i(t) = e^{-\mu_i t}, t \geq 0, \lambda_i > 0, \mu_i > 0, i = 1, 2$.

(i) Suppose that $\mu_2 < \min(\mu_1, \lambda_2)$ and $\lambda_2 + \mu_1 \leq \lambda_1 + \mu_2$. Then $X_1 \leq_{\text{icv}} X_2, X_2 \leq_{\text{st}} Y_2, Y_1 \leq_{\text{st}} Y_2$, and $\bar{F}_2(t)\bar{G}_1(t) - \bar{F}_1(t)\bar{G}_2(t) = e^{-(\lambda_2+\mu_1)t} - e^{-(\lambda_1+\mu_2)t} \geq 0, \forall t \geq 0$. Thus, the conditions of Proposition 1 of Valdés et al.

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[16] are not satisfied, but parts (a) and (b) of Theorem 2.3 are satisfied. One such choice of parameters is $\lambda_1 = 1.5, \lambda_2 = 0.5, \mu_1 = 1.25$, and $\mu_2 = 0.25$.

(ii) Suppose that $\mu_1 \leq \min(\lambda_2, \mu_2)$ and $\lambda_1 + \mu_1 = \lambda_2 + \mu_2$. Then $X_1 \leq_{\text{icv}} X_2$ and, as discussed in Example 2.1, $F_2(t)G_2(t) \geq F_1(t)G_1(t), \forall t \geq 0$. Therefore, the assumptions of Theorem 2.3(c) hold. One such choice of parameters is $\lambda_1 = 1.5, \lambda_2 = 0.5, \mu_1 = 0.25$, and $\mu_2 = 1.25$.

Romera et al. [11] proved that if $X_1 \leq_{\text{st}} X_2$ and $\bar{F}_2(t)\bar{G}_1(t) \geq \bar{F}_1(t)\bar{G}_2(t)$ for all $t \geq 0$, then $U_2 \leq_{\text{sp}} U_1$. We know that the usual stochastic order is stronger than the increasing concave order. So a natural question arises: Can we replace the usual stochastic order between X_1 and X_2 by the increasing concave order? The following theorem deals with a few situations in which the answer of this question is affirmative.

THEOREM 2.4: *Suppose that* $X_1 \leq_{\text{icv}} X_2$, $Y_2 \leq_{\text{icv}} Y_1$, *and* $Y_2, X_1, X_3, X_4, \ldots, X_n$ *(or* $Y_1, X_2, X_3, X_4, \ldots, X_n$*) have convex survival functions on* $\mathbb{R}_+$. *Then* $U_2 \leq_{\text{sp}} U_1$.

PROOF: Let $\bar{H}(\cdot)$ be the survival function of $T = \wedge\{X_3, X_4, \ldots, X_n\}$. From Romera et al. [11, Prop. 1(b)], we have

$$[U_1 > U_2] = [X_1 < \wedge\{Y_1, X_2, T\}].$$

By symmetry,

$$[U_2 > U_1] = [X_2 < \wedge\{Y_2, X_1, T\}].$$

Therefore,

$$\begin{aligned}\Delta_4 &\equiv P(U_1 > U_2) - P(U_2 > U_1)\\ &= E\left[\bar{G}_1(X_1)\bar{H}(X_1)I_{(0,\infty)}(X_2 - X_1) - \bar{G}_2(X_2)\bar{H}(X_2)I_{(0,\infty)}(X_1 - X_2)\right]. \qquad \textbf{(2.3)}\end{aligned}$$

Consider the following two cases:

Case I: $Y_2, X_1, X_3, X_4, \ldots, X_n$ have convex survival functions. Using Eq. (2.3), we can write

$$\Delta_4 = E[\psi_1(X_2)],$$

where

$$\psi_1(t) = E\left[\bar{G}_1(\hat{X}_1)\bar{H}(\hat{X}_1)I_{(0,\infty)}(t - \hat{X}_1) - \bar{G}_2(t)\bar{H}(t)I_{(0,\infty)}(\hat{X}_1 - t)\right], \qquad t \geq 0, \qquad \textbf{(2.4)}$$

and $\hat{X}_1$ is an independent copy of X_1. Clearly,

$$E[\psi_1(X_1)] = \int_0^\infty [\bar{G}_1(s) - \bar{G}_2(s)]\bar{H}(s)\bar{F}_1(s)f_1(s)ds.$$

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Since $Y_2 \leq_{\text{icv}} Y_1$, we have

$$\int_0^y [\bar{G}_1(s) - \bar{G}_2(s)]\, ds \geq 0, \qquad \forall y \geq 0.$$

The assumption that X_1 has convex survival function on $\mathbb{R}_+$ implies that $f_1(\cdot)$ is a decreasing function on $\mathbb{R}_+$. Using Lemma 7.1(b) of Barlow and Proschan [1, p. 120], it follows that $E[\psi_1(X_1)] \geq 0$. From Eq. (2.4), we have

$$\psi_1(t) = \int_0^t \bar{G}_1(s)\bar{H}(s)f_1(s)ds - \bar{G}_2(t)\bar{H}(t)\bar{F}_1(t),$$

which is an increasing function of $t \in \mathbb{R}_+$. Additionally,

$$\psi_1'(t) = [\bar{G}_1(t) + \bar{G}_2(t)]\bar{H}(t)f_1(t) + g_2(t)\bar{H}(t)\bar{F}_1(t) + \bar{G}_2(t)\bar{F}_1(t)\sum_{i=3}^{n}\left\{\left(\prod_{\substack{j=3\\ j\neq i}}^{n}\bar{F}_j(t)\right)f_i(t)\right\}.$$

The assumption that $Y_2, X_1, X_3, X_4, \ldots, X_n$ have convex survival functions on $\mathbb{R}_+$ (i.e., $g_2(\cdot)$ and $f_i(\cdot), i = 1, 3, 4, \ldots, n$ are decreasing on $\mathbb{R}_+$) implies that $\psi_1'(t)$ is a decreasing function of t on $\mathbb{R}_+$. Therefore, $\psi_1(\cdot)$ is an increasing concave function on $\mathbb{R}_+$. Now, the assumption $X_1 \leq_{\text{icv}} X_2$ implies that

$$\Delta_4 = E\big[\psi_1(X_2)\big] \geq E\big[\psi_1(X_1)\big] \geq 0;$$

that is, $U_2 \leq_{\text{sp}} U_1$.

Case II: $Y_1, X_2, X_3, \ldots, X_n$ have convex survival functions. From Eq. (2.3), we have

$$\Delta_4 = -E[\psi_2(X_1)],$$

where

$$\psi_2(t) = E[\bar{G}_2(\hat{X}_2)\bar{H}(\hat{X}_2)I_{(0,\infty)}(t - \hat{X}_2) - \bar{G}_1(t)\bar{H}(t)I_{(0,\infty)}(\hat{X}_2 - t)], \qquad t \geq 0, \tag{2.5}$$

and $\hat{X}_2$ is an independent copy of X_2. Clearly,

$$E[\psi_2(X_2)] = \int_0^\infty [\bar{G}_2(s) - \bar{G}_1(s)]\bar{H}(s)\bar{F}_2(s)f_2(s)ds.$$

On proceeding similar to Case I, we can show that $E[\psi_2(X_2)] \leq 0$ and that $\psi_2(\cdot)$ is an increasing concave function on $\mathbb{R}_+$. Since $X_1 \leq_{\text{icv}} X_2$, we get

$$\Delta_4 = -E[\psi_2(X_1)] \geq -E[\psi_2(X_2)] \geq 0;$$

that is, $U_2 \leq_{\text{sp}} U_1$. ■

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3. ALLOCATION OF TWO ACTIVE REDUNDANCIES

In this section we will deal with stochastic comparisons of Z_1 and Z_2 (defined by Eqs. (1.3) and (1.4)), thereby further extending the results of Romera et al. [11], Valdés and Zequeira [18], and Valdés et al. [16].

For $n = 2$, Valdés and Zequeira [18] proved that if $X_i =_{\text{st}} Y_i, i = 1, 2, X_1 \leq_{\text{fr}} X_2$, and $r_2(t)/r_1(t)$ is decreasing on $\mathbb{R}_+$, then $Z_1 \leq_{\text{fr}} Z_2$. The following theorem, as we will see later (Corollary 3.1), generalizes this result of Valdés and Zequeira [18].

THEOREM 3.1: *Suppose that* $X_i =_{\text{st}} Y_i, i = 1, 2, X_1 \leq_{\text{st}} X_2$, *and* $r_1(t)F_2(t) \geq r_2(t)F_1(t)$ *for all* $t \in \mathbb{R}_+$. *Then* $Z_1 \leq_{\text{fr}} Z_2$.

PROOF: We need to show that

$$h_3(t) = \frac{P(Z_2 > t)}{P(Z_1 > t)} = \frac{[1 - F_1(t)F_2(t)]^2}{[1 - F_1^2(t)][1 - F_2^2(t)]} = 1 + \frac{[F_1(t) - F_2(t)]^2}{[1 - F_1^2(t)][1 - F_2^2(t)]}$$

is increasing in $t \in \mathbb{R}_+$. For $t \in \mathbb{R}_+$, it can be easily verified that

$$\begin{aligned}\Delta_5(t) &\equiv [P(Z_1 > t)]^2 h_3'(t) \\ &= 2r_1(t)\bar{F}_1(t)[1 - F_1(t)F_2(t)][1 - F_2^2(t)][F_1(t) - F_2(t)] \\ &\quad - 2r_2(t)\bar{F}_2(t)[1 - F_1(t)F_2(t)][1 - F_1^2(t)][F_1(t) - F_2(t)].\end{aligned}$$

Since $r_1(t)F_2(t) \geq r_2(t)F_1(t)$ for all $t \in \mathbb{R}_+$ and $X_1 \leq_{\text{st}} X_2$, we conclude that

$$\Delta_5(t)F_2(t) \geq 2r_2(t)[1 - F_1(t)F_2(t)][F_1(t) - F_2(t)]^2\bar{F}_1(t)\bar{F}_2(t) \geq 0, \quad \forall t \in \mathbb{R}_+;$$

that is, $h_3(t)$ is increasing in $t \in \mathbb{R}_+$. ■

For some comments on the hypotheses of Theorem 3.1, the reader can refer to Remark 2.2(i). The proof of the following corollary is similar to that of Corollary 2.3.

COROLLARY 3.1: *Suppose that* $X_i =_{\text{st}} Y_i, i = 1, 2, X_1 \leq_{\text{st}} X_2$, *and* $r_2(t)/r_1(t)$ *is decreasing in* $t \in \mathbb{R}_+$. *Then* $Z_1 \leq_{\text{fr}} Z_2$.

THEOREM 3.2: *Suppose that* $X_i =_{\text{st}} Y_i, i = 1, 2, X_1 \leq_{\text{st}} X_2$, *and* $\tilde{r}_2(t)\bar{F}_1(t) \geq \tilde{r}_1(t)\bar{F}_2(t)$ *for all* $t \in (0, \infty)$. *Then, for* $n = 2$, $Z_1 \leq_{\text{rfr}} Z_2$.

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PROOF: We will show that

$$h_4(t) = \frac{P(Z_2 \le t)}{P(Z_1 \le t)} = \frac{1 - [1 - F_1(t)F_2(t)]^2}{1 - [1 - F_1^2(t)][1 - F_2^2(t)]}$$
$$= 1 - \frac{[F_1(t) - F_2(t)]^2}{F_1^2(t) + F_2^2(t) - F_1^2(t)F_2^2(t)}$$

is increasing in $t \in (0, \infty)$. For $t \in (0, \infty)$, consider

$$\begin{aligned}\Delta_6(t) &= [P(Z_1 \le t)]^2 h_4'(t) \\ &= 2\tilde{r}_2(t)F_1(t)F_2(t)[F_1(t) - F_2(t)][F_1(t) + F_2(t) - F_1^2(t)F_2(t)] \\ &\quad - 2\tilde{r}_1(t)F_1(t)F_2(t)[F_1(t) - F_2(t)][F_1(t) + F_2(t) - F_1(t)F_2^2(t)].\end{aligned}$$

Using $\tilde{r}_2(t)\bar{F}_1(t) \ge \tilde{r}_1(t)\bar{F}_2(t)$ for all $t \in (0, \infty)$ and $X_1 \le_{\text{st}} X_2$, it follows that

$$\Delta_6(t)\bar{F}_1(t) \ge 2\tilde{r}_1(t)F_1(t)F_2(t)[F_1(t) - F_2(t)]^2[F_1(t) + F_2(t) - F_1(t)F_2(t)] \ge 0;$$

that is, $h_4(t)$ is increasing in $t \in (0, \infty)$. ■

Remark 3.1: The set of conditions $X_1 \le_{\text{st}} X_2$ and $\tilde{r}_2(t)\bar{F}_1(t) \ge \tilde{r}_1(t)\bar{F}_2(t), \forall t \in (0, \infty)$ is equivalent to the set of conditions $X_1 \le_{\text{rfr}} X_2$ and $\tilde{r}_2(t)\bar{F}_1(t) \ge \tilde{r}_1(t)\bar{F}_2(t), \forall t \in (0, \infty)$.

As a consequence of Theorem 3.2, we have the following corollary.

COROLLARY 3.2: *Suppose that* $X_i =_{\text{st}} Y_i, i = 1, 2, X_1 \le_{\text{st}} X_2$, *and* $\tilde{r}_2(t)/\tilde{r}_1(t)$ *is decreasing in* $t \in (0, \infty)$. *Then, for* $n = 2, Z_1 \le_{\text{rfr}} Z_2$.

PROOF: It is enough to show that, under given conditions, $\tilde{r}_2(t)\bar{F}_1(t) \ge \tilde{r}_1(t)\bar{F}_2(t)$ for all $t \in (0, \infty)$ holds. Fix $t \in (0, \infty)$ and let $A > t$. Then, for some $\xi \in (t, A)$,

$$\frac{\bar{F}_2(A) - \bar{F}_2(t)}{\bar{F}_1(A) - \bar{F}_1(t)} = \frac{f_2(\xi)}{f_1(\xi)} = \frac{\tilde{r}_2(\xi)F_2(\xi)}{\tilde{r}_1(\xi)F_1(\xi)} \le \frac{\tilde{r}_2(t)}{\tilde{r}_1(t)},$$

where the inequality follows from the assumption that $\tilde{r}_2(t)/\tilde{r}_1(t)$ is decreasing in $t \in (0, \infty)$ and that $X_1 \le_{\text{st}} X_2$. Now, on taking $A \to \infty$, we get $\tilde{r}_2(t)\bar{F}_1(t) \ge \tilde{r}_1(t)\bar{F}_2(t)$ for all $t \in (0, \infty)$. ■

The following example illustrates a situation in which assumptions of Corollary 3.2 (and hence Theorem 3.2) hold.

Example 3.1: Let $F_1(t) = 1 - e^{-t}$ and $F_2(t) = (1 - e^{-t})^2, t \ge 0$. Then we have $\tilde{r}_1(t) = 1/(e^t - 1)$ and $\tilde{r}_2(t) = 2/(e^t - 1), t > 0$. We can see that $X_1 \le_{\text{st}} X_2$ and $\tilde{r}_2(t)/\tilde{r}_1(t)$ is decreasing in $t \in (0, \infty)$.

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Romera et al. [11] proved that if $X_1 \leq_{\text{fr}} X_2$ and $Y_2 \leq_{\text{fr}} Y_1$, then $Z_2 \leq_{\text{sp}} Z_1$. Valdés et al. [16] proved that if $X_1 \leq_{\text{rfr}} X_2$ and $Y_2 \leq_{\text{rfr}} Y_1$, then $Z_2 \leq_{\text{sp}} Z_1$. In the following theorem, we provide some more conditions under which $Z_2 \leq_{\text{sp}} Z_1$ holds.

THEOREM 3.3: *Suppose that one (or more) of the following conditions holds:*

(a) $X_1 \leq_{\text{st}} X_2$ *and* $Y_2 \leq_{\text{st}} X_1 \leq_{\text{st}} Y_1$;
(b) $X_1 \leq_{\text{st}} X_2$ *and* $Y_2 \leq_{\text{st}} X_2 \leq_{\text{st}} Y_1$;
(c) $X_1 \leq_{\text{st}} X_2, X_1 \leq_{\text{st}} Y_1$, *and* $Y_2 \leq_{\text{fr}} Y_1$;
(d) $X_1 \leq_{\text{st}} X_2, Y_2 \leq_{\text{st}} X_2$, *and* $Y_2 \leq_{\text{rfr}} Y_1$;
(e) $X_2 \leq_{\text{st}} X_1$ *and* $Y_1 \leq_{\text{st}} X_2 \leq_{\text{st}} Y_2$;
(f) $X_2 \leq_{\text{st}} X_1$ *and* $Y_1 \leq_{\text{st}} X_1 \leq_{\text{st}} Y_2$;
(g) $X_2 \leq_{\text{st}} X_1, X_2 \leq_{\text{st}} Y_2$, *and* $Y_1 \leq_{\text{fr}} Y_2$;
(h) $X_2 \leq_{\text{st}} X_1, Y_1 \leq_{\text{st}} X_1$, *and* $Y_1 \leq_{\text{rfr}} Y_2$.

Then, $Z_2 \leq_{\text{sp}} Z_1$.

PROOF: We will prove only assertions based on (a)–(d), as the other assertions follow on observing that Z_1 and Z_2 remain invariant if the role of X_1 is interchanged with that of X_2 and simultaneously the role of Y_1 is interchanged with that of Y_2. Let $\bar{H}(\cdot)$ and $h(\cdot)$ respectively be the survival function and the Lebesgue density function of $T = \wedge\{X_3, X_4, \ldots, X_n\}$. Romera et al. [11, Prop. 2(b)] proved that $Z_1 > Z_2$ holds if and only if one of the following two mutually exclusive inequalities is satisfied:

$$\vee(X_1, Y_2) < \wedge(X_2, Y_1, T), \qquad \vee(X_2, Y_1) < \wedge(X_1, Y_2, T).$$

Therefore,

$$\begin{aligned}
\Delta_7 &\equiv P(Z_1 > Z_2) - P(Z_2 > Z_1) \\
&= P(X_1 < X_2, Y_2 < X_2, \vee(X_1, Y_2) < Y_1, \vee(X_1, Y_2) < T) \\
&\quad - P(X_1 < X_2, Y_1 < X_2, \vee(X_1, Y_1) < Y_2, \vee(X_1, Y_1) < T) \\
&\quad + P(X_2 < X_1, Y_1 < X_1, \vee(X_2, Y_1) < Y_2, \vee(X_2, Y_1) < T) \\
&\quad - P(X_2 < X_1, Y_2 < X_1, \vee(X_2, Y_2) < Y_1, \vee(X_2, Y_2) < T).
\end{aligned}$$

Let $\hat{X}_1$ and $\hat{X}_2$ be independent copies of X_1 and X_2, respectively. Then we can write

$$\Delta_7 = E[\psi_3(X_2)] = E[\psi_4(X_1)],$$

where, for $t \geq 0$,

$$\begin{aligned}
\psi_3(t) &= \int_0^t \int_0^t [\bar{G}_1(\vee(s,y))g_2(y) - \bar{G}_2(\vee(s,y))g_1(y)]\bar{H}(\vee(s,y))f_1(s)dy\,ds \\
&\quad - \int_t^\infty \int_0^s [\bar{G}_1(\vee(t,y))g_2(y) - \bar{G}_2(\vee(t,y))g_1(y)]\bar{H}(\vee(t,y))f_1(s)dy\,ds,
\end{aligned}$$

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and

$$\psi_4(t) = \int_t^\infty \int_0^s [\bar{G}_1(\vee(t,y))g_2(y) - \bar{G}_2(\vee(t,y))g_1(y)]\bar{H}(\vee(t,y))f_2(s)\,dy\,ds - \int_0^t \int_0^t [\bar{G}_1(\vee(s,y))g_2(y) - \bar{G}_2(\vee(s,y))g_1(y)]\bar{H}(\vee(s,y))f_2(s)\,dy\,ds.$$

Clearly, $E[\psi_3(\hat{X}_1)] = E[\psi_4(\hat{X}_2)] = 0$. It can be easily verified that, for $t \geq 0$,

$$\psi_3(t) = \int_0^t [\bar{G}_1(s) - \bar{G}_2(s)]\bar{H}(s)f_1(s)ds + \int_0^t [\bar{G}_1(y)g_2(y) - \bar{G}_2(y)g_1(y)] \times \bar{H}(y)F_1(y)dy - [\bar{G}_1(t) - \bar{G}_2(t)]\bar{H}(t)\bar{F}_1(t) - \int_t^\infty [\bar{G}_1(y)g_2(y) - \bar{G}_2(y)g_1(y)]\bar{H}(y)\bar{F}_1(y)dy$$

and

$$\psi_3'(t) = 2[\bar{G}_1(t) - \bar{G}_2(t)]\bar{H}(t)f_1(t) + [\bar{G}_1(t) - \bar{F}_1(t)]\bar{H}(t)\bar{G}_2(t)\mu_2(t) + [\bar{F}_1(t) - \bar{G}_2(t)]\bar{H}(t)\bar{G}_1(t)\mu_1(t) + [\bar{G}_1(t) - \bar{G}_2(t)]\bar{F}_1(t)h(t), \qquad t \geq 0. \tag{3.1}$$

By symmetry,

$$\psi_4'(t) = 2[\bar{G}_2(t) - \bar{G}_1(t)]\bar{H}(t)f_2(t) + [\bar{G}_2(t) - \bar{F}_2(t)]\bar{H}(t)\bar{G}_1(t)\mu_1(t) + [\bar{F}_2(t) - \bar{G}_1(t)]\bar{H}(t)\bar{G}_2(t)\mu_2(t) + [\bar{G}_2(t) - \bar{G}_1(t)]\bar{F}_2(t)h(t), \qquad t \geq 0. \tag{3.2}$$

(a) Using $Y_2 \leq_{\text{st}} X_1 \leq_{\text{st}} Y_1$, we have $\psi_3'(t) \geq 0$ for all $t \in \mathbb{R}_+$; that is, $\psi_3(\cdot)$ is an increasing function on $\mathbb{R}_+$. Since $X_1 \leq_{\text{st}} X_2$ (i.e., $\hat{X}_1 \leq_{\text{st}} X_2$), it follows that

$$\Delta_7 = E[\psi_3(X_2)] \geq E[\psi_3(\hat{X}_1)] = 0;$$

that is, $Z_2 \leq_{\text{sp}} Z_1$.

(b) Using $Y_2 \leq_{\text{st}} X_2 \leq_{\text{st}} Y_1$, we have $\psi_4'(t) \leq 0$ for all $t \in \mathbb{R}_+$; that is, $\psi_4(\cdot)$ is a decreasing function on $\mathbb{R}_+$. Since $X_1 \leq_{\text{st}} X_2$ (i.e., $X_1 \leq_{\text{st}} \hat{X}_2$), it follows that

$$\Delta_7 = E[\psi_4(X_1)] \geq E[\psi_4(\hat{X}_2)] = 0;$$

that is, $Z_2 \leq_{\text{sp}} Z_1$.

(c) Using $X_1 \leq_{\text{st}} Y_1$ and $Y_2 \leq_{\text{fr}} Y_1$, from Eq. (3.1) we have

$$\psi_3'(t) \geq [\bar{G}_1(t) - \bar{G}_2(t)][2\bar{H}(t)f_1(t) + \bar{H}(t)\bar{F}_1(t)\mu_1(t) + \bar{F}_1(t)h(t)] \geq 0, \quad \forall t \in \mathbb{R}_+ \qquad (\text{since } Y_2 \leq_{\text{fr}} Y_1 \Rightarrow Y_2 \leq_{\text{st}} Y_1);$$

that is, $\psi_3(\cdot)$ is an increasing function on $\mathbb{R}_+$. Now, proceeding as in the proof of part (a), we have $\Delta_7 \geq 0$; that is, $Z_2 \leq_{\text{sp}} Z_1$.

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(d) Using $Y_2 \leq_{\text{st}} X_2$ and $Y_2 \leq_{\text{rfr}} Y_1$, from Eq. (3.2) we have

$$\begin{aligned}\psi_4'(t) &\leq [\bar{G}_2(t) - \bar{G}_1(t)][2\bar{H}(t)f_2(t) + \bar{H}(t)F_2(t)\tilde{\mu}_2(t) + \bar{F}_2(t)h(t)] \\ &\leq 0, \quad \forall t \in \mathbb{R}_+ \qquad (\text{since } Y_2 \leq_{\text{rfr}} Y_1 \Rightarrow Y_2 \leq_{\text{st}} Y_1);\end{aligned}$$

that is, $\psi_4(\cdot)$ is a decreasing function on $\mathbb{R}_+$. Now, on proceeding as in proof of part (b), we have $\Delta_7 \geq 0$; that is, $Z_2 \leq_{\text{sp}} Z_1$. ■

By assuming that $Y_i =_{\text{st}} X_i, i = 1, 2$, Valdés et al. [16] showed that $Z_1 \leq_{\text{sp}} Z_2$ will hold provided one of the following four conditions is satisfied: (i) $X_1 \leq_{\text{rfr}} X_2$; (ii) $X_2 \leq_{\text{rfr}} X_1$; (iii) $X_1 \leq_{\text{fr}} X_2$; (iv) $X_2 \leq_{\text{fr}} X_1$. In the following corollary to Theorem 3.3 we generalize this result of Valdés et al. [16].

COROLLARY 3.3: *Suppose that* $X_i =_{\text{st}} Y_i, i = 1, 2$, *and* $X_1 \leq_{\text{st}} X_2$ *or* $X_2 \leq_{\text{st}} X_1$. *Then* $Z_1 \leq_{\text{sp}} Z_2$.

PROOF: Notice that Z_2 can be obtained from Z_1 (and viceversa) by interchanging the roles of Y_1 and Y_2. Using this fact along with Theorems 3.3(a) and 3.3(e), we conclude that $Z_1 \leq_{\text{sp}} Z_2$ holds if either of the following two conditions hold: (i) $X_1 \leq_{\text{st}} X_2$ and $Y_1 \leq_{\text{st}} X_1 \leq_{\text{st}} Y_2$ or (ii) $X_2 \leq_{\text{st}} X_1$ and $Y_2 \leq_{\text{st}} X_2 \leq_{\text{st}} Y_1$. Now, on taking $X_i =_{\text{st}} Y_i, i = 1, 2$, the result follows. ■

4. CONCLUSIONS AND COMMENTS

Consider a series system consisting of n components $C_1, \ldots, C_n$. Suppose that we also have two spares, R_1 and R_2, available for active redundancy. Let the lifetimes of $C_1, \ldots, C_n, R_1$, and R_2 be denoted by independent random variables $X_1, \ldots, X_n, Y_1$, and Y_2, respectively. We consider two models for active redundancy. In the first one, we can use either R_1 with C_1, or R_2 with C_2, so that the lifetimes of two resulting systems are $U_1 = \wedge\{\vee(X_1, Y_1), X_2, X_3, \ldots, X_n\}$ and $U_2 = \wedge\{X_1, \vee(X_2, Y_2), X_3, \ldots, X_n\}$. In the second model, available spares can be used in one of the following two ways: R_1 with C_1 and R_2 with C_2, or R_1 with C_2 and R_2 with C_1, so that the lifetimes of resulting systems are $Z_1 = \wedge\{\vee(X_1, Y_1), \vee(X_2, Y_2), X_3, \ldots, X_n\}$ and $Z_2 = \wedge\{\vee(X_1, Y_2), \vee(X_2, Y_1), X_3, \ldots, X_n\}$. In the literature, the theory of stochastic orders is being used to compare the performance of two systems in each of the above models. This has been done by comparing lifetimes U_1 and U_2, and Z_1 and Z_2 with respect to the failure rate order, the usual stochastic order, the stochastic precedence order, and the increasing concave order. In this article, we provide supplements to various results available in the literature on stochastic comparisons with respect to these orders. We also compare U_1 and U_2, and Z_1 and Z_2 with respect to the reversed failure rate order. It would be worthwhile to extend the results presented in this article to k-out-of-n systems.

https://doi.org/10.1017/S0269964810000367 Published online by Cambridge University Press

Acknowledgments

The authors are grateful to the anonymous referee for his/her valuable comments and suggestions that have led to an improved version of the article. The second author would like to acknowledge the financial assistance from the Council of Scientific and Industrial Research, India, for carrying out this research work.

References

1. Barlow, R.E. & Proschan, F. (1975). *Statistical theory of reliability and life testing: Probability models.* New York: Holt, Rinehart and Winston.
2. Boland, P.J., El-Neweihi, E., & Proschan, F. (1988). Active redundancy allocation in coherent systems. *Probability in the Engineering and Informational Sciences* 2: 343–353.
3. Boland, P.J., El-Neweihi, E., & Proschan, F. (1992). Stochastic order for redundancy allocations in series and parallel systems. *Advances in Applied Probability* 24: 161–171.
4. Boland, P.J., Singh, H., & Cukic, B. (2004). The stochastic precedence ordering with applications in sampling and testing. *Journal of Applied Probability* 41: 73–82.
5. Dykstra, R., Kochar, S., & Rojo, J. (1997). Stochastic comparisons of parallel systems of heterogeneous exponential components. *Journal of Statistical Planning and Inference* 65: 203–211.
6. Kalashnikov, V.V. & Rachev, S.T. (1986). Characterization of queueing models and their stability. *Probability theory and mathematical statistics*. Amsterdam: VNU Science Press.
7. Lai, C.D. & Xie, M. (2003). Relative ageing for two parallel systems and related problems. *Mathematical and Computer Modelling* 38: 1339–1345.
8. Li, X. & Hu, X. (2008). Some new stochastic comparisons for redundancy allocations in series and parallel systems. *Statistics & Probability Letters* 78: 3388–3394.
9. Misra, N., Dhariyal, I.D., & Gupta, N. (2009). Optimal allocation of active spares in series systems and comparisons of component and system redundancies. *Journal of Applied Probability* 46: 19–34.
10. Müller, A. & Stoyan, D. (2002). *Comparison methods for stochastic models and risks.* New York: Wiley.
11. Romera, R., Valdés, J.E., & Zequeira, R.I. (2004). Active-redundancy allocation in systems. *IEEE Transactions on Reliability* 53: 313–318.
12. Sengupta, D. & Deshpande, J.V. (1994). Some results on the relative ageing of two life distributions. *Journal of Applied Probability* 31: 991–1003.
13. Shaked, M. & Shanthikumar, J.G. (1992). Optimal allocation of resources to nodes of parallel and series systems. *Advances in Applied Probability* 24: 894–914.
14. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders.* New York: Springer.
15. Singh, H. & Misra, N. (1994). On redundancy allocations in systems. *Journal of Applied Probability* 31: 1004–1014.
16. Valdés, J.E., Arango, G., Zequeira, R.I., & Brito, G. (2010). Some stochastic comparisons in series systems with active reundancy. *Statistics and Probability Letters* 80: 945–949.
17. Valdés, J.E. & Zequeira, R.I. (2003). On the optimal allocation of an active redundancy in a two-component series system. *Statistics and Probability Letters* 63: 325–332.
18. Valdés, J.E. & Zequeira, R.I. (2006). On the optimal allocation of two active redundancies in a two-component series system. *Operations Research Letters* 34: 49–52.

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