THE THERMODYNAMICS OF MAGNETIZATION

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1. *Introduction.* Recent studies by Debye (1), Bloch (2), Stoner (3) and Guggenheim (4) expressing somewhat divergent views suggest that there is still some confusion and uncertainty in the method of application of the thermodynamic argument to magnetic phenomena. The first effective contribution to the subject was made by Larmor (5) over fifty years ago, and developing this further he finally proposed in 1929 a sufficient basis for a general theory of the whole range of phenomena. At about the same time (1927), and apparently in ignorance of Larmor's earlier work, Debye (1) also gave a brief outline of the thermodynamic aspects of the subject, starting, however, from much more tentative magnetic ideas and formulae which are now known to be without adequate physical justification. In 1933, Bloch (2) elaborated the argument still further, using, however, basic ideas more akin to those employed by Larmor, but derived by a very specialized argument. Then in 1935, using the same fundamental equation as Debye, Stoner revived the discussion with a systematic survey of the theoretical results bearing on the thermodynamic aspects of the subject, attempting later (6) to justify the basic magnetic ideas by a discussion which he now admits is not entirely satisfactory.

Prompted by this work of Debye, Bloch and Stoner and the lack of generality still persisting through it, Guggenheim made a thorough survey of the whole subject—omitting Larmor's work—and in two papers (4) published in 1936 he proposed a very general reformulation of the theory including not only the thermal but also the elastic relations of the subject. Unfortunately, however, his preliminary discussion of the magnetic formulae (7, 8) is based on a misinterpretation of the now somewhat discredited form of Maxwell's theory given by Hertz and Cohn and in consequence his examination of the thermodynamic relations starts with a quite untenable form for the fundamental potentials.

2. In these circumstances it seemed advisable to re-examine the subject to determine the relations between the different ideas thus suggested and those put forward originally by Larmor, and then to attempt to obtain the complete generality which Guggenheim failed to achieve. It soon became evident that the difficulties and uncertainties in the approaches by Debye, Stoner and Guggenheim arise mainly from a too rigid adherence to the traditional interpretation of the energy relations of the magnetic field, an interpretation which exhibits the Hamiltonian function* of the field as a *total* energy function, rather than as an *available* energy function, which is the only type recognized

* The term 'Hamiltonian function' is used in this paper in a sense that is perhaps different to the conventional one. We do not necessarily imply that it is properly interpreted in terms of the momenta, as we are not concerned with it in any other capacity than as an energy function.

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in a strict dynamical theory. And if the ideas employed by these authors are interpreted in terms of a theory which fits properly into a dynamical scheme, the difficulties in their presentations are at once removed, but they then lead directly to a fundamental energy equation which is almost identical with that given by Larmor, thus confirming the completely satisfactory character of his approach to the problem.

All that remained then was to develop the basic ideas of Larmor's discussion, applied by him to the simplest case only, into a comprehensive theory covering all aspects of the interaction between the magnetization and the thermal and elastic properties of the most general type of anisotropic medium. The present paper contains the first part of this generalization and is confined to the purely thermal aspects of the subject, the processes involved being assumed to be, at least approximately, independent of the elastic properties of the medium. Distortion of the medium as a consequence of its magnetization is therefore assumed for the present to be negligible.

3. We shall throughout use a rectangular coordinate frame of reference and cartesian tensor suffix notation with the usual summation convention for repeate indices. The following symbols are those that occur most frequently:

B_α, magnetic induction vector,

H_α, magnetic force vector,

I_α, magnetic polarization intensity vector,

$I_\alpha^{(i)}$, induced magnetic polarization intensity vector,

$I_\alpha^{(p)}$, permanent magnetic polarization intensity vector.

These are related by the equations

$$I_\alpha = I_\alpha^{(i)} + I_\alpha^{(p)}, \quad B_\alpha = H_\alpha + 4\pi I_\alpha.$$

E, total intrinsic energy of the matter on the field,

E_1, intrinsic energy of elastic or thermal type,

T, absolute temperature,

S, entropy,

h, quantity of heat,

A, mechanically available or free energy,

L, Lagrangian function.

4. *The dynamical field functions and their interpretation.* It seems to be now generally agreed (6,7,8) that the magnetic contribution to the Lagrangian (force) and Hamiltonian (available energy) functions of a mechanical system which includes linear currents and a distribution of permanent and induced magnetic polarity can be represented by the space integrals* of the density functions

$$L_m = \frac{1}{4\pi}\int_0^{H_\alpha} B_\alpha dH_\alpha + \int_0^{I_\alpha^{(p)}} F_\alpha dI_\alpha^{(p)}, \tag{4·1}$$

$$A_m = \frac{1}{4\pi}\int_0^{B_\alpha} H_\alpha dB_\alpha - \int_0^{I_\alpha^{(p)}} F_\alpha dI_\alpha^{(p)}, \tag{4}$$

* In using space integrals for dynamic functions relating to quasi-static systems with o a limited number of degrees of freedom we imply little beyond the fact that these integrals : functions, of the ordinary type, of the geometrical, electrical and magnetic coordinates of ti.

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respectively. We have temporarily assumed that F_α is the effective magnetic force in the polarization process; the traditional assumption is that $F_\alpha \equiv H_\alpha$, but there are grounds for thinking that $F_\alpha \equiv B_\alpha$ is a better choice. As the polarization phenomena is known to have thermal reactions it is necessary, in order to give precision to the integrals, to assume that they are effected at constant uniform temperature.

With $F_\alpha \equiv H_\alpha$ we have the traditional expansions of these functions in the form

$$L_m = \frac{1}{8\pi} H_\alpha H_\alpha + I_\alpha H_\alpha - \int_0^{I_\alpha^{(i)}} H_\alpha dI_\alpha^{(i)}, \tag{4·3}$$

$$A_m = \frac{1}{8\pi} H_\alpha H_\alpha + \int_0^{I_\alpha^{(i)}} H_\alpha dI_\alpha^{(i)}. \tag{4·4}$$

The last formula exhibits A_m as the sum of the purely magnetic field energy—in a theory which assumes that this is determined by H_α—and the energy of effectively non-magnetic nature stored in the internal elastic connexions controlling the polarizations. This seems reasonable enough if we picture A_m as a total energy function, but difficulties arise as soon as we try to reconcile this interpretation of the two parts of A_m with the form which the expression provides for the Lagrangian function L of the field, however this function is conceived. It is, in fact, the discrepancy between these two forms which is at the bottom of the difficulties of the traditional discussion of this subject.

5. Compare this conclusion now with that derived from the alternative formulae, which assume their simplest form when we take $F_\alpha \equiv B_\alpha$, namely,

$$L_m = \frac{1}{8\pi} B_\alpha B_\alpha - \left(- I_\alpha B_\alpha + \int_0^{I_\alpha^{(i)}} B_\alpha dI_\alpha^{(i)} \right) - I_\alpha B_\alpha, \tag{5·1}$$

$$A_m = \frac{1}{8\pi} B_\alpha B_\alpha + \left(- I_\alpha B_\alpha + \int_0^{I_\alpha^{(i)}} B_\alpha dI_\alpha^{(i)} \right). \tag{5·2}$$

These formulae, unlike the previous two, admit of immediate incorporation in a dynamical scheme. All we have to do is to assume that the purely magnetic energy is distributed throughout the field with a density $T_m = B_\alpha B_\alpha / 8\pi$ when reckoned as kinetic energy. The remaining part V_m of the function A_m is then

$$V_m = - I_\alpha B_\alpha + \int_0^{I_\alpha^{(i)}} B_\alpha dI_\alpha^{(i)},$$

and consists essentially of two terms, the first being the positional (potential) energy of the polarizations relative to the magnetic field and the second the elastic or thermal energy stored in the media as energy of effectively non-magnetic character.

With this interpretation of the two terms of A_m we have at once that

$$L_m = T_m - V_m - I_\alpha B_\alpha,$$

which is recognized immediately as a Lagrangian function modified in the Routhian

elements giving rise to the field. But, besides achieving a certain degree of compactness, especially when the system involves a continuous distribution of magnetism, we have thereby laid the foundations for the 'field-theory' interpretations of our results and their extension to the more complex systems involving radiation. In this general case it is usual to assume that one of the field vectors has the character of a continuously distributed elastic displacement coordinate. Cf. Larmor, *Mathematical and physical papers*, vol. 2, 106.

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manner by ignoring the coordinates defining the kinetic details of the electron motions to which the magnetism is probably due. The assumption of a polar constitution for the magnetism thus appears as equivalent to the ignoration of these coordinates.

Our interpretation of the formulae (5·1) and (5·2) thus conforms naturally to all the physical assumptions of ultimate dynamical significance which underlie the theory of the magnetic field. And it provides perhaps the most conclusive theoretical evidence for the choice of the induction vector B_α as the sole force intensity vector of the theory.

6. *The energy equation of the thermodynamic theory.* In the approach to the thermodynamic argument we now concentrate our attention on the energy changes which take place in a particular magnetic body or part of such body when it moves in an imposed magnetic field due in the main to other magnets and currents, but to which it also makes its own contribution. If we denote by E the total energy intrinsic to the chosen mass of material, the equation for the variation of this energy in any small displacement of the mass, during which heat of amount δh is transferred to it from outside sources, is, in its general form,

$$\delta E = \delta h + \Sigma Q_s \delta q_s, \tag{6·1}$$

where $\Sigma Q_s \delta q_s$ represents the work done on the mass by applied forces of mechanical origin, the force Q_s acting in the coordinate q_s of the mass.

The symbol E represents all the energy intrinsic to the molecules of the matter under review; this includes both kinetic energy of motions (both thermal and organized) and potential energy of position or configuration, not only of the molecules relative to one another, but also to any field applied from the rest of the magnetic system of which the mass forms part. This concept of the potential energy possessed by a system in virtue of the action on it of certain more or less well-defined forces is introduced into physical theory with the object of avoiding too detailed a specification of the forces or motions to which the energy is due. And the fact that we can so frame our discussions in the most general cases in terms of such energy functions and real mechanical forces alone implies that the ultimate and detailed specification of the forces and motions to which the energy belongs is irrelevant to the description of the dynamical processes which we can follow in detail.

7. To interpret this energy equation for our present magnetic problem all we need to know is that certain mechanical forces are generally necessary to hold a magnet in a field of force, and that therefore these forces will do work when the magnet to which they are applied is displaced in the field. The assumption that the positional (potential) energy of the magnetic polarizations I_α in the field B_α is measured per unit volume by $-I_\alpha B_\alpha$ implies at once that the work done by these mechanical forces acting to balance the presumed magnetic attractions is distributed over the matter with a density $-I_\alpha \delta B_\alpha$, δB_α being the change produced by the displacement in the value of B_α associated with I_α. The idea is briefly that in the displacement intrinsic changes of the equilibrium configuration of the material arising from molecular actions produced by the polarization I_α are to be prevented by constraint so that there is no change of internal energy of any kind in the media.

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The equation of energy thus assumes the form

$$\delta E = \delta h - \int (I_\alpha \delta B_\alpha)\, dv, \tag{7·1}$$

the integral in the last term being extended over the mass of the material under review. This equation is equivalent to that used by Larmor,

$$\delta E = \delta h - I_\alpha \delta B_\alpha,$$

where all the quantities are reckoned as per unit volume of the material.

8. The method of approach used by Debye and Guggenheim is slightly different. Recognizing that the work done by the mechanical forces represents an increase in the mechanically available energy A, Debye replaces the last term in (6·1) by the variation of the function A_m given in (4·4)*

$$\delta A_m = \frac{1}{4\pi} H_\alpha \delta B_\alpha = \frac{1}{4\pi} H_\alpha \delta H_\alpha + H_\alpha \delta I_\alpha. \tag{8·1}$$

Assuming then that the variation of the purely magnetic field energy, represented by the first term, is independent of the changes of energy in the matter, he rejects this first term and arrives at the equation

$$\delta E = \delta H + H_\alpha \delta I_\alpha. \tag{8·2}$$

Guggenheim, apparently dissatisfied with the argument for rejecting the first term, decides that in order to obtain complete generality the only satisfactory form of the equation is one which retains the complete expression (8·1) for δA and he writes therefore ((4), p. 73)

$$\delta E = \delta h + \frac{1}{4\pi} H_\alpha \delta B_\alpha. \tag{8·3}$$

While one may perhaps sympathize with Guggenheim's view of Debye's argument, this method of avoiding the difficulty seems to be even more unsatisfactory. The correct approach is in fact to be obtained not by this method but by a slight modification of the original argument in which, however, the form (5·2) is used for A_m rather than the form (4·4) used by Debye. If we write

$$\delta A_m = \frac{1}{4\pi} B_\alpha \delta B_\alpha - I_\alpha \delta B_\alpha, \tag{8·4}$$

we can divide the variation of B_α into two parts; an intrinsic variation at each point of space due to a field changing in time; this accounts for the whole of the first term and part of the second; and there is then the change in B_α due to the change of position of I_α, which accounts for the remainder of the second term. But the variation of A due to a field varying in time represents work supplied by the electromotive forces in the originating current circuits over and above that required to account for the heat developed in those circuits. Unless we are prepared therefore to introduce these applied electromotive forces as external forces into our equation of energy we must reject this part of the variation as properly accounted for in this way. This, of course,

* For the rest of the argument of this paper we follow the usual procedure and ignore the permanent magnetism except as providing a static origin for the field.

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is equivalent to assuming that the general field in which the material moves is maintained constant as regards its space distribution, and the argument then leads properly to Larmor's equation

$$\delta E = \delta h - I_\alpha \delta B_\alpha, \tag{8·5}$$

in which δB_α is a variation produced simply by the displacement of I_α.

We now see to what extent Debye's argument is at fault, and also that Guggenheim's equation (8·3) is incomplete to the extent that he omits the action of the applied electromotive forces as a possible external source of energy for his system.

9. *The thermodynamic potentials.* To make further progress in the thermodynamic discussions we must now restrict our system of interest still further. Thermodynamics is essentially a theory of the intrinsic and therefore localized changes of the equilibrium configuration of matter produced by changes in the physical conditions under which it exists, and it becomes tractable only when applied to a mass of substance such that the physical conditions in it are sufficiently uniform to admit of definition in terms of a finite number of coordinate variables. The condition of uniformity may be approximately satisfied by a finite magnet whose properties are under investigation, but generally speaking the magnetization will vary from point to point, so that we have to resort to the analyses of the behaviour of the small elements separately*.

We imagine that our energy equation, which we shall continue to quote as relating to unit volume, refers in effect simply to the densities in a small volume of matter in which the conditions as to temperature and magnetic state are defined by the absolute temperature T and the magnetic field strength B_α, which may be assumed to be uniform throughout it. The choice of the small element also has the advantage that the magnetic field strength B_α operative through it may be assumed to be due entirely to the surrounding 'external' magnetic system, the small contribution of the element itself being negligible.

10. The energy equation (8·5) per unit volume for the small element is of course a simple consequence of the principle of the conservation of energy and is of the same universal validity. But this principle goes further and asserts that δE is the exact differential of a function E of the instantaneous conditions prevailing in the element and which is independent therefore of how these conditions are attained.

The heat δh, absorbed during the transformation will in general also depend on the extent of the variation in T and B_α, but unlike δE this variation is not the complete differential of a unique function of these coordinates. There is, however, a closely related quantity, the entropy S, which changes in the transformation by $\delta h/T$ and which is a definite function of the conditions of the element at each instant. We may therefore write

$$\delta h = T \frac{\partial S}{\partial B_\alpha} \delta B_\alpha + T \frac{\partial S}{\partial T} \delta T, \tag{10·1}$$

* Guggenheim attempts to waive this restriction and to obtain results relating to the complete field. But his assumption, that two equations of the type

$$\delta E = \int (\mathbf{V} \delta \mathbf{B})\, dv, \quad \partial E/\partial \mathbf{B} = \int \mathbf{V}\, dv$$

are equivalent, is tenable only when the value of $\delta \mathbf{B}$ is the same at all points of space: it is only then that the expression $\partial E/\partial \mathbf{B}$ has any meaning.

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and then (8·5) takes the form

$$\delta E = \left(T\frac{\partial S}{\partial B_\alpha} - I_\alpha\right)\delta B_\alpha + T\frac{\partial S}{\partial T}\delta T, \tag{10·2}$$

and the conditions that this differential function is complete lead immediately to the two sets of relations

$$\frac{\partial I_\alpha}{\partial B_\beta} = \frac{\partial I_\beta}{\partial B_\alpha}, \tag{10·3}$$

$$\frac{\partial S}{\partial B_\alpha} = \frac{\partial I_\alpha}{\partial T}. \tag{10·4}$$

11. The equations (10·3) suggest at once that there exists a potential function

$$A_m = -\int_0^{B_\alpha} I_\alpha dB_\alpha, \tag{11·1}$$

in which the integration is to be effected at constant temperature T, which is such that

$$I_\alpha = -\frac{\partial A_m}{\partial B_\alpha}. \tag{11·2}$$

This function defines the part of the available energy (Hamiltonian function) which is directly dependent on the polarizations. There may be another part of this function dependent solely on the temperature, and if we call this part A_0 we have $A = A_m + A_0$ as the complete expression for the mechanically available energy associated, per unit volume, with the element.

Interpreted in terms of A equation (10·4) is equivalent to $S = -\partial A/\partial T$, and with this value for S the energy equation (8·5) becomes

$$\delta E = -T\frac{\partial \delta A}{\partial T} + \delta A. \tag{11·3}$$

On integration at constant temperature, in conformity with the definition of A, this implies that

$$E = -T^2\frac{\partial}{\partial T}\left(\frac{A}{T}\right), \tag{11·4}$$

an equation showing the relation between the mechanically available energy A and the total intrinsic energy E. When the effect produced by the mechanical forces depends on the temperature some of the work done by these forces, measured by A, will be dissipated into heat. To maintain the temperature therefore this heat, measured by $T\partial A/\partial T$, will have to be extracted. It is only the excess of A over this amount of heat that is stored as additional internal energy E in the body.

12. The equation (8·5) of energy is of course equivalent to the equation

$$\delta E_1 = \delta(E + I_\alpha B_\alpha) = \delta h + B_\alpha \delta I_\alpha \tag{12·1}$$

used by Debye and Stoner, and their intrinsic energy function E_1 is equally a definite function of the physical coordinates defining the instantaneous conditions in the element. The equation may therefore be employed with equal success to derive many of the subsidiary relations of the theory, which are frequently quite independent of the particular form of intrinsic energy function used in their derivation. If, for example, we introduce the energy function

$$E_m = -\int^{I_\alpha} B_\alpha dI_\alpha, \tag{12·2}$$

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together with a corresponding function E_0 depending on the temperature, and write $T\delta S$, for δh, we can derive in exactly the same way as above, the relation

$$E_1 = -T^2 \frac{\partial}{\partial T}\left(\frac{E_m + E_0}{T}\right). \tag{12·3}$$

The relation between the potentials E and E_1 is obvious. The total intrinsic energy E includes not only the internal energy E_1, but also the potential energy of the polarizations relative to the external magnetic field, the view being that unless and until we can be sure of the exact nature of the magnetic forces themselves it is safer to express the whole of the relations of the theory in terms of this energy rather than of the forces to which it is due. The different ways in which the energies enter into the processes envisaged on our variation can be seen if we apply the two equations

$$\delta E = \delta h - I_\alpha \delta B_\alpha, \quad \delta h = \delta E_1 - B_\alpha \delta I_\alpha,$$

when the complete variation is divided into two stages. In the first stage $\delta h = 0$ and the internal constitution of the element is held fixed so that the increase δE is represented simply by the work done on the element by the mechanical forces. In the second stage the element is fixed in its new position but the constraints on the internal constitution are removed. Now δE can increase only through the absorption of heat from outside sources. But during the readjustment which follows the removal of the constraints and which takes place more or less automatically the intrinsic energy E_1, in the narrower sense, will increase both by the transfer of heat and by the work done by the magnetic forces working against the internal elastic or constitutional controls. But work done by the magnetic forces is at the expanse of the magnetic energy so this magnetic part of the increase of E_1 is accompanied by a corresponding decrease of the magnetic energy and there is therefore no alteration of total represented by the complete energy function. It is only the excess of the increase δE_1 over that part of it which is derived from magnetic sources which has to be supplied as heat and can therefore be counted as an increase of E.

13. *The equations of magnetic state.* The most significant result in these discussions is obtained by interpreting A_m as a function of the ratios $B'_\alpha = B_\alpha/T$, and the temperature T. Calling the new function A'_m, and the complete function A', we have

$$I_\alpha = -\frac{\partial A'_m}{\partial B_\alpha} = -\frac{1}{T}\frac{\partial A'_m}{\partial B'_\alpha}, \tag{13·1}$$

and then

$$E_1 = -T^2 \frac{\partial}{\partial T}\left(\frac{A'_m + A_0}{T}\right) = -T^2 \frac{\partial}{\partial T}\left(\frac{A'}{T}\right). \tag{13·2}$$

This relation shows that whenever A'_m is proportional to T, that is whenever

$$A'_m = -T\phi(B'_\alpha), \tag{13·3}$$

the magnetic contribution of A'_m to E_1 vanishes altogether, whilst its contribution to the total intrinsic energy E reduces to $-I_\alpha B_\alpha$. With such a formula for A'_m we have associated a law of induction or magnetic equation of state

$$I_\alpha = \phi'_\alpha(B'_\beta) = \phi'_\alpha\left(\frac{B_\beta}{T}\right), \tag{13·4}$$

the generalized form of a well-known result relating to isotropic media.

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The same conclusions follow equally from the equation (12·3) when the energy function E_m is proportional to T, or when

$$E_m = -\int^{I_\alpha} B_\alpha dI_\alpha = -T\psi(I_\alpha). \tag{13·5}$$

With this relation there is associated an equation of state

$$B_\alpha = T\psi'_\alpha(I_\beta), \tag{13·6}$$

which is, of course, the inverse of (13·4). The difference between A_m and E_m is, of course, $I_\alpha B_\alpha$, and under these conditions is also proportional to T, whether interpreted as a function of B'_α or I_α.

14. The physical significance of this special case where E_1 is independent explicitly of the magnetic condition of the medium is best seen by following the succession of events as the magnetic material is moved slowly into the field whilst its temperature is kept constant. The equation $\delta h = \delta E_1 - B_\alpha \delta I_\alpha$

shows that when the temperature and therefore also E_1 are kept constant the energy $B_\alpha \delta I_\alpha$ must be developed simply as extra thermal energy in the body and then extracted as such. This is obviously the case in an ideal paramagnetic gas, where the internal energy is entirely thermal, and where the forces controlling the orientation of the molecules have also a thermal origin; but the law of Curie suggests that a similar state of affairs exists in all paramagnetics at high temperatures. At lower temperatures they may pass into the ferromagnetic condition where the orientation of the molecules is controlled mainly by their interaction with neighbouring molecules which for this purpose form elastic systems (crystalline lattices) and the energy associated with these elastic controls forms part, and in some cases perhaps the main part of E_1, with a different law of variation with the temperature from the thermal constituent which is always present. There is an analogy in the case of a gas where potential energy of configuration of the molecules is introduced as additional internal energy to explain the deviation of its behaviour from that of an ideal gas.

15. When the magnetic substance remains isotropic the directional character of the field vectors can be neglected and the relations expressed simply as equations between their magnitudes. The general form of the magnetic equation of state (13·4) then reduces to

$$I = f(B/T), \tag{15·1}$$

the well-known Langevin-Curie law. By assuming the molecules of a gas to be simple magnetic bi-poles and applying the principles of statistical mechanics Langevin was able to determine a form for the function $f(u)$ applicable to such substances. His result is that

$$f(u) = \alpha\left(\coth u - \frac{1}{u}\right), \tag{15·2}$$

and holds when the axes of the magnets are otherwise free to point in any direction. When they are restricted to point only parallel or anti-parallel to the field a similar calculation gives

$$f(u) = \alpha \tanh u. \tag{15·3}$$

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These special results are of very limited application except in those regions where they both approximate to the linear law $f(u) = \alpha u$. It was, however, on the basis of the first of these formulae that Weiss developed his modified theory and thereby succeeded in coordinating many of the characteristics of para- and ferro-magnetic phenomena.

16. Weiss's modification of the theory for the ideal magnetic gas rests on the formally simple assumption that the effect of the elastic interactions between the molecules to which the part of E_1 that is not purely thermal energy relates, is equivalent to the effect of a local magnetic field, which he assumed to be proportional to the intensity of magnetization. He replaces the general law of induction (15·1) therefore by

$$I = f\left(\frac{B + aI}{T}\right).$$

At high temperatures where $(B + aI)/T$ is small this is equivalent to

$$I = \frac{\alpha}{T - T_0} B, \quad T_0 = a\alpha,$$

the Curie-Weiss law of induction which is found to be obeyed in many paramagnetics over a wide range of temperatures.

The physical significance of this modification will be seen if we return to formula (12·2) above, and separate out from the function E_m the part of it which is directly proportional to the temperature and which predominates at high temperatures. Writing then

$$E_m = -T\phi(I_\beta) + \phi_0(I_\beta, T),$$

where ϕ_0 contains all the terms in E_m not directly proportional to T, and is such that ϕ_0/T tends to zero at high temperatures, we see that the contribution of E_m to E_1 amounts to

$$E_1 = -T^2 \frac{\partial}{\partial T}\left(\frac{\phi_0}{T}\right),$$

whilst the associated law of induction is now

$$\frac{B_\alpha}{T} = \phi'_\alpha(I_\beta) - \frac{1}{T}\phi'_{0\alpha}(I_\beta, T).$$

This can be written in the form $I_\alpha = \psi\left(\frac{B_\beta + F_\beta}{T}\right)$,

where $F_\beta = \phi'_{0\alpha}(I_\beta)$.

This shows that the essence of the generalization achieved by Weiss lies, not necessarily in the assumption of local magnetic fields, but mainly in the much more general assumption of the existence of a part of the internal energy of the elastic controls, which the medium exercises over the degree of magnetization induced in it, which is not derived directly from thermal sources and is not therefore directly proportional to the temperature. This constituent of E_1 when it first becomes significant is probably independent directly of the temperature but at lower temperatures affairs are more complicated*.

* In Weiss's case when a is independent of T we have $\phi_0 = \frac{1}{2}aI^2$.

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17. While on this subject of magnetic equations of state reference may be made to another type of equation given by Guggenheim. Introducing two new functions ξ, ζ defined by the equations

$$\xi(B,H) = \frac{T}{H}\left(\frac{\partial H}{\partial T}\right)_B, \quad \zeta(B,H) = \frac{T}{B}\left(\frac{\partial B}{\partial T}\right)_H,$$

the suffix in each case denoting the variable to be kept constant in the differentiation, he derives eventually the equation

$$\frac{1}{H}\frac{\partial}{\partial B}\left(\frac{1}{\xi}\right)_H = \frac{1}{B}\frac{\partial}{\partial H}\left(\frac{1}{\zeta}\right)_B$$

to represent what he describes as the most general equation of state (for isotropic media).

The solution of this equation is, however,

$$B - H = 4\pi\phi(B,T),$$

so that it simply expresses in a complicated sort of way the original assumption that I is some function of B and T. It lays down no restrictions on the type of functionality involved and cannot therefore be properly described as an equation of state.

18. *Conclusion.* In summing up the present discussion we may say that:

(i) The traditional interpretation of the dynamical relations of the magnetic field, based on the assumption that H_α is the fundamental vector of the field, is incompatible with a strict dynamical theory of these affairs. The alternative form of the relations, based on B_α as the field intensity vector, appears, however, to be entirely satisfactory in this respect.

(ii) The alternative approaches to the thermodynamic relations, formulated since, and without reference to, Larmor's original discussion, prove, when properly interpreted in terms of the new dynamical field functions, to be identical with this discussion, interpreted however with the induction vector B_α as the magnetic force intensity.

(iii) The whole discussion can be formulated with complete generality to cover the whole range of static phenomena under the most generally conceivable conditions which the physical science of matter in bulk admits, with the single exception, in which this discussion is not of course alone, that it still fails to allow for the important effects of hysteresis.

Among other results the discussion provides complete generalizations, applicable to crystalline media, for the Curie-Langevin and Curie-Weiss equations of magnetic state, the latter being obtained by simple energy considerations independently of any hypothesis about local molecular fields.

The only remaining details of interest are those dealing with the specific heats of the material and its adiabatic relations. These follow, however, automatically once the fundamental relations are agreed so there is little point in setting them out here. The discussion of the interaction between the phenomena here under review and the elastic properties of the media, dealt with briefly by Stoner and at greater length by Guggenheim, raises a number of questions of principle and is better dealt with in a separate paper.

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