ESAIM: COCV 20 (2014) 460–487
DOI: 10.1051/cocv/2013071

ESAIM: Control, Optimisation and Calculus of Variations
www.esaim-cocv.org

REGULARITY RESULTS FOR AN OPTIMAL DESIGN PROBLEM WITH A VOLUME CONSTRAINT

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Abstract. Regularity results for minimal configurations of variational problems involving both bulk and surface energies and subject to a volume constraint are established. The bulk energies are convex functions with p-power growth, but are otherwise not subjected to any further structure conditions. For a minimal configuration (u, E), Hölder continuity of the function u is proved as well as partial regularity of the boundary of the minimal set E. Moreover, full regularity of the boundary of the minimal set is obtained under suitable closeness assumptions on the eigenvalues of the bulk energies.

Mathematics Subject Classification. 49N15, 49N60, 49N99.

Received May 29, 2013.
Published online March 7, 2014.

1. Introduction and statements

In this paper we study minimal energy configurations of a mixture of two materials in a bounded, connected open set $\Omega \subset \mathbb{R}^n$, when the perimeter of the interface between the materials is penalized. Precisely, the energy is given by

$$\mathcal{I}(u, E) := \int_\Omega \left(F(\nabla u) + \chi_E G(\nabla u)\right) \mathrm{d}x + P(E, \Omega), \tag{1.1}$$

where $E \subset \Omega$ is a set of finite perimeter, $u \in W^{1,p}(\Omega)$, $p > 1$, χ_E is the characteristic function of the set E and $P(E, \Omega)$ denotes the perimeter of E in Ω. We assume that F, $G\colon \mathbb{R}^n \to \mathbb{R}$ are C^1 integrands satisfying, for $p > 1$ and positive constants ℓ, L, α, $\beta > 0$ and $\mu \geq 0$, the following growth and uniform strong p-convexity hypotheses:

(F1) $$0 \leq F(\xi) \leq L(\mu^2 + |\xi|^2)^{\frac{p}{2}},$$

(F2) $$\int_\Omega F(\xi + \nabla\varphi)\,\mathrm{d}x \geq \int_\Omega \left(F(\xi) + \ell(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2\right) \mathrm{d}x,$$

Keywords and phrases. Regularity, nonlinear variational problem, free interfaces.

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Article published by EDP Sciences

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https://doi.org/10.1051/cocv/2013071 Published online by Cambridge University Press

and

$$0 \le G(\xi) \le \beta L(\mu^2 + |\xi|^2)^{\frac{p}{2}}, \tag{G1}$$

$$\int_\Omega G(\xi + \nabla\varphi)\,\mathrm{d}x \ge \int_\Omega \left(G(\xi) + \alpha\ell(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2\right)\mathrm{d}x \tag{G2}$$

for every $\xi \in \mathbb{R}^n$ and $\varphi \in C_0^1(\Omega)$.

We are interested in the following constrained problem

$$\min\left\{\mathcal{I}(u,E) :\ u = u_0 \text{ on } \partial\Omega,\ |E| = d\right\}, \tag{P}$$

where $u_0 \in W^{1,p}(\Omega)$ and $0 < d < |\Omega|$ are prescribed. Note that the strong convexity of F and G, expressed by (F2) and (G2), by virtue of Theorem 1.4 below, ensures the existence of solutions of the problem (P).

Energies with surface terms competing with a volume term appear in a plethora of phenomena in materials science such as models for optimal design [4], phase transitions [18], liquid crystals [19], epitaxy [12] (see also [11]).

Our first regularity result is the following:

Theorem 1.1. *Let F and G satisfy assumptions* (F1)–(F2) *and* (G1)–(G2)*, respectively. Assume, in addition, that F is p-homogeneous,* i.e., *$F(t\xi) = t^pF(\xi)$, for all $t \ge 0$. If (u,E) is a minimizer of problem (P), then $u \in C^{0,\frac{1}{p'}}_{\mathrm{loc}}(\Omega)$, where p' denotes the Hölder's conjugate exponent of p,* i.e., *$p' = \frac{p}{p-1}$. Moreover, $\mathcal{H}^{n-1}((\partial E \setminus \partial^* E) \cap \Omega) = 0$.*

Previous results in this direction have been obtained in [4] and [20]. Precisely, Ambrosio and Buttazzo [4] and Lin [20] considered problems of the form

$$\int_\Omega \left(\sigma_E(x)|\nabla u|^2\right)\mathrm{d}x + P(E,\Omega) \tag{1.2}$$

with $u = 0$ on $\partial\Omega$ and $\sigma_E(x) := a\chi_E + b\chi_{\Omega\setminus E}$ for a and b positive constants. It was proven in [4] that minimizers of (1.2) exist and that if (u,E) is a minimal configuration then u is locally Hölder continuous in Ω and, up to a set of $\mathcal{H}^{n-1}$ measure zero, there is no difference between the theoretic measure boundary of E and its topological boundary. Recently, in [8], it has been proven that there exists $\gamma = \gamma(n)$ such that, for a minimal configuration (u,E) of (1.2) if $1 < a/b < \gamma(n)$, then u is locally Hölder continuous in Ω and ∂^*E, the reduced boundary of E, is a $C^{1,\alpha}$-hypersurface. Moreover, Lin [20] showed that if (u,E) is a minimizer of (1.2) among all configurations such that u and ∂E are prescribed on $\partial\Omega$, then $u \in C^{0,1/2}(\Omega)$ and ∂^*E, the reduced boundary of E, is a $C^{1,\alpha}$-hypersurface away from a singular set Σ of $\mathcal{H}^{n-1}$ measure zero. In [21], Lin and Kohn establish a partial regularity result for the boundary of the minimal set of the problem

$$\mathcal{I}(u,E) := \int_\Omega \left(F(x,u,\nabla u) + \chi_{_E}G(x,u,\nabla u)\right)\mathrm{d}x + P(E,\Omega), \tag{1.3}$$

subject to the following constraints

$$u = \Phi \text{ on } \partial\Omega \text{ and } |E| = d,$$

requiring that F and G satisfy severe structure assumptions and have quadratic growth. A more detailed analysis of the minimal configurations of (P) was carried out in the two dimensional case by Larsen in [19]. However, also in this case only partial regularity of ∂^*E is obtained.

All minimum problems considered in the above mentioned papers have bulk energies of Dirichlet type with quadratic growth, *i.e.*, of the form $|\cdot|^2$. Here, in Theorem 1.1 we treat constrained problems, we do not require any additional structure assumption on the bulk energies, and we assume p-growth (not necessarily $p = 2$) with respect to the gradient.

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We point out that the Hölder exponent $\frac{1}{p'}$ in Theorem 1.1 is critical, in the sense that the two terms in the energy functional (1.1) locally have the same dimension $n-1$ (under appropriate scalings). Actually, we will show that $u \in C^{0,\frac{1}{p'}+\delta}_{\mathrm{loc}}(\Omega)$, for some $\delta > 0$, under suitable conditions on the eigenvalues of F and G, that, together with a result in [23] (see Thm. 2.3 in Sect. 2), allows us to conclude that $\partial^* E$ is a $C^{1,\widetilde{\delta}}$ hypersurface, for some $0 < \widetilde{\delta} < 1$. More precisely, we have

Theorem 1.2. *Let F and G satisfy assumptions* (F1)–(F2) *and* (G1)–(G2)*, respectively. There exist $\gamma = \gamma(n,p,\frac{\ell}{L}) < 1$ and $\widetilde{\sigma} = \widetilde{\sigma}(n,p) > 0$ such that if*

$$\left(\frac{\beta}{\alpha+1}\right)\left(\frac{\beta+1}{\alpha+1}\right)^{\widetilde{\sigma}} \leq \gamma \tag{1.4}$$

and if (u,E) is a minimizer of problem (P), then $u \in C^{0,\frac{1}{p'}+\delta}_{\mathrm{loc}}(\Omega)$ for some positive δ depending on n,p,α,β. Moreover $\partial^* E$ is a $C^{1,\widetilde{\delta}}$-hypersurface in Ω, for some $\widetilde{\delta} < \frac{1}{2}$ depending on n,p,α,β, and $\mathcal{H}^s((\partial E \setminus \partial^* E) \cap \Omega) = 0$ for all $s > n-8$.

Note that in Theorem 1.2 we do not impose the p-homogeneity of F, condition that we required in Theorem 1.1 only to avoid heavy technicalities.

Consider the prototype integrands

$$F(\xi) := L(\mu^2 + |\xi|^2)^{\frac{p}{2}} \qquad \text{and} \qquad G(\xi) := \beta L(\mu^2 + |\xi|^2)^{\frac{p}{2}}.$$

In this case the parameter α in assumption (G2) coincides with β in assumption (G1), and condition (1.4) reduces to

$$\beta \leq \frac{\gamma}{1-\gamma},$$

with $\gamma = \gamma(n,p) < 1$.

The functional (1.2) is a particular case of (1.1), setting

$$F(\xi) := b|\xi|^2 \qquad \text{and} \qquad G(\xi) := (a-b)|\xi|^2, \qquad a > b.$$

In this case, the parameters α, β in (G2) and (G1) are given by

$$\beta = \alpha := \frac{a-b}{b},$$

and condition (1.4) becomes

$$1 < \frac{a}{b} \leq \frac{1}{1-\gamma}.$$

So, Theorem 1.2 gives back Theorem 2 in [8] as a particular case.

Further, without imposing any condition on the eigenvalues of the integrands, we are still able to obtain the following partial regularity result:

Theorem 1.3. *Assume that* (F1)–(F2) *and* (G1)–(G2) *hold and let (u,E) be a minimizer of problem (P). Then there exists an open set $\Omega_0 \subset \Omega$ with full measure such that $u \in C^{0,\eta}(\Omega_0)$, for every positive $\eta < 1$. In addition, $\partial^* E \cap \Omega_0$ is a $C^{1,\widetilde{\eta}}$-hypersurface in Ω_0, for every $0 < \widetilde{\eta} < \frac{1}{2}$, and $\mathcal{H}^s((\partial E \setminus \partial^* E) \cap \Omega_0) = 0$ for all $s > n-8$.*

In the study of regularity properties, the constraint $|E| = d$ introduces extra difficulties, since one can work only with variations which keep the volume constant. The next theorem allows us to circumvent this extra difficulties, ensuring that every minimizer of the constrained problem (P) is also a minimizer of a suitable unconstrained energy functional with a volume penalization.

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Theorem 1.4. *There exists $\lambda_0 > 0$ such that if (u, E) is a minimizer of the functional*

$$\mathcal{I}_\lambda(v, A) := \int_\Omega (F(\nabla v) + \chi_A G(\nabla v))\,\mathrm{d}x + P(A, \Omega) + \lambda\big||A| - d\big| \tag{1.5}$$

for some $\lambda \geq \lambda_0$ and among all configurations (v, A) such that $v = u_0$ on $\partial\Omega$, then $|E| = d$ and (u, E) is a minimizer of problem (P). Conversely, if (u, E) is a minimizer of the problem (P), then it is a minimizer of (1.5), for all $\lambda \geq \lambda_0$.

Theorem 1.4 is a straightforward modification of a result due to Esposito and Fusco (see [8], Thm. 1). Since several modifications are needed, we present its proof in Section 3 for the reader's convenience. Similar arguments have been used in Fonseca, Fusco, Leoni and Millot [11] (see also Alt and Caffarelli [1]).

From the point of view of regularity, the extra term $\lambda\big||A| - d\big|$ is a higher order, negligible perturbation, in the sense that if $x_0 \in \partial^* E \cap \partial\Omega$ then $|E \cap B_\varrho(x_0)|$ decays as ϱ^n as $\varrho \to 0^+$ while the leading term $\int_{B_\varrho(x_0)} (F(\nabla u) + \chi_E G(\nabla u))\,\mathrm{d}x + P(E, B_\varrho(x_0))$ decays as ϱ^{n-1}.

The proof of Theorem 1.1 is based on a decay estimate for the gradient of the minimizer u, obtained by blowing-up the minimizer u in small balls. We establish that the minimizers of the rescaled problems converge to a Hölder continuous function v, and we show that u and v are "close enough" (with respect to the norm in the Sobolev space $W^{1,p}$) in order to ensure that u inherits the regularity estimates of v.

Theorems 1.2 and 1.3 are obtained by a comparison argument between the minimizer of (P) and the minimizer of a suitable convex scalar functional with p-growth, for which regularity results are well known. Also here, we show that the two minimizers are "close" enough to share the same good regularity properties. We remark that in this comparison argument we need that u is a real valued function. In fact, in the vectorial setting (see [22]) minimizers of regular variational functionals may have singularities and only partial regularity results are known (see for example [3, 7, 13]).

This paper is organized as follows: in Section 2 we fix the notation and collect standard preliminary results. The proof of Theorem 1.4 is given in Section 3, since the result is needed in the proofs of the other theorems. The proofs of Theorems 1.1, 1.2 and 1.3 are presented in Sections 4, 5 and 6, respectively.

2. Notations and preliminary results

In this paper we follow usual convention and denote by c a general constant that may vary from expression to expression, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use in $\mathbb{R}^n$ is the standard Euclidean norm, and it will be denoted by $|\cdot|$. In particular, for vectors $\xi, \eta \in \mathbb{R}^n$ we write $\langle \xi, \eta \rangle$ for the inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ is the corresponding Euclidean norm. When $a, b \in \mathbb{R}^n$ we write $a \otimes b$ for the tensor product defined as the matrix that has the element $a_r b_s$ in its rth row and sth column. Observe that $(a \otimes b)x = (b \cdot x)a$ for $x \in \mathbb{R}^n$, and $|a \otimes b| = |a||b|$. When $F\colon \mathbb{R}^n \to \mathbb{R}$ is C^1, we write

$$D_\xi F(\xi)[\eta] := \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} F(\xi + t\eta)$$

if $\xi, \eta \in \mathbb{R}^n$. It is convenient to express the convexity and growth conditions of the integrands in terms of an auxiliary function defined for all $\xi \in \mathbb{R}^n$ as

$$V(\xi) = V_{p,\mu}(\xi) := \left(\mu^2 + |\xi|^2\right)^{\frac{p-2}{4}} \xi, \tag{2.1}$$

where $\mu \geq 0$ and $p \geq 1$. We recall the following lemmas.

Lemma 2.1. *Let $1 < p < \infty$ and $0 \leq \mu \leq 1$. There exists a constant $c = c(n, N, p) > 0$ such that*

$$c^{-1}\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}} \leq \frac{|V_{p,\mu}(\xi) - V_{p,\mu}(\eta)|^2}{|\xi - \eta|^2} \leq c\left(\mu^2 + |\xi|^2 + |\eta|^2\right)^{\frac{p-2}{2}}$$

for all $\xi, \eta \in \mathbb{R}^n$.

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For the proof we refer to [17], Lemma 8.3. The next lemma can be found in ([15], Lem. 2.1) and ([2], Lem. 2.1) for $p \geq 2$ and $1 < p < 2$, respectively.

Lemma 2.2. *For $1 < p < \infty$ and all ξ, $\eta \in \mathbb{R}^n$ one has*

$$\frac{1}{c} \leq \frac{\int_0^1 (\mu^2 + |\xi + t\eta)|^2)^{\frac{p-2}{2}} \mathrm{d}t}{(\mu^2 + |\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}} \leq c,$$

where c depends only on p.

It is well-known that for convex C^1 integrands, the assumptions (F1) and (G1) yield the upper bounds

$$|D_\xi F(\xi)| \leq c_1(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \qquad |D_\xi G(\xi)| \leq c_2(\mu^2 + |\xi|^2)^{\frac{p-1}{2}} \tag{2.2}$$

for all $\xi \in \mathbb{R}^n$, where we can use $c_1 := 2^p L$ and $c_2 := 2^p \beta L$ (see [17]).

Also, if F and G satisfy (F2) and (G2), respectively, then the following strong p-monotonicity conditions hold:

$$\begin{aligned} \langle D_\xi F(\xi) - D_\xi F(\eta), \xi - \eta \rangle &\geq c(p)\ell |V(\xi) - V(\eta)|^2 \\ \langle D_\xi G(\xi) - D_\xi G(\eta), \xi - \eta \rangle &\geq c(p)\alpha\ell |V(\xi) - V(\eta)|^2 \end{aligned} \tag{2.3}$$

for all ξ, $\eta \in \mathbb{R}^n$ and some $c(p) > 0$. In fact, (F2) and (G2) are equivalent to the convexity of the functions

$$\xi \to \widetilde{F}(\xi) := F(\xi) - \ell(\mu^2 + |\xi|^2)^{\frac{p}{2}}$$

and

$$\xi \to \widetilde{G}(\xi) := G(\xi) - \alpha\ell(\mu^2 + |\xi|^2)^{\frac{p}{2}},$$

respectively (see for example [17], p. 164). Hence, the convexity of $\widetilde{F}$ implies

$$F(\xi) - \ell(\mu^2 + |\xi|^2)^{\frac{p}{2}} \geq F(\eta) - \ell(\mu^2 + |\eta|^2)^{\frac{p}{2}} + \langle D_\xi F(\eta), \xi - \eta \rangle - \ell p \langle (\mu^2 + |\eta|^2)^{\frac{p}{2}-1}\eta, \xi - \eta \rangle$$

and

$$F(\eta) - \ell(\mu^2 + |\eta|^2)^{\frac{p}{2}} \geq F(\xi) - \ell(\mu^2 + |\xi|^2)^{\frac{p}{2}} + \langle D_\xi F(\xi), \eta - \xi \rangle - \ell p \langle (\mu^2 + |\xi|^2)^{\frac{p}{2}-1}\xi, \eta - \xi \rangle.$$

Summing previous inequalities and using Lemmas 2.2 and 2.1, we obtain

$$\begin{aligned} \langle D_\xi F(\xi) - D_\xi F(\eta), \xi - \eta \rangle &\geq \ell p \langle (\mu^2 + |\xi|^2)^{\frac{p}{2}-1}\xi, \xi - \eta \rangle - \ell p \langle (\mu^2 + |\eta|^2)^{\frac{p}{2}-1}\eta, \xi - \eta \rangle \\ &\geq \ell p \langle (\mu^2 + |\xi|^2)^{\frac{p}{2}-1}\xi - (\mu^2 + |\eta|^2)^{\frac{p}{2}-1}\eta, \xi - \eta \rangle \\ &\geq c(p)\ell \int_0^1 (\mu^2 + |\xi + t(\eta - \xi)|^2)^{\frac{p}{2}-1} \, \mathrm{d}t |\xi - \eta|^2 \\ &\geq c(p)\ell |V(\xi) - V(\eta)|^2, \end{aligned}$$

i.e., the first inequality in (2.3). The second inequality in (2.3) can be derived arguing similarly.

Further, if F and G are C^2, then (F2) and (G2) are equivalent to the following standard strong p-ellipticity conditions

$$\langle D^2 F(\xi)\eta, \eta \rangle \geq c_3(\mu^2 + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2, \qquad \langle D^2 G(\xi)\eta, \eta \rangle \geq c_4(\mu^2 + |\xi|^2)^{\frac{p-2}{2}} |\eta|^2$$

for all ξ, $\eta \in \mathbb{R}^n$, where c_i are positive constants of form $c_3 = c(p)\ell$ and $c_4 = c(p)\alpha\ell$, respectively.

The next lemma establishes that the uniform strong p-convexity assumptions (F2) and (G2) yield growth conditions from below for the functions F and G.

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Lemma 2.3. *Suppose that $H : \mathbb{R}^n \to [0, +\infty)$ is a C^1 function such that*

$$0 \le H(\xi) \le \widetilde{L}(\mu^2 + |\xi|^2)^{\frac{p}{2}} \tag{2.4}$$

for all $\xi \in \mathbb{R}^n$, where $p > 1$, $0 \le \mu \le 1$, $\widetilde{L} > 0$. Assume, in addition, that

$$\int_Q H(\xi + \nabla\varphi)\,\mathrm{d}x \ge \int_Q H(\xi) + \widetilde{\ell}(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2\,\mathrm{d}x \tag{2.5}$$

for all $\xi \in \mathbb{R}^n$, $\varphi \in C_0^1(Q)$, $Q \subset \mathbb{R}^n$ and for some positive constant $\widetilde{\ell}$. Then there exists a positive constant $c(p, \widetilde{L}, \widetilde{\ell}, \mu)$ such that

$$H(\xi) \ge \frac{\widetilde{\ell}}{2}(\mu^2 + |\xi|^2)^{\frac{p}{2}} - c(p, \widetilde{L}, \widetilde{\ell}, \mu) \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.6}$$

Proof. We use again the fact that assumption (2.5) is equivalent to the convexity of the function

$$\xi \to K(\xi) := H(\xi) - \widetilde{\ell}(\mu^2 + |\xi|^2)^{\frac{p}{2}}.$$

Hence

$$K(\xi) \ge K(0) + \langle D_\xi K(0), \xi\rangle,$$

or, equivalently,

$$H(\xi) \ge \widetilde{\ell}(\mu^2 + |\xi|^2)^{\frac{p}{2}} + H(0) - \widetilde{\ell}\mu^p + \langle D_\xi H(0), \xi\rangle \tag{2.7}$$

for all $\xi \in \mathbb{R}^n$. By (2.4) and (2.2), we have that

$$H(0) \ge 0 \qquad \text{and} \qquad |D_\xi H(0)| \le 2^p \widetilde{L}\mu^{p-1},$$

and by Young's inequality

$$\begin{aligned}\left|\langle D_\xi H(0), \xi\rangle\right| &= \left|\langle (\widetilde{\ell})^{-\frac{1}{p}} D_\xi H(0), (\widetilde{\ell})^{\frac{1}{p}}\xi\rangle\right| \le c(\varepsilon)(\widetilde{\ell})^{-\frac{1}{p-1}}|D_\xi H(0)|^{\frac{p}{p-1}} + \varepsilon\widetilde{\ell}|\xi|^p \\ &\le c(\varepsilon)2^{\frac{p^2}{p-1}}(\widetilde{\ell})^{-\frac{1}{p-1}}\widetilde{L}^{\frac{p}{p-1}}\mu^p + \varepsilon\widetilde{\ell}(\mu^2 + |\xi|^2)^{\frac{p}{2}}. \end{aligned} \tag{2.8}$$

Inserting (2.8) in (2.7), we get

$$H(\xi) \ge \widetilde{\ell}(\mu^2 + |\xi|^2)^{\frac{p}{2}} - c(\varepsilon)2^{\frac{p^2}{p-1}}(\widetilde{\ell})^{-\frac{1}{p-1}}\widetilde{L}^{\frac{p}{p-1}}\mu^p - \varepsilon\widetilde{\ell}(\mu^2 + |\xi|^2)^{\frac{p}{2}} - \widetilde{\ell}\mu^p$$

and, choosing $\varepsilon = \frac{1}{2}$, we conclude that

$$H(\xi) \ge \frac{\widetilde{\ell}}{2}(\mu^2 + |\xi|^2)^{\frac{p}{2}} - c_p\left(\frac{\widetilde{L}^p}{\widetilde{\ell}}\right)^{\frac{1}{p-1}}\mu^p - \widetilde{\ell}\mu^p. \qquad \square$$

As already mentioned in the Introduction, we will compare the minimizer u of the problem (P) with the minimizer of a suitable regular convex variational integral. In order to take advantage of the comparison argument, we will need the following regularity result (see [10], Thm. 2.2)

Theorem 2.1. *Let $H : \mathbb{R}^n \to [0, +\infty)$ be a continuous function such that*

$$0 \le H(\xi) \le \widetilde{L}(\mu^2 + |\xi|^2)^{\frac{p}{2}}$$

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for all $\xi \in \mathbb{R}^n$, *where* $p > 1$, $0 \le \mu \le 1$, $\widetilde{L} > 0$. *Suppose, in addition, that*

$$\int_Q H(\xi + \nabla\varphi)\,\mathrm{d}x \ge \int_Q H(\xi) + \widetilde{\ell}(\mu^2 + |\xi|^2 + |\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2\,\mathrm{d}x$$

for all $\xi \in \mathbb{R}^n$, $\varphi \in C_0^1(Q)$, $Q \subset \mathbb{R}^n$ *and for some positive constant* $\widetilde{\ell}$. *If* $v \in W^{1,p}(\Omega)$ *is a local minimizer of the functional*

$$\mathcal{H}(w,\Omega) := \int_\Omega H(\nabla w)\,\mathrm{d}x,$$

i.e.,

$$\mathcal{H}(v, B_r(x_0)) = \min\left\{\mathcal{H}(w, B_r(x_0)) \ : \ w \in v + W_0^{1,p}(B_r(x_0))\right\} \qquad \textit{for all } B_r(x_0) \subset \Omega,$$

then v *is locally Lipschitz in* Ω, *and*

$$\operatorname*{ess\,sup}_{B_{\frac{R}{2}}(x_0)} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}} \le c(n, \widetilde{L}, \widetilde{\ell}, p) \fint_{B_R(x_0)} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}}\,\mathrm{d}x \tag{2.9}$$

for every $B_{\frac{R}{2}}(x_0) \subset B_R(x_0) \subset \Omega$.

In what follows, we will need a more explicit dependence on the eigenvalues of H of the constant in (2.9). Actually, a careful inspection of the proof of Theorem 2.2 in [10] reveals that the constant in estimate (2.9) is of the type

$$c(n, \widetilde{L}, \widetilde{\ell}, p) = c\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{2n}{p}} \tag{2.10}$$

where $c = c(n,p) \ge 1$.

The following is a technical iteration lemma (see [17], Lem. 7.3)

Lemma 2.4. *Let* φ *be a nonnegative, nondecreasing function and assume that there exist* $\vartheta \in (0,1)$, $\bar{R} > 0$, *and* $0 < \beta < \gamma$ *such that*

$$\varphi(\vartheta r) \le \vartheta^\gamma \varphi(r) + b r^\beta$$

for all $0 < r \le \bar{R}$. *Then we have*

$$\varphi(\rho) \le C\left\{\left(\frac{\rho}{r}\right)^\beta \varphi(r) + b\rho^\beta\right\},$$

for every $0 < \rho < r \le \bar{R}$, *with* $C = C(\vartheta, \beta, \gamma)$.

The next result relates the decay estimate for the gradient of a Sobolev function with its Hölder regularity properties (see [14], Thm. 1.1, p. 64 and [16], Thm. 7.19)

Theorem 2.2 (Morrey's Lemma)**.** *Let* $u \in W^{1,1}(\Omega)$ *and suppose that there exist positive constants* K, $0 < \alpha \le 1$ *such that*

$$\int_{B_r(x)} |\nabla u|\,\mathrm{d}x \le K r^{n-1+\alpha},$$

for all balls $B(x,r) \subset \Omega$, $x \in \Omega$, $r > 0$. *Then* $u \in C^{0,\alpha}(\Omega)$.

Given a Borel set E in $\mathbb{R}^n$, $P(E,\Omega)$ denotes the perimeter of E in Ω, defined as

$$P(E,\Omega) = \sup\left\{\int_E \mathrm{div}\phi\,\mathrm{d}x : \ \phi \in C_0^1(\Omega;\mathbb{R}^n),\ |\phi| \le 1\right\}.$$

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It is known that, for a set of finite perimeter E, one has

$$P(E,\Omega)=\mathcal{H}^{n-1}(\partial^* E)$$

where

$$\partial^* E=\left\{x\in\Omega:\ \limsup_{\rho\to 0^+}\frac{P(E,B_\rho(x))}{\rho^{n-1}}>0\right\}$$

is the reduced boundary of E (for more details we refer to [5]).

Given a set $E\subset\Omega$ of finite perimeter in Ω, for every ball $B_r(x)\Subset\Omega$ we measure how far E is from being an area minimizer in the ball by setting

$$\psi(E,B_r(x)):=P(E,B_r(x))-\min\left\{P(A,B_r(x)):\ A\Delta E\Subset B_r(x),\ \chi_A\in BV(\mathbb{R}^n)\right\}.$$

The following regularity result, due to Tamanini (see [23]), asserts that if the excess $\psi(E,B_r(x))$ decays fast enough when $r\to 0$, then E has essentially the same regularity properties of an area minimizing set.

Theorem 2.3. *Let Ω be an open subset of $\mathbb{R}^n$ and let E be a set of finite perimeter satisfying, for some $\delta\in(0,\frac{1}{2})$,*

$$\psi(E,B_r(x))\le cr^{n-1+\delta}$$

for every $x\in\Omega$ and every $r\in(0,r_0)$, with $c=c(x),r_0=r_0(x)$ local positive constants. Then $\partial^* E$ is a $C^{1,\delta}$-hypersurface in Ω and $\mathcal{H}^s\left((\partial E\setminus\partial^* E)\cap\Omega\right))=0$ for all $s>n-8$.

3. Proof of Theorem 1.4

This section is devoted to the proof of Theorem 1.4, which follows closely that of Theorem 1 in [8]. Since several modifications are needed, we present it here for the convenience of the reader.

Proof of Theorem 1.4.

Step 1. We prove the first part of Theorem 1.4 arguing by contradiction. Assume that there exist a sequence $\{\lambda_h\}_{h\in\mathbb{N}}$ such that $\lambda_h\to\infty$ as $h\to\infty$, and a sequence of configurations $\{(u_h,E_h)\}$ minimizing $\mathcal{I}_{\lambda_h}$ such that $u_h=u_0$ on $\partial\Omega$ and $|E_h|\neq d$ for all h. Notice that

$$\mathcal{I}_{\lambda_h}(u_h,E_h)\le\mathcal{I}(u_0,E_0)=:\Theta,\tag{3.1}$$

where $E_0\subset\Omega$ is a fixed set of finite perimeter such that $|E_0|=d$. Assume that $|E_h|<d$ for a (not relabeled) subsequence (if $|E_h|>d$ the proof is similar). We claim that, for h sufficiently large, there exists a configuration $(\tilde{u}_h,\widetilde{E}_h)$ such that $\mathcal{I}_{\lambda_h}(\tilde{u}_h,\widetilde{E}_h)<\mathcal{I}_{\lambda_h}(u_h,E_h)$, thus proving that $|E_h|=d$ for all h sufficiently large, say $\lambda\ge\lambda_0$.

By our assumptions on F and G, it follows that the sequence $\{u_h\}$ is bounded in $W^{1,p}(\Omega)$, the perimeters of the sets E_h are bounded, and $|E_h|\to d$. Therefore, without loss of generality, we may assume, possibly extracting a subsequence (not relabeled), that there exists a configuration (u,E) such that $u_h\to u$ weakly in $W^{1,p}(\Omega)$, $\chi_{E_h}\to\chi_E$ a.e. in Ω, and E is a set of finite perimeter in Ω with $|E|=d$.

Step 2. *Construction of $(\tilde{u}_h,\widetilde{E}_h)$.* Fix a point $x\in\partial^* E\cap\Omega$ (such a point exists since E has finite perimeter in Ω, $0<|E|<|\Omega|$, and Ω is connected). By De Giorgi's structure theorem for sets of finite perimeter (see [5], Thm. 3.59), the sets $E_r=(E-x)/r$ converge locally in measure to the half space $H=\{z\cdot\nu_E(x)>0\}$, *i.e.*, $\chi_{E_r}\to\chi_H$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, where $\nu_E(x)$ is the generalized inner normal to E at x (see [5], Def. 3.54). Let $y\in B_1(0)\setminus H$ be the point $y:=-\nu_E(x)/2$. Given $\varepsilon>0$ and small (to be chosen at the end of the proof), since $\chi_{E_r}\to\chi_H$ in $L^1(B_1(0))$ there exists $r>0$ such that

$$B_{2r}(x)\subset\Omega,\qquad |E_r\cap B_{1/2}(y)|<\varepsilon,\qquad |E_r\cap B_1(y)|\ge|E_r\cap B_{1/2}(0)|>\frac{\omega_n}{2^{n+2}},$$

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where ω_n denotes the measure of the unit ball of $\mathbb{R}^n$. Therefore, setting $x_r := x + ry \in \Omega$, we have

$$B_r(x_r) \subset \Omega, \qquad |E \cap B_{r/2}(x_r)| < \varepsilon r^n, \qquad |E \cap B_r(x_r)| > \frac{\omega_n r^n}{2^{n+2}}.$$

Assume, without loss of generality, that $x_r = 0$, and in the sequel denote the open ball centered at the origin and with radius $r > 0$ by B_r. From the convergence of $\{E_h\}$ to E we have that, for all h sufficiently large,

$$|E_h \cap B_{r/2}| < \varepsilon r^n, \qquad |E_h \cap B_r| > \frac{\omega_n r^n}{2^{n+2}}. \tag{3.2}$$

Define the bi-Lipschitz map $\phi : B_r \to B_r$ by

$$\Phi(x) := \begin{cases} \big(1 - \sigma(2^n - 1)\big)x & \text{if } |x| < \dfrac{r}{2}, \\ x + \sigma\left(1 - \dfrac{r^n}{|x|^n}\right)x & \text{if } \dfrac{r}{2} \le |x| < r, \\ x & \text{if } |x| \ge r, \end{cases} \tag{3.3}$$

for some fixed $0 < \sigma < 1/2^n$, to be determined later, such that, setting

$$\widetilde{E}_h := \Phi(E_h), \qquad \tilde{u}_h := u_h \circ \Phi^{-1},$$

we have $|\widetilde{E}_h| < d$. We obtain

$$\begin{aligned} \mathcal{I}_{\lambda_h}(u_h, E_h) - \mathcal{I}_{\lambda_h}(\tilde{u}_h, \widetilde{E}_h) \;&=\; \left[\int_{B_r} \Big(F(\nabla u_h) + \chi_{E_h} G(\nabla u_h)\Big)\,\mathrm{d}x - \int_{B_r} \Big(F(\nabla \tilde{u}_h) + \chi_{\widetilde{E}_h} G(\nabla \tilde{u}_h)\Big)\,\mathrm{d}y\right] \\ &\quad + \big(P(E_h, \overline{B}_r) - P(\widetilde{E}_h, \overline{B}_r)\big) + \lambda_h\big(|\widetilde{E}_h| - |E_h|\big) \\ &=: I_{1,h} + I_{2,h} + I_{3,h}. \end{aligned} \tag{3.4}$$

Step 3. *Estimate of* $I_{1,h}$. We start by evaluating the gradient and the Jacobian determinant of Φ in the annulus $B_r \setminus B_{r/2}$. If $r/2 < |x| < r$, then we have

$$\frac{\partial \Phi_i}{\partial x_j}(x) = \left(1 + \sigma - \frac{\sigma r^n}{|x|^n}\right)\delta_{ij} + n\sigma r^n \frac{x_i x_j}{|x|^{n+2}} \qquad \text{for all } i, j = 1, \dots n$$

and thus, if $\eta \in \mathbb{R}^n$,

$$(\nabla \Phi \circ \eta) \cdot \eta = \left(1 + \sigma - \frac{\sigma r^n}{|x|^n}\right)|\eta|^2 + n\sigma r^n \frac{(x \cdot \eta)^2}{|x|^{n+2}}$$

from which it follows that

$$|\nabla \Phi \circ \eta| \ge \left(1 + \sigma - \frac{\sigma r^n}{|x|^n}\right)|\eta|.$$

From this inequality we easily deduce an estimate on the norm of $\nabla \Phi^{-1}$, precisely,

$$\begin{aligned} \big\|\nabla \Phi^{-1}\big(\Phi(x)\big)\big\| &= \max_{|\eta|=1} \left|\nabla \Phi^{-1} \circ \left(\frac{\nabla \Phi \circ \eta}{|\nabla \Phi \circ \eta|}\right)\right| = \max_{|\eta|=1} \frac{1}{|\nabla \Phi \circ \eta|} \\ &\le \left(1 + \sigma - \frac{\sigma r^n}{|x|^n}\right)^{-1} \le \big(1 - (2^n - 1)\sigma\big)^{-1} \qquad \text{for all } x \in B_r \setminus B_{r/2}. \end{aligned} \tag{3.5}$$

Concerning the Jacobian, we write, for $x \in B_r \setminus B_{r/2}$,

$$\Phi(x) = \varphi(|x|)\frac{x}{|x|}, \tag{3.6}$$

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where

$$\varphi(t) = t\left(1+\sigma-\frac{\sigma r^n}{t^n}\right), \qquad \text{for all } t\in[r/2,r].$$

Let I denote the identity map in $\mathbb{R}^n$. Recalling that if $A = I+a\otimes b$ for some vectors $a,\ b\in\mathbb{R}^n$, then $\det A = 1+a\cdot b$, a straightforward calculation gives for all $x\in B_r\setminus B_{r/2}$

$$J\Phi(x) = \varphi'(|x|)\left(\frac{\varphi(|x|)}{|x|}\right)^{n-1} = \left(1+\sigma+\frac{(n-1)\sigma r^n}{|x|^n}\right)\left(1+\sigma-\frac{\sigma r^n}{|x|^n}\right)^{n-1}. \tag{3.7}$$

We have

$$\begin{aligned}\left(1+\sigma-\frac{\sigma r^n}{|x|^n}\right)^{n-1} &= (1+\sigma)^{n-1}\left(1-\frac{\frac{\sigma r^n}{|x|^n}}{1+\sigma}\right)^{n-1} \geq (1+\sigma)^{n-1}\left(1-(n-1)\frac{\frac{\sigma r^n}{|x|^n}}{1+\sigma}\right)\\ &= (1+\sigma)^{n-2}\left(1+\sigma-(n-1)\frac{\sigma r^n}{|x|^n}\right) \geq 1+\sigma-(n-1)\frac{\sigma r^n}{|x|^n}.\end{aligned} \tag{3.8}$$

Since $x\in B_r\setminus B_{\frac{r}{2}}$, by (3.7) and (3.8) we have

$$\begin{aligned}J\Phi(x) &\geq \left(1+\sigma+(n-1)\frac{\sigma r^n}{|x|^n}\right)\left(1+\sigma-(n-1)\frac{\sigma r^n}{|x|^n}\right) = (1+\sigma)^2-(n-1)^2\frac{\sigma^2 r^{2n}}{|x|^{2n}}\\ &\geq (1+\sigma)^2-4^n(n-1)^2\sigma^2 = 1+2\sigma-\left(4^n(n-1)^2-1\right)\sigma^2 > 1+\sigma\end{aligned} \tag{3.9}$$

provided that we chose

$$\sigma < \frac{1}{4^n(n-1)^2-1}.$$

On the other hand, from (3.7) we get also

$$J\Phi(x) \leq 1+2^n n\sigma. \tag{3.10}$$

Let us now turn to the estimate of $I_{1,h}$. Performing the change of variable $y=\Phi(x)$ in the second integral defining $I_{1,h}$, and observing that $\chi_{\widetilde{E}_h}(\Phi(x)) = \chi_{E_h}(x)$, we get

$$\begin{aligned}I_{1,h} &= \int_{B_r}\Big[F(\nabla u_h(x)) - F\big(\nabla u_h(x)\circ\nabla\Phi^{-1}\big(\Phi(x)\big)\big)J\Phi(x)\\ &\qquad +\chi_{E_h}(x)\Big[G(\nabla u_h(x)) - G\big(\nabla u_h(x)\circ\nabla\Phi^{-1}\big(\Phi(x)\big)\big)J\Phi(x)\Big]\,\mathrm{d}x\\ &=: A_{1,h}+A_{2,h},\end{aligned} \tag{3.11}$$

where $A_{1,h}$ stands for the above integral evaluated in $B_{r/2}$ and $A_{2,h}$ for the same integral evaluated in $B_r\setminus B_{r/2}$. Recalling the definition of Φ in (3.3) and the growth assumptions on F, G in (F1) and (G1), respectively, we get

$$\begin{aligned}A_{1,h} &= \int_{B_{r/2}}\left[F(\nabla u_h(x)) - F\big(\nabla u_h(x)\circ\big(1-\sigma(2^n-1)\big)^{-1}I\big)\big(1-\sigma(2^n-1)\big)^n\right]\mathrm{d}x\\ &\quad+\int_{B_{r/2}}\chi_{E_h}(x)\left[G(\nabla u_h(x)) - G\big(\nabla u_h(x)\circ\big(1-\sigma(2^n-1)\big)^{-1}I\big)\big(1-\sigma(2^n-1)\big)^n\right]\mathrm{d}x\\ &\geq -\int_{B_{r/2}}F(\nabla u_h(x)\circ\big(1-\sigma(2^n-1)\big)^{-1}I\big)\big(1-\sigma(2^n-1)\big)^n\,\mathrm{d}x\end{aligned}$$

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$$
\begin{aligned}
&-\int_{B_{r/2}} \chi_{E_h}(x) G(\nabla u_h(x) \circ \big(1-\sigma(2^n-1)\big)^{-1} I\big)\big(1-\sigma(2^n-1)\big)^n \,\mathrm{d}x \\
&\geq -c(p,\beta,L)\int_{B_{r/2}} (1+\chi_{E_h}(x))\big|\nabla u_h(x) \circ \big(1-\sigma(2^n-1)\big)^{-1} I\big|^p \big(1-\sigma(2^n-1)\big)^n \,\mathrm{d}x \\
&\quad -c(p,\beta,L)\mu^p\big(1-\sigma(2^n-1)\big)^n r^n \\
&\geq -c(p,\beta,L)\int_{B_{r/2}} (1+\chi_{E_h}(x))\big|\nabla u_h(x)\big|^p \big(1-\sigma(2^n-1)\big)^{n-p} \,\mathrm{d}x \\
&\quad -c(p,\beta,L)\mu^p\big(1-\sigma(2^n-1)\big)^n r^n \\
&= -c\big(1-\sigma(2^n-1)\big)^{n-p}\int_{B_{r/2}} \big|\nabla u_h(x)\big|^p \,\mathrm{d}x - c\mu^p\big(1-\sigma(2^n-1)\big)^n r^n \\
&\geq -C(n,p,\beta,L,\sigma,\mu)(\Theta + r^n)
\end{aligned}
$$

where we used (3.1). Recalling (3.5), (3.10) and (3.1) we have

$$
\begin{aligned}
A_{2,h} &= \int_{B_r\setminus B_{r/2}} \big[F(\nabla u_h(x)) - F\left(\nabla u_h(x)\circ\nabla\Phi^{-1}\big(\Phi(x)\big)\right) J\Phi(x)\big]\,\mathrm{d}x \\
&\quad + \int_{B_r\setminus B_{r/2}} \chi_{E_h}(x)\big[G(\nabla u_h(x)) - G\left(\nabla u_h(x)\circ\nabla\Phi^{-1}\big(\Phi(x)\big)\right) J\Phi(x)\big]\,\mathrm{d}x \\
&\geq -c(p,\beta,L)\int_{B_r\setminus B_{r/2}} (1+\chi_{E_h}(x))|\nabla u_h(x)|^p\big(1-(2^n-1)\sigma\big)^{-p}\big(1+2^n n\sigma\big)\,\mathrm{d}x \\
&\quad -c(p,\beta,L)\mu^p\big(1+2^n n\sigma\big)r^n \\
&\geq -C(n,p,\beta,L,\sigma)\int_{B_r\setminus B_{r/2}} |\nabla u_h(x)|^p\,\mathrm{d}x - c(p,\beta,L)\mu^p\big(1+2^n n\sigma\big)r^n \\
&\geq -C(n,p,\beta,L,\sigma,\mu)(\Theta + r^n).
\end{aligned}
$$

Thus, from the above estimates we conclude that

$$
I_{1,h} \geq -C(n,p,\beta,L,\sigma,\mu)(\Theta + r^n). \tag{3.12}
$$

Step 4. *Estimate of* $I_{2,h}$. We use the area formula for maps between rectifiable sets. To this aim, for $x \in \partial^* E_h$ denote by $T_{h,x} : \pi_{h,x} \to \mathbb{R}^n$ the tangential differential at x of Φ along the approximate tangent space $\pi_{h,x}$ to $\partial^* E_h$, which is defined by $T_{h,x}(\tau) = \nabla\Phi(x)\circ\tau$ for all $\tau \in \pi_{h,x}$. We recall (see [5], Def. 2.68) that the $(n-1)$-dimensional jacobian of $T_{h,x}$ is given by

$$
J_{n-1}T_{h,x} = \sqrt{\det\big(T_{h,x}^* \circ T_{h,x}\big)},
$$

where $T_{h,x}^*$ is the adjoint of the map $T_{h,x}$. To estimate $J_{n-1}T_{h,x}$, fix $x \in \partial^* E_h \cap (B_r \setminus B_{r/2})$. Denote by $\{\tau_1,\dots,\tau_{n-1}\}$ an orthonormal base for $\pi_{h,x}$, and by L the $n\times(n-1)$ matrix representing $T_{h,x}$ with respect to the fixed base in $\pi_{h,x}$ and the standard base $\{e_1,\dots,e_n\}$ in $\mathbb{R}^n$. From (3.6) we have

$$
L_{ij} = \nabla\Phi_i\cdot\tau_j = \frac{\varphi(|x|)}{|x|}e_i\cdot\tau_j + \left(\varphi'(|x|) - \frac{\varphi(|x|)}{|x|}\right)\frac{x_i}{|x|}\frac{x\cdot\tau_j}{|x|}, \qquad i=1,\dots,n,\ j=1,\dots,n-1.
$$

Thus, for $j,l=1,\dots,n-1$, we obtain

$$
(L^*\circ L)_{jl} = \frac{\varphi^2(|x|)}{|x|^2}\sum_{i=1}^n (e_i\cdot\tau_j)(e_i\cdot\tau_l) + \left(\varphi'^2(|x|) - \frac{\varphi^2(|x|)}{|x|^2}\right)\frac{(x\cdot\tau_j)(x\cdot\tau_l)}{|x|^2}.
$$

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Since $J_{n-1}T_{h,x}$ is invariant by rotation, in order to evaluate $\det(L^*\circ L)$ we may assume, without loss of generality, that $\tau_j = e_j$, for all $j = 1,\dots,n-1$. We deduce that

$$L^*\circ L = \frac{\varphi^2(|x|)}{|x|^2} I^{(n-1)} + \left(\varphi'^2(|x|) - \frac{\varphi^2(|x|)}{|x|^2}\right)\frac{x'\otimes x'}{|x|^2},$$

where $I^{(n-1)}$ denotes the identity map on $\mathbb{R}^{n-1}$ and $x' = (x_1,\dots,x_{n-1})$. With a calculation similar to the one performed to obtain (3.7), from the equality above we easily get that

$$\det(L^*\circ L) = \left(\frac{\varphi^2(|x|)}{|x|^2}\right)^{n-1}\left[1 + \frac{|x|^2}{\varphi^2(|x|)}\left(\varphi'^2(|x|) - \frac{\varphi^2(|x|)}{|x|^2}\right)\frac{|x'|^2}{|x|^2}\right]$$

and so, using (3.6) we can estimate for $x\in\partial^* E_h\cap(B_r\setminus B_{r/2})$

$$\begin{aligned} J_{n-1}T_{h,x} = \sqrt{\det(L^*\circ L)} &= \left(\frac{\varphi(|x|)}{|x|}\right)^{n-1}\sqrt{1 + \frac{|x|^2}{\varphi^2(|x|)}\left(\varphi'^2(|x|) - \frac{\varphi^2(|x|)}{|x|^2}\right)\frac{|x'|^2}{|x|^2}} \\ &\le \left(\frac{\varphi(|x|)}{|x|}\right)^{n-2}\varphi'(|x|) \le \varphi'(|x|) \le 1+\sigma+2^n(n-1)\sigma. \end{aligned} \tag{3.13}$$

To estimate $I_{2,h}$, we use the area formula for maps between rectifiable sets ([5], Thm. 2.91), and we get

$$\begin{aligned} I_{2,h} &= P(E_h,\overline{B}_r) - P(\widetilde{E}_h,\overline{B}_r) = \int_{\partial^* E_h\cap\overline{B}_r} \mathrm{d}\mathcal{H}^{n-1} - \int_{\partial^* E_h\cap\overline{B}_r} J_{n-1}T_{h,x}\,\mathrm{d}\mathcal{H}^{n-1} \\ &= \int_{\partial^* E_h\cap\overline{B}_r\setminus B_{r/2}} (1 - J_{n-1}T_{h,x})\,\mathrm{d}\mathcal{H}^{n-1} + \int_{\partial^* E_h\cap B_{r/2}} (1 - J_{n-1}T_{h,x})\,\mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

Notice that the last integral in the above formula is nonnegative since Φ is a contraction in $B_{r/2}$, hence $J_{n-1}T_{h,x} < 1$ in $B_{r/2}$, while from (3.13) and (3.1) we have

$$\int_{\partial^* E_h\cap\overline{B}_r\setminus B_{r/2}} (1 - J_{n-1}T_{h,x})\,\mathrm{d}\mathcal{H}^{n-1} \ge -2^n n P(E_h,\overline{B}_r)\sigma \ge -2^n n\Theta\sigma,$$

thus concluding that

$$I_{2,h} \ge -c(n)\Theta\sigma. \tag{3.14}$$

Step 5. *Estimate of* $I_{3,h}$. We recall (3.2), (3.3), (3.7) to obtain

$$\begin{aligned} I_{3,h} &= \lambda_h\int_{E_h\cap B_r\setminus B_{r/2}} (J\Phi(x) - 1)\,\mathrm{d}x + \lambda_h\int_{E_h\cap B_{r/2}} (J\Phi(x) - 1)\,\mathrm{d}x \\ &\ge \lambda_h C_1(n)\left(\frac{\omega_n}{2^{n+2}} - \varepsilon\right)\sigma r^n - \lambda_h\left[1 - \left(1 - (2^n-1)\sigma\right)^n\right]\varepsilon r^n \\ &\ge \lambda_h\sigma r^n\left[C_1(n)\frac{\omega_n}{2^{n+2}} - C_1(n)\varepsilon - (2^n-1)n\varepsilon\right]. \end{aligned}$$

Therefore, if we choose $0 < \varepsilon < \varepsilon(n)$, with $\varepsilon(n)$ depending only on the dimension, we have that

$$I_{3,h} \ge \lambda_h C_2(n)\sigma r^n \tag{3.15}$$

for some positive $C_2(n)$.

Step 6. *Conclusion of Step 1.* Estimate (3.15), together with (3.4), (3.12) and (3.14), yields

$$\mathcal{I}_{\lambda_h}(u_h,E_h) - \mathcal{I}_{\lambda_h}(\tilde{u}_h,\widetilde{E}_h) \ge \lambda_h\sigma C_3 r^n - C(n,p,\sigma,\mu)(\Theta + r^n) > 0$$

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if λ_h is sufficiently large. This contradicts the minimality of (u_h, E_h), thus concluding the proof of the first part of Theorem 1.4.

Step 7. Conversely, if (u, E) is a minimizer of $\mathcal{I}$ and λ_0 is as determined on Step 1, then for $\lambda > \lambda_0$ Steps 1–5 ensure the existence of a minimizer (u_λ, E_λ) of $\mathcal{I}_\lambda$ with $|E_\lambda| = d$. Hence, by the minimality,

$$\mathcal{I}(u, E) \leq \mathcal{I}(u_\lambda, E_\lambda) = \mathcal{I}_\lambda(u_\lambda, E_\lambda) \leq \mathcal{I}_\lambda(u, E) = \mathcal{I}(u, E)$$

i.e.,

$$\mathcal{I}(u, E) = \mathcal{I}_\lambda(u_\lambda, E_\lambda) \qquad \square$$

and so (u, E) also minimizes $\mathcal{I}_\lambda$.

4. Proof of Theorem 1.1

This section is devoted to the proof of our first regularity result, stated in Theorem 1.1. The proof is obtained by establishing that the bulk energy and the perimeter of the free interface both decay on balls of radius ρ as ρ^{n-1}, for $\rho \to 0^+$. We divide it in two steps: In the first we prove the decay estimate for the perimeter, and in the second we address the decay of the bulk energies.

Proof of Theorem 1.1. Let (u, E) be a solution of the problem (P).

Step 1. *First decay estimate.* Fix $x_0 \in \Omega$ and let $R \leq \operatorname{dist}(x_0, \partial\Omega)$. Assume, without loss of generality that $0 < R < 1$. Here we want to prove that there exists a constant $c_0 = c_0(n, p, \lambda_0, \alpha, \beta, \ell, L)$ such that

$$\int_{B_r(x_0)\cap E} |\nabla u|^p \, \mathrm{d}x + P(E, B_r(x_0)) \leq c_0 r^{n-1}, \tag{4.1}$$

for every $0 < r < R$.

First, consider $x_0 \in \partial E \cap \Omega$ and set $\widetilde{E} := E \setminus B_r(x_0)$ where $0 < r < R$. For λ_0 determined in Theorem 1.4, we have

$$\mathcal{I}_{\lambda_0}(u, E) \leq \mathcal{I}_{\lambda_0}(u, \widetilde{E}),$$

i.e.,

$$\begin{aligned} &\int_\Omega \Big(F(\nabla u) + \chi_E G(\nabla u)\Big)\, \mathrm{d}x + P(E, \Omega) \\ &\leq \int_\Omega \Big(F(\nabla u) + \chi_{\widetilde{E}} G(\nabla u)\Big)\, \mathrm{d}x + P(\widetilde{E}, \Omega) \, + \lambda_0 |\, |\widetilde{E}| \, - \, d\,|. \end{aligned}$$

Therefore,

$$\int_\Omega (\chi_E - \chi_{\widetilde{E}}) G(\nabla u)\, \mathrm{d}x + P(E, B_r(x_0)) \leq P(B_r(x_0)) + \lambda_0 |\, |\widetilde{E}| \, - \, d\,|,$$

and so

$$\int_{B_r(x_0)} \chi_E G(\nabla u)\, \mathrm{d}x + P(E, B_r(x_0)) \leq c(n) r^{n-1} + c(n)\lambda_0 r^n \leq c(n, \lambda_0) r^{n-1},$$

since $r < 1$. Lemma 2.3 implies that

$$\alpha \frac{\ell}{2} \int_{B_r(x_0)} \chi_E |\nabla u|^p \, \mathrm{d}x - c(p, \mu, \alpha, \beta, \ell, L) |B_r(x_0) \cap E| + P(E, B_r(x_0)) \leq c(n, \lambda_0) r^{n-1},$$

or, equivalently,

$$\alpha \frac{\ell}{2} \int_{B_r(x_0)} \chi_E |\nabla u|^p \, \mathrm{d}x + P(E, B_r(x_0)) \leq c(n, \lambda_0) r^{n-1} + c(p, \mu, \alpha, \beta, \ell, L) |B_r(x_0) \cap E|$$

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$$\leq c(n,p,\mu,\alpha,\beta,\ell,L,\lambda_0)r^{n-1}.$$

Therefore

$$\min\left\{\alpha\frac{\ell}{2},1\right\}\left[\int_{B_r(x_0)}\chi_E|\nabla u|^p\,\mathrm{d}x+P(E,B_r(x_0))\right]\leq c(n,p,\mu,\alpha,\beta,\ell,L,\lambda_0)r^{n-1}.$$

This inequality yields that

$$\int_{B_r(x_0)\cap E}|\nabla u|^p\,\mathrm{d}x+P(E,B_r(x_0))\leq c_0 r^{n-1}, \tag{4.2}$$

where we set $c_0:=c(n,p,\lambda_0,\alpha,\beta,\ell,L)$.

If $x_0\notin\partial E\cap\Omega$, or $B_r(x_0)\cap E$ is not empty and we argue exactly as before, or $B_r(x_0)\subset\Omega\setminus E$ and estimate (4.1) is trivially satisfied.

Step 2. *Second decay estimate.* Here we want to prove that there exist $\tau\in\left(0,\frac{1}{2}\right)$ and $\delta\in(0,1)$ such that for every $M>0$ there exists $h_0\in\mathbb{N}$ such that $\forall B(x_0,r)\subset\Omega$ we have

$$\int_{B_r(x_0)}|\nabla u|^p\,\mathrm{d}x\leq h_0 r^{n-1}\qquad\text{or}\qquad\int_{B_{\tau r}(x_0)}|\nabla u|^p\,\mathrm{d}x\leq M\tau^{n-\delta}\int_{B_r(x_0)}|\nabla u|^p\,\mathrm{d}x. \tag{4.3}$$

In order to prove (4.3), we argue by contradiction. Fix $\tau\in(0,1/2)$, $\delta\in(0,1)$ and choose $M>\tau^{\delta-n}$. Suppose that for every $h\in\mathbb{N}$, there exists a ball $B_{r_h}(x_h)\subset\Omega$ such that

$$\int_{B_{r_h}(x_h)}|\nabla u|^p\,\mathrm{d}x>hr_h^{n-1} \tag{4.4}$$

and

$$\int_{B_{\tau r_h}(x_h)}|\nabla u|^p\,\mathrm{d}x>M\tau^{n-\delta}\int_{B_{r_h}(x_h)}|\nabla u|^p\,\mathrm{d}x. \tag{4.5}$$

Note that estimates (4.2) and (4.4) yield

$$\int_{B_{r_h}(x_h)\cap E}|\nabla u|^p\,\mathrm{d}x+P(E,B_{r_h}(x_h))\leq c_0 r_h^{n-1}<\frac{c_0}{h}\int_{B_{r_h}(x_h)}|\nabla u|^p\,\mathrm{d}x, \tag{4.6}$$

and so

$$\int_{B_{r_h}(x_h)\cap E}|\nabla u|^p\,\mathrm{d}x<\frac{c_0}{h}\int_{B_{r_h}(x_h)}|\nabla u|^p\,\mathrm{d}x. \tag{4.7}$$

Substep 2.a. *Blow-up.* Set

$$\varsigma_h^p=⨍_{B_{r_h}(x_h)}|\nabla u|^p\,\mathrm{d}x$$

and, for $y\in B_1(0)$, introduce the sequence of rescaled functions defined as

$$v_h(y):=\frac{u(x_h+r_h y)-a_h}{\varsigma_h r_h},\quad\text{where}\quad a_h:=⨍_{B_{r_h}(x_h)}u(x)\,\mathrm{d}x.$$

We have

$$\nabla v_h(y)=\frac{1}{\varsigma_h}\nabla u(x_h+r_h y)$$

and a change of variable yields

$$⨍_{B_1}|\nabla v_h(y)|^p\,\mathrm{d}y=\frac{1}{\varsigma_h^p}⨍_{B_{r_h}(x_h)}|\nabla u(x)|^p\,\mathrm{d}x=1. \tag{4.8}$$

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Therefore, there exist a subsequence of v_h (not relabeled) and $v \in W^{1,p}(B_1)$ such that

$$v_h \rightharpoonup v \qquad \text{weakly in } W^{1,p}(B_1), \text{ and } v_h \to v \qquad \text{strongly in } L^p(B_1).$$

Moreover, the lower semicontinuity of the norm implies

$$\fint_{B_1} |\nabla v(y)|^p \, \mathrm{d}y \le \liminf_{h\to\infty} \fint_{B_1} |\nabla v_h(y)|^p \, \mathrm{d}y = 1. \tag{4.9}$$

Substep 2.b. *We claim that* $v_h \to v$ *in* $W^{1,p}_{\mathrm{loc}}(B_1)$. Consider the sets

$$E_h^* := \frac{E - x_h}{r_h} \cap B_1.$$

Since $P(E_h^*, B_1) = \frac{1}{r_h^{n-1}} P(E, B_{r_h}(x_h))$, by (4.6) we have that $P(E_h^*, B_1)$ are bounded, and so, up to the extraction of a subsequence (not relabeled), $\chi_{E_h^*} \to \chi_{E^*}$ in L^1 (and weakly in $BV(B_1)$) for some set of finite perimeter $E^* \subset B_1$.

Using the minimality of (u, E) with respect to $(u + \varphi, E)$, for $\varphi \in W^{1,p}_0(B_{r_h}(x_h))$ we obtain

$$\int_{B_{r_h}(x_h)} \Big(F(\nabla u(x)) + \chi_E G(\nabla u(x)) \Big) \, \mathrm{d}x \le \int_{B_{r_h}(x_h)} \Big(F(\nabla u(x) + \nabla\varphi(x)) + \chi_E G(\nabla u(x) + \nabla\varphi(x)) \Big) \, \mathrm{d}x,$$

or, equivalently, using the change of variable $x = x_h + r_h y$, we get

$$\begin{aligned} &\int_{B_1} \Big(F(\varsigma_h \nabla v_h(y)) + \chi_{E_h^*} G(\varsigma_h \nabla v_h(y)) \Big) \, \mathrm{d}y \\ &\le \int_{B_1} \Big(F(\varsigma_h \nabla v_h(y) + \nabla\psi(y)) + \chi_{E_h^*} G(\varsigma_h \nabla v_h(y) + \nabla\psi(y)) \Big) \, \mathrm{d}x \end{aligned} \tag{4.10}$$

for every $\psi \in W^{1,p}_0(B_1)$. Let $\eta \in C^\infty_0(B_1)$, $0 \le \eta \le 1$. Choosing $\psi_h(y) = \varsigma_h \eta (v - v_h)$ as test function in (4.10), we get

$$\begin{aligned} &\int_{B_1} \Big(F(\varsigma_h \nabla v_h(y)) + \chi_{E_h^*} G(\varsigma_h \nabla v_h(y)) \Big) \, \mathrm{d}y \\ &\le \int_{B_1} \Big(F\big(\varsigma_h \eta \nabla v(y) + \varsigma_h (1-\eta) \nabla v_h(y) + \nabla\eta \varsigma_h (v - v_h)\big) \Big) \, \mathrm{d}y \\ &\quad + \int_{B_1} \chi_{E_h^*} \Big(G\big(\varsigma_h \eta \nabla v(y) + \varsigma_h (1-\eta) \nabla v_h(y) + \nabla\eta \varsigma_h (v - v_h)\big) \Big) \, \mathrm{d}y \\ &\le \int_{B_1} \Big(F\big(\varsigma_h \eta \nabla v(y) + \varsigma_h (1-\eta) \nabla v_h(y)\big) \Big) \, \mathrm{d}y + \int_{B_1} \chi_{E_h^*} \Big(G\big(\varsigma_h \eta \nabla v(y) + \varsigma_h (1-\eta) \nabla v_h(y)\big) \Big) \, \mathrm{d}y \\ &\quad + \int_{B_1} \Big\langle D_\xi F\big(\varsigma_h \eta \nabla v(y) + \varsigma_h (1-\eta) \nabla v_h(y) + \nabla\eta \varsigma_h (v - v_h)\big), \nabla\eta \varsigma_h (v - v_h) \Big\rangle \, \mathrm{d}y \\ &\quad + \int_{B_1} \chi_{E_h^*} \Big\langle D_\xi G\big(\varsigma_h \eta \nabla v(y) + \varsigma_h (1-\eta) \nabla v_h(y) + \nabla\eta \varsigma_h (v - v_h)\big), \nabla\eta \varsigma_h (v - v_h) \Big\rangle \, \mathrm{d}y \\ &\le \int_{B_1} \Big(F\big(\varsigma_h \eta \nabla v(y) + \varsigma_h (1-\eta) \nabla v_h(y)\big) \Big) \, \mathrm{d}y + \int_{B_1} \chi_{E_h^*} \Big(G\big(\varsigma_h \eta \nabla v(y) + \varsigma_h (1-\eta) \nabla v_h(y)\big) \Big) \, \mathrm{d}y \\ &\quad + c \int_{B_1} \big(\mu^2 + |\varsigma_h \nabla v_h|^2 + |\varsigma_h \nabla v|^2 + |\varsigma_h (v - v_h)|^2 \big)^{\frac{p-1}{2}} |\varsigma_h (v - v_h)| \, \mathrm{d}y \end{aligned} \tag{4.11}$$

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where, in the last inequality, we used (2.2). Hence, using Hölder's inequality and the convexity of F and G in estimate (4.11), we obtain

$$\begin{aligned}
&\int_{B_1}\Big(F(\varsigma_h\nabla v_h(y))+\chi_{E_h^*}G(\varsigma_h\nabla v_h(y))\Big)\,\mathrm{d}y\\
&\le\int_{B_1}\Big((1-\eta)F(\varsigma_h\nabla v_h(y))\,\mathrm{d}y+\eta F(\varsigma_h\nabla v(y))\Big)\,\mathrm{d}y\\
&\quad+\int_{B_1}\chi_{E_h^*}\Big((1-\eta)G(\varsigma_h\nabla v_h(y))+\eta G(\varsigma_h\nabla v(y))\Big)\,\mathrm{d}y\\
&\quad+c\int_{B_1}|\varsigma_h(v-v_h)|^p\,\mathrm{d}x+\varsigma_h^p\left(\int_{B_1}\mu^p+|\nabla v_h|^p+|\nabla v|^p\,\mathrm{d}x\right)^{\frac{p-1}{p}}\left(\int_{B_1}|v-v_h|^p\,\mathrm{d}x\right)^{\frac{1}{p}},
\end{aligned}\tag{4.12}$$

since we may suppose that $\varsigma_h>1$ for h large. In fact, by (4.4) and the definition of ς_h, we have

$$\varsigma_h^p\ge\frac{h}{r_h},\tag{4.13}$$

and so $\varsigma_h\to+\infty$ as $h\to+\infty$. By virtue of (4.8), from estimate (4.12) we infer that

$$\begin{aligned}
&\int_{B_1}\eta\Big(F(\varsigma_h\nabla v_h(y))+\chi_{E_h^*}G(\varsigma_h\nabla v_h(y))\Big)\,\mathrm{d}y\\
&\le\int_{B_1}\Big(\eta F(\varsigma_h\nabla v(y))\Big)\,\mathrm{d}y+\int_{B_1}\chi_{E_h^*}\Big(\eta G(\varsigma_h\nabla v(y))\Big)\,\mathrm{d}y\\
&\quad+c\varsigma_h^p\int_{B_1}|v-v_h|^p+c\varsigma_h^p\left(\int_{B_1}|v-v_h|^p\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned}\tag{4.14}$$

Note that, by changing variable in (4.7), we have

$$r_h^n\int_{B_1\cap E_h^*}|\nabla u(x_h+r_hy)|^p\,\mathrm{d}y<\frac{c_0}{h}r_h^n\int_{B_1}|\nabla u(x_h+r_hy)|^p\,\mathrm{d}y,$$

and thus, by the definition of v_h,

$$\int_{B_1\cap E_h^*}|\nabla v_h(y)|^p\,\mathrm{d}y<\frac{c_0}{h}\int_{B_1}|\nabla v_h(y)|^p\,\mathrm{d}y$$

and, by the use of (4.8), we get

$$\int_{B_1\cap E_h^*}|\nabla v_h(y)|^p\,\mathrm{d}y<\frac{c_0\omega_n}{h}.\tag{4.15}$$

Since $\chi_{E_h^*}\to\chi_{E^*}$ weakly in $BV(B_1)$, by Fatou's Lemma and (4.15) we obtain

$$\int_{B_1\cap E^*}|\nabla v(y)|^p\,\mathrm{d}y\le\liminf_h\int_{B_1\cap E_h^*}|\nabla v_h(y)|^p\,\mathrm{d}y=0.\tag{4.16}$$

Using assumption (G1) and the homogeneity of F in (4.14), we get

$$\begin{aligned}
\int_{B_1}\eta\varsigma_h^pF(\nabla v_h(y))\,\mathrm{d}y&\le\int_{B_1}\eta\varsigma_h^pF(\nabla v(y))\,\mathrm{d}y+c\int_{B_1}\chi_{E_h^*}\big(|\varsigma_h\nabla v(y)|^p+|\varsigma_h\nabla v_h(y)|^p\big)\,\mathrm{d}y\\
&\quad+c\varsigma_h^p\int_{B_1}|v-v_h|^p\,\mathrm{d}y+c\varsigma_h^p\left(\int_{B_1}|v-v_h|^p\,\mathrm{d}y\right)^{\frac{1}{p}},
\end{aligned}$$

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i.e.,

$$\int_{B_1} \eta F(\nabla v_h(y))\,\mathrm{d}y \le \int_{B_1} \eta F(\nabla v(y))\,\mathrm{d}y + \int_{B_1} \chi_{E_h^*}(|\nabla v(y)|^p + |\nabla v_h(y)|^p)\,\mathrm{d}y + c\int_{B_1} |v - v_h|^p\,\mathrm{d}y + c\left(\int_{B_1} |v - v_h|^p\,\mathrm{d}y\right)^{\frac{1}{p}}. \tag{4.17}$$

Passing to the limit as $h \to +\infty$ in (4.17), by virtue of (4.16), the strong convergence of v_h to v in L^p and the lower semicontinuity of F, we obtain

$$\int_{B_1} \eta F(\nabla v(y))\,\mathrm{d}y \le \liminf_h \int_{B_1} \eta F(\nabla v_h(y))\,\mathrm{d}y \le \int_{B_1} \eta F(\nabla v(y))\,\mathrm{d}y,$$

that is,

$$\lim_h \int_{B_1} \eta F(\nabla v_h(y))\,\mathrm{d}y = \int_{B_1} \eta F(\nabla v(y))\,\mathrm{d}y. \tag{4.18}$$

By the strong p-convexity of F and Lemma 2.1, we have

$$\begin{aligned} &\int_{B_1} \eta |V(\nabla v_h(y)) - V(\nabla v(y))|^2\,\mathrm{d}y \\ &\le c(p,\ell)\int_{B_1} \eta\Big(F(\nabla v_h(y)) - F(\nabla v(y))\Big) - \langle D_\xi F(\nabla v_h(y)), \eta(\nabla v_h(y) - \nabla v(y))\rangle\,\mathrm{d}y. \end{aligned} \tag{4.19}$$

By the minimality of (u, E), we get

$$\int_{B_{r_h}(x_h)} \Big\langle D_\xi F(\nabla u(y)) + \chi_E D_\xi G(\nabla u(y)), \nabla\varphi\Big\rangle\,\mathrm{d}x = 0$$

for every $\varphi \in W_0^{1,p}(B_{r_h}(x_h))$ or, equivalently,

$$\int_{B_1} \Big\langle D_\xi F(\varsigma_h \nabla v_h(y)) + \chi_{E_h^*} D_\xi G(\varsigma_h \nabla v_h(y)), \nabla\psi\Big\rangle\,\mathrm{d}x = 0$$

for every $\psi \in W_0^{1,p}(B_1)$, or still

$$\int_{B_1} \Big\langle D_\xi F(\varsigma_h \nabla v_h(y)), \nabla\psi\Big\rangle\,\mathrm{d}y = -\int_{B_1} \Big\langle \chi_{E_h^*} D_\xi G(\varsigma_h \nabla v_h(y)), \nabla\psi\Big\rangle\,\mathrm{d}x. \tag{4.20}$$

Then, choosing $\psi := \eta(v_h - v)$ with $\eta \in C_0^\infty(B_1)$ as test function in (4.20), we obtain

$$\begin{aligned} &\int_{B_1} \Big\langle D_\xi F(\varsigma_h \nabla v_h(y)), \eta(\nabla v_h - \nabla v)\Big\rangle\,\mathrm{d}y \\ &= -\int_{B_1} \Big\langle D_\xi F(\varsigma_h \nabla v_h(y)), \nabla\eta(v_h - v)\Big\rangle\,\mathrm{d}y \\ &\quad - \int_{B_1} \chi_{E_h^*} \Big\langle D_\xi G(\varsigma_h \nabla v_h(y)), \eta(\nabla v_h - \nabla v)\Big\rangle\,\mathrm{d}y \\ &\quad - \int_{B_1} \chi_{E_h^*} \Big\langle D_\xi G(\varsigma_h \nabla v_h(y)), \nabla\eta(v_h - v)\Big\rangle\,\mathrm{d}y. \end{aligned} \tag{4.21}$$

Using estimates (2.2) for $D_\xi F$ and $D_\xi G$, (4.21) yields

$$\left|\int_{B_1} \Big\langle D_\xi F(\varsigma_h \nabla v_h(y)), \eta(\nabla v_h - \nabla v)\Big\rangle\,\mathrm{d}y\right|$$

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$$\leq c(p,\beta,L)\int_{B_1}|\varsigma_h\nabla v_h(y)|^{p-1}|\nabla\eta||v_h-v|\,\mathrm{d}y$$
$$+c(p,\beta,L)\int_{B_1\cap E_h^*}|\varsigma_h\nabla v_h(y)|^{p-1}|\eta||\nabla v_h-\nabla v|\,\mathrm{d}y. \tag{4.22}$$

By the homogeneity of F, Hölder's inequality, (4.8) and (4.15), (4.22) implies that

$$\left|\int_{B_1}\Big\langle D_\xi F(\nabla v_h(y)),\eta(\nabla v_h-\nabla v)\Big\rangle\,\mathrm{d}y\right|$$
$$\leq c(p,\beta,L,||\nabla\eta||_\infty)\left(\int_{B_1}|\nabla v_h(y)|^p\,\mathrm{d}y\right)^{\frac{p-1}{p}}\left(\int_{B_1}|v_h-v|^p\,\mathrm{d}y\right)^{\frac{1}{p}}$$
$$+c(p,\beta,L,||\eta||_\infty)\left(\int_{B_1\cap E_h^*}|\nabla v_h(y)|^p\,\mathrm{d}y\right)^{\frac{p-1}{p}}\left(\int_{B_1}|\nabla v_h|^p+|\nabla v|^p\,\mathrm{d}y\right)^{\frac{1}{p}}$$
$$\leq c(n,p,\beta,L,||\nabla\eta||_\infty)\left(\int_{B_1}|v_h-v|^p\,\mathrm{d}y\right)^{\frac{1}{p}}+c(p,\beta,L,||\eta||_\infty)\left(\frac{c_0}{h}\right)^{\frac{p-1}{p}}. \tag{4.23}$$

Since v_h converge strongly to v in $L^p(B_1)$, passing to the limit as $h\to\infty$ in (4.23), we get

$$\lim_{h\to+\infty}\left|\int_{B_1}\langle D_\xi F(\nabla v_h(y)),\eta(\nabla v_h-\nabla v)\rangle\,\mathrm{d}y\right|=0. \tag{4.24}$$

Passing to the limit in (4.19) and using (4.18) and (4.24), we obtain

$$\lim_{h\to+\infty}\int_{B_1}\eta|V(\nabla v_h(y))-V(\nabla v(y))|^2\,\mathrm{d}y=0,$$

which, by Lemma 2.1, implies that

$$\lim_{h\to+\infty}\int_{B_1}\eta(\mu^2+|\nabla v_h(y)|^2+|\nabla v(y)|^2)^{\frac{p-2}{2}}|\nabla v_h(y)-\nabla v(y)|^2\,\mathrm{d}y=0. \tag{4.25}$$

In the case $p\geq 2$, one can easily check that (4.25) implies

$$v_h\to v\qquad\text{strongly in } W^{1,p}_{\mathrm{loc}}(B_1).$$

In the case $1<p<2$, it suffices to observe that Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ yields

$$\int_{B_1}\eta|\nabla v_h-\nabla v|^p\,\mathrm{d}x$$
$$\leq\left(\int_{B_1}\eta(\mu^2+|\nabla v_h|^2+|\nabla v|^2)^{\frac{p-2}{2}}|\nabla v_h-\nabla v|^2\,\mathrm{d}x\right)^{\frac{p}{2}}\left(\int_{B_1}\eta(\mu^2+|\nabla v_h|^2+|\nabla v|^2)^{\frac{p}{2}}\,\mathrm{d}x\right)^{\frac{2-p}{2}}$$
$$\leq c\left(\int_{B_1}\eta(\mu^2+|\nabla v_h|^2+|\nabla v|^2)^{\frac{p-2}{2}}|\nabla v_h-\nabla v|^2\,\mathrm{d}x\right)^{\frac{p}{2}}, \tag{4.26}$$

where we used (4.8). Hence, also in this case, by (4.25) we conclude that

$$v_h\to v\qquad\text{strongly in } W^{1,p}_{\mathrm{loc}}(B_1),$$

and this asserts the claim.

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Substep 2.c. *Reaching a contradiction.* Notice that (4.5) can be written as

$$\fint_{B_{\tau r_h}(x_h)} |\nabla u|^p \,\mathrm{d}x > M\tau^{-\delta} \fint_{B_{r_h}(x_h)} |\nabla u|^p \,\mathrm{d}x,$$

or, equivalently,

$$\frac{1}{\varsigma_h^p} \fint_{B_{\tau r_h}(x_h)} |\nabla u|^p \,\mathrm{d}x > M\tau^{-\delta}, \tag{4.27}$$

by the definition of ς_h. By the change of variable $x = x_h + r_h y$ and the definition of v_h, from (4.27) we infer that

$$\frac{1}{\varsigma_h^p} \fint_{B_\tau} |\varsigma_h \nabla v_h|^p \,\mathrm{d}x > M\tau^{-\delta},$$

i.e.,

$$\fint_{B_\tau} |\nabla v_h|^p \,\mathrm{d}x > M\tau^{-\delta}. \tag{4.28}$$

By virtue of the strong convergence of v_h to v in $W^{1,p}_{\mathrm{loc}}(B_1)$ and (4.9), we have that

$$\lim_h \fint_{B_\tau} |\nabla v_h|^p = \fint_{B_\tau} |\nabla v|^p \leq \frac{1}{\tau^n}. \tag{4.29}$$

Clearly, (4.29) contradicts (4.28) because $M > \tau^{\delta - n}$.

Step 3. *Conclusion.* We conclude that if (u, E) is a solution of (P), then there exist $\tau \in \left(0, \frac{1}{2}\right)$ and $\delta \in (0,1)$ such that, setting $M = 1$, there exists $h_0 \in \mathbb{N}$ with the property that whenever $B_r(x) \subset \Omega$, then

$$\int_{B_r(x_0)} |\nabla u|^p \,\mathrm{d}x \leq h_0 r^{n-1} \qquad \text{or} \qquad \int_{B_{\tau r}(x_0)} |\nabla u|^p \,\mathrm{d}x \leq \tau^{n-\delta} \int_{B_r(x_0)} |\nabla u|^p \,\mathrm{d}x.$$

Hence,

$$\int_{B_{\tau r}(x_0)} |\nabla u|^p \,\mathrm{d}x \leq \tau^{n-\delta} \int_{B_r(x_0)} |\nabla u|^p \,\mathrm{d}x + h_0 r^{n-1},$$

and using Lemma 2.4 with $\varphi(\rho) := \int_{B_\rho(x_0)} |\nabla u|^p \,\mathrm{d}x$, $\gamma = n - \delta$ and $\beta = n - 1$, we obtain that

$$\int_{B_\rho(x_0)} |\nabla u|^p \,\mathrm{d}x \leq c \left\{ \left(\frac{\rho}{r}\right)^{n-1} \int_{B_r(x_0)} |\nabla u|^p \,\mathrm{d}x + h_0 \rho^{n-1} \right\},$$

for every $0 < \rho < r \leq R$, and so

$$\int_{B_\rho(x_0)} |\nabla u|^p \,\mathrm{d}x \leq C\rho^{n-1}.$$

By Hölder's inequality

$$\int_{B_\rho(x_0)} |\nabla u| \,\mathrm{d}x \leq c \left(\int_{B_\rho(x_0)} |\nabla u|^p \,\mathrm{d}x \right)^{\frac{1}{p}} \rho^{\frac{n}{p'}} \leq C\rho^{\frac{n-1}{p} + \frac{n}{p'}} = C\rho^{n - \frac{1}{p}}.$$

Theorem 2.2 yields that, at least, u is locally Hölder continuous with exponent $\frac{1}{p'}$. At this point, the equivalence between reduced boundary and topological boundary can be obtained arguing as in ([4], Thm. 2.2), with the obvious modifications. □

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5. Proof of Theorem 1.2 – Full regularity

This section is devoted to the proof of the full regularity result stated in Theorem 1.2. The key point is to prove that if the ratio $\frac{\beta}{\alpha+1}$, where α and β are the parameters appearing in hypotheses (G1) and (G2), is sufficiently small then $\int_{B_\rho}|\nabla u|^p$ decays as $\rho^{n-1+\delta}$.

Proof of Theorem 1.2.

Step 1. Let (u,E) be a minimal configuration of problem (P). We first show that $u\in C^{0,1/p'+\delta}_{\rm loc}(\Omega)$ for some positive δ, with $p'=\frac{p}{p-1}$. Fix $x\in\Omega$ and a ball $B_r(x)\subset\subset\Omega$. Assume, without loss of generality, that $x=0$ and $r<1$. In what follows, we will omit the dependence on the center simply denoting by B_r the ball $B_r(0)$. By Theorem 1.4, we have that (u,E) is a minimizer of problem (1.5) for λ sufficiently large. Let v be the minimizer of

$$w\in W^{1,p}(B_r)\mapsto\int_{B_r}(F+G)(\nabla w)\,{\rm d}x,$$

satisfying the boundary condition $v=u$ on ∂B_r. Then

$$\int_{B_r}\Big(D_\xi F(\nabla u)+\chi_{_E}D_\xi G(\nabla u)\Big)\cdot\nabla\varphi\,{\rm d}x\;=\;0 \tag{5.1}$$

and

$$\int_{B_r}D_\xi(F+G)(\nabla v)\cdot\nabla\varphi\,{\rm d}x\;=\;0 \tag{5.2}$$

for all $\varphi\in W^{1,p}_0(B_r)$. Note that assumptions (F1)–(F2) and (G1)–(G2) imply that the integrand $F+G$ satisfies

(H1) $$0\le(F+G)(\xi)\le\widetilde{L}(\mu^2+|\xi|^2)^{\frac{p}{2}},$$

(H2) $$\int_\Omega(F+G)(\xi+\nabla\varphi)\,{\rm d}x\ge\int_\Omega\Big((F+G)(\xi)+\widetilde{\ell}(\mu^2+|\xi|^2+|\nabla\varphi|^2)^{\frac{p-2}{2}}|\nabla\varphi|^2\Big)\,{\rm d}x,$$

and (see (2.3))

(H3) $$\langle D_\xi(F+G)(\xi)-D_\xi(F+G)(\eta),\xi-\eta\rangle\ge c(p)\widetilde{\ell}|V(\xi)-V(\eta)|^2,$$

with growth and coercivity constants $\widetilde{L}$, $\widetilde{\ell}$ such that

$$\tilde{L}\le(\beta+1)L\qquad\text{and}\qquad\tilde{\ell}\ge(\alpha+1)\ell.$$

By virtue of (H1) and (H2), we can apply Theorem 2.1 and (2.10) to $H=F+G$, to obtain that for all $0<\varrho<\frac{r}{2}$

$$\begin{aligned}\int_{B_\varrho}\big(\mu^2+|\nabla v|^2\big)^{\frac{p}{2}}\,{\rm d}x&\le|B_\varrho|\sup_{B_\varrho}\big(\mu^2+|\nabla v|^2\big)^{\frac{p}{2}}\le c_n\varrho^n\sup_{B_{\frac{r}{2}}}(\mu^2+|\nabla v|^2)^{\frac{p}{2}}\\&\le c\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^\sigma\left(\frac{\varrho}{r}\right)^n\int_{B_r}(\mu^2+|\nabla v|^2)^{\frac{p}{2}}\,{\rm d}x,\end{aligned} \tag{5.3}$$

for some constants $c=c(n,p)\ge1$ and $\sigma=\frac{2n}{p}$. On the other hand if $\frac{r}{2}\le\varrho<r$, one easily gets that

$$\int_{B_\varrho}(\mu^2+|\nabla v|^2)^{\frac{p}{2}}\,{\rm d}x\le2^n\frac{\varrho^n}{r^n}\int_{B_r}(\mu^2+|\nabla v|^2)^{\frac{p}{2}}\,{\rm d}x.$$

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Therefore estimate (5.3) holds for every $0 < \varrho < r$. Subtracting (5.2) from (5.1), we obtain

$$\int_{B_r} \Big(D_\xi(F+G)(\nabla u) - D_\xi(F+G)(\nabla v) \Big) \cdot \nabla\varphi \, \mathrm{d}x - \int_{B_r \setminus E} D_\xi G(\nabla u) \cdot \nabla\varphi \, \mathrm{d}x \, = 0,$$

or, equivalently,

$$\int_{B_r} \Big(D_\xi(F+G)(\nabla u) - D_\xi(F+G)(\nabla v) \Big) \cdot \nabla\varphi \, \mathrm{d}x = \int_{B_r \setminus E} D_\xi G(\nabla u) \cdot \nabla\varphi \, \mathrm{d}x. \tag{5.4}$$

Next, we treat separately the cases $p \geq 2$ and $1 < p < 2$.

Case $p \geq 2$. Set $\varphi := u - v$ in (5.4). In (5.4) we use (H3) and Lemma 2.1 in the left hand side, the second condition in (2.2) and Hölder's inequality in the right hand side, thus obtaining

$$\begin{aligned} c_{p,n}\widetilde{\ell} \int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x &\leq c_p \beta L \int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p-1}{2}} |\nabla u - \nabla v| \, \mathrm{d}x \\ \leq c_p \beta L \left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{1}{2}} & \left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \\ \leq c_p \beta L \left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{1}{2}} & \left(\int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} \end{aligned}$$

where, in the last inequality, we used that $p \geq 2$. Hence

$$\int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x \leq c_{p,n} \left(\frac{\beta L}{\widetilde{\ell}} \right)^2 \int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x. \tag{5.5}$$

By virtue of (5.5), one has that for $0 < \rho < r$

$$\begin{aligned} \int_{B_\varrho} |\nabla u - \nabla v|^p \, \mathrm{d}x &\leq c_p \int_{B_\varrho} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x \\ &\leq c_{p,n} \left(\frac{\beta L}{\widetilde{\ell}} \right)^2 \int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x, \end{aligned} \tag{5.6}$$

therefore, from (5.3) and (5.6), we get

$$\begin{aligned} \left(\int_{B_\varrho} |\nabla u|^p \, \mathrm{d}x \right)^{1/p} &\leq \left(\int_{B_\varrho} |\nabla u - \nabla v|^p \, \mathrm{d}x \right)^{1/p} + \left(\int_{B_\varrho} |\nabla v|^p \, \mathrm{d}x \right)^{1/p} \\ &\leq c_{n,p} \left(\frac{\beta L}{\widetilde{\ell}} \right)^{\frac{2}{p}} \left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{1/p} \\ &\quad + c_{n,p} \left(\frac{\widetilde{L}}{\widetilde{\ell}} \right)^{\frac{\sigma}{p}} \left(\frac{\varrho}{r} \right)^{n/p} \left(\int_{B_r} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{1/p}. \end{aligned} \tag{5.7}$$

By Lemma 2.3 applied for $H = F + G$ and by the minimality of v, we have

$$\begin{aligned} \int_{B_r} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}} \, \mathrm{d}x &\leq \frac{2}{\widetilde{\ell}} \int_{B_r} (F+G)(\nabla v) \, \mathrm{d}x + c(n, p, \widetilde{L}, \widetilde{\ell}, \mu) r^n \\ &\leq \frac{2}{\widetilde{\ell}} \int_{B_r} (F+G)(\nabla u) \, \mathrm{d}x + c(n, p, \widetilde{L}, \widetilde{\ell}, \mu) r^n \end{aligned}$$

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$$\leq \frac{2\widetilde{L}}{\widetilde{\ell}} \int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x + c(n, p, \widetilde{L}, \widetilde{\ell}, \mu) r^n, \tag{5.8}$$

where in last line we used the growth assumption (H1). Combining (5.7) and (5.8), we obtain, for all $0 < \varrho < r$, that

$$\begin{aligned}
\left(\int_{B_\varrho} |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} &\leq c_{n,p} \left(\frac{\beta L}{\widetilde{\ell}} \right)^{\frac{2}{p}} \left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\quad + c_{n,p} \left(\frac{\widetilde{L}}{\widetilde{\ell}} \right)^{\frac{\sigma+1}{p}} \left(\frac{\varrho}{r} \right)^{\frac{n}{p}} \left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{1}{p}} + c(n, p, \widetilde{L}, \widetilde{\ell}, \mu) \varrho^{\frac{n}{p}} \\
&= c_{n,p} \left[\left(\frac{\beta L}{\widetilde{\ell}} \right)^{\frac{2}{p}} + \left(\frac{\widetilde{L}}{\widetilde{\ell}} \right)^{\frac{\sigma+1}{p}} \left(\frac{\varrho}{r} \right)^{\frac{n}{p}} \right] \left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\quad + c(n, p, \widetilde{L}, \widetilde{\ell}, \mu) \varrho^{\frac{n}{p}},
\end{aligned}$$

and therefore the following estimate holds

$$\begin{aligned}
\int_{B_\varrho} |\nabla u|^p \, \mathrm{d}x &\leq c_{n,p} \left[\left(\frac{\beta L}{\widetilde{\ell}} \right)^{\frac{2}{p}} + \left(\frac{\widetilde{L}}{\widetilde{\ell}} \right)^{\frac{\sigma+1}{p}} \left(\frac{\varrho}{r} \right)^{\frac{n}{p}} \right]^p \int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}} \, \mathrm{d}x \\
&\quad + c(n, p, \widetilde{L}, \widetilde{\ell}, \mu) \varrho^n.
\end{aligned} \tag{5.9}$$

for every $0 < \rho < r$.

Case $1 < p < 2$. As before, we choose $\varphi = u - v$ in (5.4). In (5.4) we use (H3) and Lemma 2.1 in the left hand side, the second condition in (2.2) and Hölder's inequality in the right hand side to obtain

$$\begin{aligned}
&c_{n,p} \widetilde{\ell} \int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x \leq c_p \beta L \int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p-1}{2}} |\nabla u - \nabla v| \, \mathrm{d}x \\
&\leq c_p \beta L \int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-1}{2}} |\nabla u - \nabla v| \, \mathrm{d}x \\
&\leq c_p \beta L \left(\int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}
\end{aligned}$$

and so

$$\int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x \leq c_{n,p} \left(\frac{\beta L}{\widetilde{\ell}} \right)^2 \int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \, \mathrm{d}x. \tag{5.10}$$

On the other hand, for $1 < p < 2$, Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$ yields

$$\begin{aligned}
&\int_{B_\varrho} |\nabla u - \nabla v|^p \, \mathrm{d}x \\
&\leq \left(\int_{B_\varrho} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p-2}{2}} |\nabla u - \nabla v|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_{B_\varrho} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{2-p}{2}} \\
&\leq c_{n,p} \left(\frac{\beta L}{\widetilde{\ell}} \right)^p \left(\int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_{B_\varrho} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}} \, \mathrm{d}x \right)^{\frac{2-p}{2}}
\end{aligned}$$

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$$= c_{n,p}\left(\frac{\beta L}{\widetilde{\ell}}\right)^p \int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}}\,\mathrm{d}x, \tag{5.11}$$

where we used (5.10). From (5.3) and (5.11), we get

$$\begin{aligned}
\left(\int_{B_\varrho} |\nabla u|^p\,\mathrm{d}x\right)^{1/p} &\le \left(\int_{B_\varrho} |\nabla u - \nabla v|^p\,\mathrm{d}x\right)^{1/p} + \left(\int_{B_\varrho} |\nabla v|^p\,\mathrm{d}x\right)^{1/p} \\
&\le c_{n,p}\left(\frac{\beta L}{\widetilde{\ell}}\right)\left(\int_{B_r} (\mu^2 + |\nabla u|^2 + |\nabla v|^2)^{\frac{p}{2}}\,\mathrm{d}x\right)^{1/p} \\
&\quad + c_{n,p}\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma}{p}}\left(\frac{\varrho}{r}\right)^{n/p}\left(\int_{B_r} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}}\,\mathrm{d}x\right)^{1/p} \\
&\le c_{n,p}\left(\frac{\beta L}{\widetilde{\ell}}\right)\left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}}\,\mathrm{d}x\right)^{1/p} \\
&\quad + c_{n,p}\left[\left(\frac{\beta L}{\widetilde{\ell}}\right) + \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma}{p}}\left(\frac{\varrho}{r}\right)^{n/p}\right]\left(\int_{B_r} (\mu^2 + |\nabla v|^2)^{\frac{p}{2}}\,\mathrm{d}x\right)^{1/p}.
\end{aligned} \tag{5.12}$$

By virtue of (5.8), that holds for all $p > 1$, from estimate (5.12) we obtain

$$\begin{aligned}
\left(\int_{B_\varrho} |\nabla u|^p\,\mathrm{d}x\right)^{1/p} &\le c_{n,p}\left(\frac{\beta L}{\widetilde{\ell}}\right)\left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}}\,\mathrm{d}x\right)^{1/p} \\
&\quad + c_{n,p}\left[\left(\frac{\beta L}{\widetilde{\ell}}\right) + \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma}{p}}\left(\frac{\varrho}{r}\right)^{n/p}\right]\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{1}{p}}\left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}}\,\mathrm{d}x\right)^{1/p} \\
&\quad + c(n,p,\widetilde{L},\widetilde{\ell},\mu) r^{\frac{n}{p}} \\
&\le c_{n,p}\left[\left(\frac{\beta L}{\widetilde{\ell}}\right)\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{1}{p}} + \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}}\left(\frac{\varrho}{r}\right)^{n/p}\right]\left(\int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}}\,\mathrm{d}x\right)^{1/p} \\
&\quad + c(n,p,\widetilde{L},\widetilde{\ell},\mu) r^{\frac{n}{p}}.
\end{aligned} \tag{5.13}$$

Therefore

$$\begin{aligned}
\int_{B_\varrho} |\nabla u|^p\,\mathrm{d}x &\le c_{n,p}\left[\left(\frac{\beta L}{\widetilde{\ell}}\right)\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{1}{p}} + \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}}\left(\frac{\varrho}{r}\right)^{n/p}\right]^p \int_{B_r} (\mu^2 + |\nabla u|^2)^{\frac{p}{2}}\,\mathrm{d}x \\
&\quad + c(n,p,\widetilde{L},\widetilde{\ell},\mu) r^n.
\end{aligned} \tag{5.14}$$

Hence, both estimates (5.9) and (5.14) can be written as

$$\int_{B_\varrho} |\nabla u|^p\,\mathrm{d}x \le c_{n,p}\left[\zeta + \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}}\left(\frac{\rho}{r}\right)^{n/p}\right]^p \int_{B_r} |\nabla u|^p\,\mathrm{d}x + c(n,p,\widetilde{L},\widetilde{\ell},\mu) r^n, \tag{5.15}$$

where

$$\zeta := \begin{cases} \left(\frac{\beta L}{\widetilde{\ell}}\right)^{\frac{2}{p}} & \text{if } p \ge 2, \\ \left(\frac{\beta L}{\widetilde{\ell}}\right)\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{1}{p}} & \text{if } 1 < p < 2. \end{cases} \tag{5.16}$$

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We find the largest $\zeta < 1$ for which there exists $\vartheta < 1$ such that

$$c_{n,p}\left(\zeta + \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}} \vartheta^{n/p}\right)^p = \vartheta^{n-1}.$$

This equality is equivalent to

$$\zeta = \frac{\vartheta^{(n-1)/p}}{c^{\frac{1}{p}}} - \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}} \vartheta^{n/p} =: f(\vartheta),$$

where, for simplicity, we set $c = c_{n,p}$, $c > 1$. Note that such $\vartheta, \zeta \in [0,1)$ exist. Indeed

$$\frac{df}{d\vartheta}(\vartheta) = \frac{1}{p}\vartheta^{\frac{n}{p}-1}\left(\frac{n-1}{c^{\frac{1}{p}}}\vartheta^{-\frac{1}{p}} - n\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}}\right),$$

and so

$$\frac{df}{d\vartheta}(\vartheta) = 0 \qquad \Leftrightarrow \qquad \vartheta = \frac{1}{c}\left(\frac{n-1}{n}\right)^p \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{-(\sigma+1)}.$$

Set

$$\vartheta_0 := \frac{1}{c}\left(\frac{n-1}{n}\right)^p \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{-(\sigma+1)} \qquad \text{and} \qquad \zeta_0 := f(\vartheta_0).$$

Since $\widetilde{\ell} \le \widetilde{L}$ and $c \ge 1$ it follows that $\vartheta_0 \in (0,1)$ and

$$f(\vartheta_0) = \max_{\vartheta\in[0,1]} f(\vartheta).$$

Moreover,

$$\begin{aligned}\zeta_0 &= \frac{\vartheta_0^{(n-1)p}}{c^{\frac{1}{p}}} - \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}} \vartheta_0^{\frac{n}{p}} = \vartheta_0^{\frac{n}{p}}\left(\frac{\vartheta_0^{-\frac{1}{p}}}{c^{\frac{1}{p}}} - \left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}}\right)\\ &= \vartheta_0^{\frac{n}{p}}\left(\frac{n}{n-1} - 1\right)\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}} = \vartheta_0^{\frac{n}{p}}\left(\frac{1}{n-1}\right)\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}}\end{aligned}$$

We write

$$\zeta_0 = \frac{(n-1)^{(n-1)}}{n^n}\frac{1}{c^{\frac{n}{p}}}\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{\sigma+1}{p}(1-n)} = \tilde{c}_{n,p}\left(\frac{\widetilde{\ell}}{\widetilde{L}}\right)^{\widetilde{\sigma}}$$

with $\tilde{c}_{n,p} := \dfrac{(n-1)^{(n-1)}}{n^n}\dfrac{1}{c^{\frac{n}{p}}}$ and $\widetilde{\sigma} := \frac{\sigma+1}{p}(n-1)$. Note that $\zeta_0 \in (0,1)$. In case in which $p \ge 2$, we need

$$\left(\frac{\beta L}{\widetilde{\ell}}\right)^{\frac{2}{p}} < \zeta_0. \tag{5.17}$$

Recalling that $\widetilde{\ell} \ge (\alpha+1)\ell$ and $\widetilde{L} \le (\beta+1)L$, in order to have (5.17) it suffices to impose that

$$\left(\frac{\beta}{\alpha+1}\right)^{\frac{2}{p}}\left(\frac{\beta+1}{\alpha+1}\right)^{\widetilde{\sigma}} < \tilde{c}_{n,p}\left(\frac{\ell}{L}\right)^{\widetilde{\sigma}+\frac{2}{p}}. \tag{5.18}$$

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In fact

$$(\beta L)^{\frac{2}{p}}\Big((\beta+1)L\Big)^{\tilde{\sigma}} < \tilde{c}_{n,p}\Big((\alpha+1)\ell\Big)^{\tilde{\sigma}+\frac{2}{p}} \quad \Rightarrow \quad (\beta L)^{\frac{2}{p}}\widetilde{L}^{\tilde{\sigma}} < \tilde{c}_{n,p}(\widetilde{\ell})^{\tilde{\sigma}+\frac{2}{p}}$$

$$\Leftrightarrow \quad \left(\frac{\beta L}{\widetilde{\ell}}\right)^{\frac{2}{p}} < \zeta_0 = \tilde{c}_{n,p}\left(\frac{\widetilde{\ell}}{\widetilde{L}}\right)^{\tilde{\sigma}},$$

and inequality (5.18) is clearly fulfilled if the ratio $\frac{\beta}{\alpha+1}$ is sufficiently small.

Similarly, in case in which $1 < p < 2$, we need

$$\left(\frac{\beta L}{\widetilde{\ell}}\right)\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{1}{p}} < \zeta_0. \tag{5.19}$$

In order to have (5.19), it suffices to impose

$$\frac{\beta}{\alpha+1}\left(\frac{\beta+1}{\alpha+1}\right)^{\tilde{\sigma}+\frac{1}{p}} < \tilde{c}_{n,p}\left(\frac{\ell}{L}\right)^{\tilde{\sigma}+1+\frac{1}{p}}. \tag{5.20}$$

In fact

$$\beta(\beta+1)^{\tilde{\sigma}+\frac{1}{p}}L^{\tilde{\sigma}+1+\frac{1}{p}} < \widetilde{c}_{n,p}((\alpha+1)\ell)^{\tilde{\sigma}+1+\frac{1}{p}} \quad \Rightarrow \quad (\beta L)\widetilde{L}^{\tilde{\sigma}+\frac{1}{p}} < \tilde{c}_{n,p}(\widetilde{\ell})^{\tilde{\sigma}+1+\frac{1}{p}}$$

$$\Leftrightarrow \quad \left(\frac{\beta L}{\widetilde{\ell}}\right)\left(\frac{\widetilde{L}}{\widetilde{\ell}}\right)^{\frac{1}{p}} < \tilde{c}_{n,p}\left(\frac{\widetilde{\ell}}{\widetilde{L}}\right)^{\tilde{\sigma}} = \zeta_0$$

and inequality (5.20) is clearly fulfilled if the ratio $\frac{\beta}{\alpha+1}$ is sufficiently small.

Then, choosing α, β such that (5.17) (if $p \geq 2$) or (5.19) (if $1 < p < 2$) are satisfied, in view of (5.15) there exist $\vartheta \in (0, \vartheta_0)$ and $\bar{\delta} > 0$, depending on $\alpha, \beta, n, p, \ell, L$, such that

$$\int_{B_{\vartheta r}} |\nabla u|^p \,\mathrm{d}x \leq \vartheta^{n-1+p\bar{\delta}} \int_{B_r} |\nabla u|^p \,\mathrm{d}x + c(n,p,\mu,\widetilde{L},\widetilde{\ell})r^n.$$

Since $r < 1$, the term r^n can be majorized by $r^{n-1+p\delta}$, for every $0 < \delta < \min\{\bar{\delta}, \frac{1}{p}\}$, and from the previous estimate, we deduce that

$$\int_{B_{\vartheta r}} |\nabla u|^p \,\mathrm{d}x \leq \vartheta^{n-1+p\bar{\delta}} \int_{B_r} |\nabla u|^p \,\mathrm{d}x + c(n,p,\mu,\widetilde{L},\widetilde{\ell})r^{n-1+p\delta}.$$

This estimate, by virtue of Lemma 2.4, yields that for all $0 < \varrho < r < 1$

$$\int_{B_\varrho} |\nabla u|^p \,\mathrm{d}x \leq c(n)\left(\frac{\varrho}{r}\right)^{n-1+p\delta} \int_{B_r} |\nabla u|^p \,\mathrm{d}x + c\rho^{n-1+p\delta} \tag{5.21}$$

So, for $0 < \varrho \ll 1$

$$\int_{B_\varrho} |\nabla u|^p \,\mathrm{d}x \leq c\rho^{n-1+p\delta}$$

and by Hölder's inequality

$$\int_{B_\varrho} |\nabla u| \,\mathrm{d}x \leq c\left(\int_{B_\varrho} |\nabla u|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \varrho^{\frac{n}{p'}} \leq c\varrho^{\frac{n-1+p\delta+n(p-1)}{p}} = c\rho^{n-1+\delta+\frac{1}{p'}}.$$

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By Theorem 2.2, the previous inequality implies that $u \in C^{0,\frac{1}{p}+\delta}_{\rm loc}(\Omega)$ whenever (5.18) (if $p \geq 2$) or (5.20) (if $1 < p < 2$) hold true.

Step 2. Fix a point $x \in \Omega$ and let $\bar{r} > 0$ be such that $\operatorname{dist}(x, \partial\Omega) > \bar{r}$. Consider $0 < r < r_0 \leq \bar{r}$ and denote by A any set of finite perimeter such that $E \Delta A \subset\subset B_r(x)$. From Theorem 1.4 we have that

$$\mathcal{I}_{\lambda_0}(u, E) \leq \mathcal{I}_{\lambda_0}(u, A),$$

and thus

$$\begin{aligned}&\int_\Omega \Big(F(\nabla u) + \chi_{_E} G(\nabla u)\Big)\,\mathrm{d}x + P(E,\Omega) + \lambda_0\big||E| - d\big| \\ &\leq \int_\Omega \Big(F(\nabla u) + \chi_{_A} G(\nabla u)\Big)\,\mathrm{d}x + P(A,\Omega) + \lambda_0\big||A| - d\big|.\end{aligned}$$

Using that $E \Delta A \subset\subset B_r(x)$, we deduce that

$$\begin{aligned}P(E, B_r(x)) - P(A, B_r(x)) &\leq \int_{B_r} \Big(\chi_{_A}(x) - \chi_{_E}(x)\Big) G(\nabla u)\,\mathrm{d}x + \lambda_0\big||A| - |E|\big| \\ &\leq \beta L \int_{B_r} |\nabla u|^p + c r^n,\end{aligned}$$

where we invoked assumption (G2). By the decay estimate (5.21), we infer that

$$P(E, B_r(x)) - P(A, B_r(x)) \leq c r^{n-1+p\delta} + c r^n \leq c r^{n-1+p\delta}$$

since $r < 1$. As δ can be replaced by any smaller number, we can choose $p\delta < \frac{1}{2}$ and the result follows from Theorem 2.3. □

6. Partial regularity – Proof of Theorem 1.3

In this section, we prove that a partial regularity result holds without imposing any bounds on α and β, as stated in Theorem 1.3.

Proof of Theorem 1.3. Set

$$\Omega_0 := \left\{ x \in \Omega : \limsup_{\rho \to 0} \frac{1}{\rho^{n-1+p\delta}} \int_{B_\rho} |\nabla u|^p = 0 \right\},$$

for an arbitrary $0 < \delta < \frac{1}{p}$. Note that (see Thm. 3, Sect. 2.4.3, in [9]), $|\Omega \setminus \Omega_0| = 0$. Fix a point $x \in \Omega_0$ and let r_0 be such that $\operatorname{dist}(x, \partial\Omega) > r_0$. Since $x \in \Omega_0$, for every $\varepsilon > 0$ there exists a radius $R = R(\varepsilon) < r_0$ such that

$$\frac{1}{r^{n-1+p\delta}} \int_{B_r} |\nabla u|^p < \varepsilon \tag{6.1}$$

for all $0 < r \leq R(\varepsilon)$. By (5.15) and (5.17), for all $0 < \varrho < r$ we have

$$\int_{B_\varrho} |\nabla u|^p\,\mathrm{d}x \leq c_{n,p} \left[\zeta + \left(\frac{\widetilde{L}}{\widetilde{\ell}} \right)^{\frac{\sigma+1}{p}} \left(\frac{\rho}{r} \right)^{n/p} \right]^p \int_{B_r} |\nabla u|^p\,\mathrm{d}x + c(n, p, \widetilde{L}, \widetilde{\ell}, \mu) r^n.$$

Inserting (6.1) in previous inequality, we get

$$\int_{B_\varrho} |\nabla u|^p\,\mathrm{d}x \leq c\, \varepsilon^{\frac{1}{2}} \left[1 + \left(\frac{\varrho}{r} \right)^{n/p} \right]^p r^{\frac{n-1+p\delta}{2}} \left(\int_{B_r} |\nabla u|^p\,\mathrm{d}x \right)^{\frac{1}{2}} + c r^n,$$

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where $c = c(n, p, \mu, \alpha, \beta, \ell, L)$. By Young's inequality, we deduce that

$$\begin{aligned}\int_{B_\varrho} |\nabla u|^p \,\mathrm{d}x &\le c\,\varepsilon^{\frac{1}{2}} \left[1 + \left(\frac{\varrho}{r}\right)^{n/p}\right]^p \left\{\int_{B_r} |\nabla u|^p \,\mathrm{d}x + r^{n-1+p\delta}\right\} + cr^n \\ &\le c\,\varepsilon^{\frac{1}{2}} \left[1 + \left(\frac{\varrho}{r}\right)^{n/p}\right]^p \int_{B_r} |\nabla u|^p \,\mathrm{d}x + cr^{n-1+p\delta}. \end{aligned} \tag{6.2}$$

for every $0 < \rho < r \le R(\varepsilon)$, since we may suppose, without loss of generality, that $r < 1$. Therefore, in particular, writing (6.2) for $\rho = \frac{r}{2}$, we get

$$\int_{B_{\frac{r}{2}}} |\nabla u|^p \,\mathrm{d}x \le c\,\varepsilon^{\frac{1}{2}} \left[1 + \left(\frac{1}{2}\right)^{n/p}\right]^p \int_{B_r} |\nabla u|^p \,\mathrm{d}x + cr^{n-1+p\delta}. \tag{6.3}$$

Choosing ε in (6.3) such that

$$\varepsilon^{\frac{1}{2}} < \frac{2^{1-p}}{c\left(1 + 2^{\frac{n}{p}}\right)^p},$$

we obtain

$$\begin{aligned}\int_{B_{\frac{r}{2}}} |\nabla u|^p \,\mathrm{d}x &\le \frac{2^{1-p}}{\left(1 + 2^{\frac{n}{p}}\right)^p} \left[1 + \left(\frac{1}{2}\right)^{n/p}\right]^p \int_{B_r} |\nabla u|^p \,\mathrm{d}x + cr^{n-1+p\delta} \\ &= \frac{2^{1-p}}{\left(1 + 2^{\frac{n}{p}}\right)^p} \frac{\left(1 + 2^{\frac{n}{p}}\right)^p}{2^n} \int_{B_r} |\nabla u|^p \,\mathrm{d}x + cr^{n-1+p\delta} \\ &= \left(\frac{1}{2}\right)^{n-1+p} \int_{B_r} |\nabla u|^p \,\mathrm{d}x + cr^{n-1+p\delta}. \end{aligned} \tag{6.4}$$

From (6.4), thanks to Lemma 2.4 applied with $\varphi(r) := \int_{B_r} |\nabla u|^p \,\mathrm{d}x$ and $\vartheta = \frac{1}{2}$, we obtain that

$$\int_{B_\varrho} |\nabla u|^p \,\mathrm{d}x \le c\rho^{n-1+p\delta} \tag{6.5}$$

for all $0 < \varrho < r < r_0$ and some $c = c(n, p, \alpha, \beta, \ell, L)$. Hence, by virtue of Theorem 2.2 and Hölder inequality, we deduce that $u \in C^{0,\frac{1}{p'}+\delta}(\Omega_0)$, for every $0 < \delta < \frac{1}{p}$.

Let us denote by A any set of finite perimeter such that $E \Delta A \subset\subset B_\rho(x)$. From Theorem 1.4 we have that

$$\mathcal{I}_{\lambda_0}(u, E) \le \mathcal{I}_{\lambda_0}(u, A),$$

therefore, by assumption (G1) and the decay estimate (6.5), we deduce that

$$\begin{aligned}P(E, B_\rho(x)) - P(A, B_\rho(x)) &\le \int_{B_\rho} (\chi_A(x) - \chi_E(x)) G(\nabla u) \,\mathrm{d}x + \lambda_0 \big||A| - |E|\big| \\ &\le \beta L \int_{B_\rho} |\nabla u|^p \,\mathrm{d}x + \lambda_0 \big||A| - |E|\big| \le c\rho^{n-1+p\delta} + c\lambda_0 \rho^n \end{aligned}$$

and the conclusion follows again by Theorem 2.3 applied to Ω_0 in place of Ω. □

Acknowledgements. The authors warmly thank the Center for Nonlinear Analysis (NSF Grants Nos. DMS- 0405343 and DMS-0635983), where part of this research was carried out. The research of I. Fonseca was partially funded by the National Science Foundation under Grant No. DMS- 0905778.

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