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On functional Nörlund methods

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1. *Introduction.* Integral transformations analogous to the Nörlund means have been introduced and investigated by Kuttner, Knopp and Vanderburg (6), (5), (4). It is known that with any regular Nörlund mean (N, p) there is associated a function

$$p(z) = \sum_{n=0}^{\infty} p_n z^n \quad (p_0 \neq 0)$$

regular for $|z| < 1$, and if we have two Nörlund means (N, p) and (N, r), where $(N, pr$ is regular, while the function $r(z) = \sum_{n=0}^{\infty} r_n z^n$ $(r_0 \neq 0)$ is regular for $|z| \leqslant 1$ and different) from zero at $z = 1$, then $q(z) = r(z)p(z)$ belongs to a regular Nörlund mean (N, q). Concerning Nörlund means Peyerimhoff (7) and Miesner (3) have recently obtained the relation between the convergence fields of the Nörlund means (N, p) and (N, r) on the one hand and the convergence field of the Nörlund mean (N, q) on the other hand.

The object of the present paper is to obtain results for integral transforms analogous to those of Peyerimhoff and Miesner. In order to exhibit clearly the analogy between the results of Miesner and those of the present paper, we first state Miesner's main theorem, which substantially includes Peyerimhoff's Theorem 1 (7).

Definition. The Nörlund mean (N, p) of the sequence $\{s_n\}$ of complex numbers is defined by

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{1}$$

for $n \geqslant N$ where

$$P_n = \sum_{v=0}^{n} p_v \neq 0 \quad \text{for} \quad n \geqslant N. \tag{2}$$

If σ_n is convergent to s, the sequence $\{s_n\}$ is said to be limitable (N, p) to the value s. The set of sequences limitable to 0 will be denoted by $o(N, p)$.

Necessary and sufficient conditions in order that the Nörlund mean (N, p) generated by $\{p_n\}$ be regular are

$$p_n = o(P_n) \quad \text{for} \quad n \to \infty \tag{3}$$

and

$$\sum_{v=0}^{n} |p_v| = O(P_n) \quad \text{for} \quad n \to \infty. \tag{4}$$

Miesner's main Theorem is

Theorem A. *Let* $r(z) = \sum_{n=0}^{\infty} r_n z^n$ *be absolutely convergent for* $|z| \leqslant 1$, *with* $r(z) \neq 0$ *for* $z = 0$ *and* $|z| = 1$, *and having inside the unit circle the roots* $\alpha_1, \alpha_2, \ldots, \alpha_k$ *with the multiplicities* $\gamma_1, \gamma_2, \ldots, \gamma_k$ $(\gamma_i > 0)$. *Let* $p(z) = \sum_{n=0}^{\infty} p_n z^n$ *be a function regular for* $|z| < 1$, *and*

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suppose that the root α_i of $r(z)$ is a root of $p(z)$ with multiplicity $\lambda_i \geqslant 0$ $(i = 1, 2, \ldots, k)$. Define $q(z)$ by $q(z) = r(z)\,p(z)$. If $\{p_n\}$ satisfies the conditions (2), (3) *and* (4), *then $s_n \in o(N, q)$ if and only if*

$$s_n = t_n + \sum_{i=1}^{k} \alpha_i^{-n} \sum_{j=\lambda_i}^{\lambda_i+\gamma_i-1} C_{ij} A_n^j \tag{5}$$

with t_n in $o(N, p)$, $A_n^j = \binom{n+j}{n}$, and $C_{ij} = const.$

Miesner also obtained a similar result for absolute summability. In the following we shall deal with the ordinary summability of functional Nörlund methods.

2. *Functional Nörlund methods.* Let S be the class of all (complex valued) functions $f(t)$ of a real variable t defined for all positive t and bounded and measurable in every finite interval $(0, A)$, $A > 0$. We suppose throughout that $p(t)$ is Lebesgue-integrable in any (relevant) finite interval. We write

$$p_1(t) = \beta + \int_0^t p(u)\,du,$$

and similarly for other letters in place of p. We assume that, for sufficiently large t,

$$p_1(t) \neq 0. \tag{6}$$

The (N, p, β) summability of $f(t)$ $(f(t) \in S)$ to s is defined as meaning

$$\frac{1}{p_1(t)}\left[\beta f(t) + \int_0^t p(t-u) f(u)\,du\right] \to s \quad \text{as} \quad t \to \infty; \tag{7}$$

in other words, that

$$\beta f(t) + \int_0^t p(t-u) f(u)\,du = p_1(t)\,(s + o(1)). \tag{8}$$

The set of functions $f(t) \in S$ which are summable (N, p, β) to 0 will be denoted by $o(N, p, \beta)$.

We remark that, in view of (6), the expression on the left of (7) must be defined for all sufficiently large t; it need not however exist for all t.

We recall that, with the usual notation for convolutions, the integral on the left of (8) can be written $(p * f)_t$. We will several times have to make use of the result that convolutions are associative and commutative, i.e.

$$a * b = b * a,$$

$$a * (b * c) = (a * b) * c.$$

The method (N, p, β) is said to be regular over the set S, if $f(t) \in S$, $f(t) \to s$ implies $f(t) \to s (N, p, \beta)$. Sufficient conditions for the regularity of the method (N, p, β) over the set S are

(i) $p(t) = o(p_1(t))$,

(ii) $\int_0^t |p(u)|\,du = O(p_1(t))$.

From now on we will suppose throughout that

$$p(t) = o(p_1(t)). \tag{9}$$

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The lemmas which follow are true only under this assumption. We note some immediate consequences of (9) of which we shall have to make use. Given constant $c > 0$, there is a T (depending on c) such that

$$\left|\frac{p(t)}{p_1(t)}\right| \leqslant c \quad (t \geqslant T).$$

Integrating, we see that, for $u \geqslant t \geqslant T$,

$$|\log p_1(u) - \log p_1(t)| = \left|\int_t^u \frac{p(t)}{p_1(t)}\,dt\right| \leqslant \int_t^u \left|\frac{p(t)}{p_1(t)}\right| dt \leqslant c(u-t).$$

Thus, in particular

$$-c(u-t) \leqslant \operatorname{re}\{\log p_1(u) - \log p_1(t)\} \leqslant c(u-t).$$

Putting these last inequalities in other words we see that, given any $c > 0$ we have, for $u \geqslant t \geqslant T = T(c)$,

$$|p_1(u)| \leqslant |p_1(t)|\, e^{c(u-t)}, \tag{10}$$

$$|p_1(u)| \geqslant |p_1(t)|\, e^{-c(u-t)}. \tag{11}$$

It also follows that, for any $c > 0$ and for large u

$$|p_1(u)| = O(e^{cu}); \tag{12}$$

also, for sufficiently large u,

$$|p_1(u)| \geqslant k e^{-cu}, \tag{13}$$

where $k = k(c)$ is a strictly positive constant. We deduce at once from (9) and (12) that

$$|p(u)| = O(e^{cu}) \tag{14}$$

(but we cannot deduce a corresponding result from (13)).

We will write

$$P(s) = \beta + \int_0^\infty e^{-st} p(t)\,dt; \tag{15}$$

it follows from (14), since $c > 0$ is arbitrary, that the integral (15) converges absolutely for $\operatorname{re} s > 0$. We will again use a similar notation with other letters in place of p.

3. Lemma 1. *Let s_0 be a complex constant, $\operatorname{re} s_0 > 0$. Let K be an integer, $K \geqslant 0$. Suppose that*

$$\psi(t) = o(p_1(t)).$$

Then as $t \to \infty$,

$$\int_t^\infty (u-t)^k e^{-s_0(u-t)} \psi(u)\,du = o(p_1(t)). \tag{16}$$

Proof. Write $\gamma_0 = \operatorname{re} s_0$, and take c any constant with $0 < c < \gamma_0$. Given any ϵ, there is a t_0 such that

$$|\psi(t)| \leqslant \epsilon |p_1(t)| \quad (t > t_0).$$

We may suppose that $t_0 > T(c)$ (with the notation used in (10)). Then, for $t > t_0$, the integral on the left of (16) does not in modulus exceed

$$\epsilon \int_t^\infty (u-t)^k e^{-\gamma_0(u-t)} |p_1(u)|\,du,$$

which is

$$\leqslant \epsilon |p_1(t)| \int_t^\infty (u-t)^k e^{-(\gamma_0 - c)(u-t)}\,du \quad \text{(by (10))}$$

$$= \text{const.}\, \epsilon |p_1(t)| \quad (\text{since } c < \gamma_0).$$

This gives the result.

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In particular, by (9) we may take

$$\psi(t) = p(t).$$

Hence, with s_0, k as above,

$$\int_t^\infty (u-t)^k e^{-s_0(u-t)} p(u)\, du = o(p_1(t)). \tag{17}$$

LEMMA 2. *Let s_0 be a given complex number with $\operatorname{re} s_0 > 0$. Let $A(t)$ be a polynomial of degree $n \geqslant 0$. Then the function*

$$A(t)\, e^{s_0 t} \tag{18}$$

is summable (N, p, β) if and only if $P(s)$ has a zero of order at least $n+1$ at $s = s_0$. If summable at all, it is summable (N, p, β) to zero.

Proof. The statement that (18) is summable (N, p, β) means that $X(t)$ should tend to a limit as $t \to \infty$, where

$$X(t) = \frac{1}{p_1(t)} \left\{ \beta A(t)\, e^{s_0 t} + \int_0^t A(t-u)\, e^{s_0(t-u)} p(u)\, du \right\}. \tag{19}$$

Thus by (17)

$$X(t) = \frac{1}{p_1(t)} \left\{ \beta A(t)\, e^{s_0 t} + \int_0^\infty A(t-u)\, e^{s_0(t-u)} p(u)\, du \right\} + o(1). \tag{20}$$

Differentiating (15) j times, we have for $j \geqslant 1$,

$$P^{(j)}(s) = \int_0^\infty (-u)^j e^{-su} p(u)\, du.$$

Hence, expanding $(t-u)^k$ by the binomial theorem we get

$$\int_0^\infty (t-u)^k e^{s_0(t-u)} p(u)\, du = e^{s_0 t} \sum_{j=0}^{k} (k_j)\, t^{k-j} P^{(j)}(s_0) - \beta e^{s_0 t} t^k. \tag{21}$$

Write $A(t) = a_0 + a_1 t + \ldots + a_n t^n$ (since $A(t)$ was assumed to be of degree n, $a_n \neq 0$).

From (21), $X(t)$ can be written as

$$\begin{aligned} X(t) = \frac{e^{s_0 t}}{p_1(t)} \{ & a_n P(s_0)\, t^n + (n a_n P'(s_0) + a_{n-1} P(s_0))\, t^{n-1} \\ & + (\tfrac{1}{2} n(n-1)\, a_n P''(s_0) + (n-1)\, a_{n-1} P'(s_0) + a_{n-2} P(s_0))\, t^{n-2} + \ldots \} + o(1). \end{aligned} \tag{22}$$

First, suppose that $P(s)$ has a zero of order $\geqslant n+1$ at $s = s_0$; in other words,

$$P(s_0) = P'(s_0) = \ldots = P^{(n)}(s_0) = 0. \tag{23}$$

Then all the terms in the $\{\}$ in (22) vanish, so that

$$X(t) \to 0.$$

Conversely, suppose that $X(t)$ tends to some limit. Then from (22)

$$X(t) = \frac{e^{s_0 t}}{p_1(t)}\, t^n [a_n P(s_0) + o(1)] + o(1).$$

Since $|e^{s_0 t}/p_1(t)| \to \infty$ (by (14)) and since $a_n \neq 0$, it follows that we must have $P(s_0) = 0$. Substituting $P(s_0) = 0$ in (22), we deduce that

$$X(t) = \frac{e^{s_0 t}}{p_1(t)}\, t^{n-1} [n a_n P'(s_0) + o(1)] + o(1),$$

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so that for $X(t)$ to tend to a limit we must have $P'(s_0) = 0$ (assuming $n \geqslant 1$). Substituting $P(s_0) = P'(s_0) = 0$ in (22) we get

$$X(t) = \frac{e^{s_0 t}}{p_1(t)} t^{n-2} \left[\frac{n(n-1)}{2} a_n P''(s_0) + o(1) \right] + o(1).$$

Thus for $X(t)$ to tend to a limit we must have $P''(s_0) = 0$ whenever $n \geqslant 2$. Continuing in this way we obtain (23).

Corollary I. Let s_0, $A(t)$ be as in Lemma 2, and suppose that $P(s)$ has a zero of order n at $s = s_0$. (If $n = 0$, this is taken as meaning no zero at all.) Then, with $X(t)$ defined by (19),

$$X(t) = \frac{a_n P^{(n)}(s_0)\, e^{s_0 t}}{p_1(t)} + o(1). \tag{24}$$

For, all the terms but one inside the bracket in (22) vanish.

4. In order to obtain a result analogous to Lemma 7 (i) of (7) we have to construct $q(t)$ in such a way that $Q(s)$ is equal to $P(s)$ multiplied by a function having a simple zero at an assigned point $s = s_0$ (with $\operatorname{re} s_0 > 0$) and no other zero in $\operatorname{re} s > 0$. Integrating (15) by parts, and using (12), we see that, for $\operatorname{re} s > 0$,

$$P(s) = s \int_0^\infty e^{-st} p_1(t)\, dt. \tag{25}$$

Hence the first thing one naturally thinks of is taking

$$q(t) = p(t) - s_0 p_1(t),$$

since this gives us (using (25))

$$Q(s) = P(s)\,(1 - s_0/s). \tag{26}$$

However, this definition is not satisfactory, since it would be possible for $p(t)$ to satisfy (9) without $q(t)$ satisfying the corresponding equation. As an example to demonstrate this, take $\beta = 0$ and

$$p(t) = 2 + (1+i)\,(2+i)\,t^i. \tag{27}$$

Then

$$p_1(t) = t[2 + (2+i)\,t^i], \tag{28}$$

and also,

$$p_2(t) = \int_0^t p_1(u)\, du = t^2(1 + t^i). \tag{29}$$

Now $|t^i| = 1$, $|i+2| = \sqrt{5}$. So as t varies, the expression inside [ ] in (28) describes a circle of centre 2, radius $\sqrt{5}$. Hence

$$|p_1(t)| \geqslant t(\sqrt{5} - 2).$$

Since $p(t) = O(1)$, (9) clearly holds. But the expression in [ ] in (29) describes a circle of centre 1, radius 1, and thus vanishes for some arbitrary large t. It follows from the definition that

$$|q(t)| \geqslant |s_0|\,(\sqrt{5} - 2)\,t + O(1).$$

But

$$\begin{aligned} q_1(t) &= p_1(t) - s_0 p_2(t) \\ &= p_1(t) = O(t) \end{aligned}$$

for some arbitrary large t; so that we do not have

$$q(t) = o(q_1(t)).$$

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We note that in (26) the extra factor $(1-s_0/s)$ has a pole on re $s = 0$, i.e. on the boundary of the region we are considering. This suggests that it might be all right if instead of this we had a factor regular in the whole of re $s \geqslant 0$. We therefore adopt the following definition. Let

$$\tilde{r}(t) = \beta e^{-t} + \int_0^t e^{-t+u} p(u)\, du = \beta e^{-t} + \tilde{\tilde{r}}(t), \quad \tilde{R}(s) = \int_0^\infty e^{-st} \tilde{r}(t)\, dt. \tag{30}$$

We now have (the inversion in the order of integration being justified by absolute convergence whenever re $s > 0$, by virtue of (14))

$$\begin{aligned} \tilde{R}(s) &= \beta \int_0^\infty e^{-(s+1)t}\, dt + \int_0^\infty e^{-st} dt \int_0^t e^{-(t-u)} p(u)\, du \\ &= \beta \int_0^\infty e^{-(s+1)t}\, dt + \int_0^\infty e^{-su} p(u)\, du \int_u^\infty e^{-(s+1)(t-u)}\, dt \\ &= \frac{P(s)}{s+1}. \end{aligned} \tag{31}$$

Remarks. (*a*) We have proved (31) from first principles. But $\tilde{\tilde{r}}(t)$ is the convolution $e^{-t} * p(t)$ and it is known that (under appropriate conditions) the Laplace transform of a convolution is the product of the Laplace transform of the two functions. Also, the Laplace transform of e^{-t} is obviously $1/(s+1)$. It is this that suggests the definition of $\tilde{r}$.

(*b*) We could equally well define

$$\tilde{\tilde{r}}(t) = \int_0^t e^{-\delta(t-u)} p(u)\, du, \quad \tilde{r}(t) = \beta e^{-\delta t} + \tilde{\tilde{r}}(t),$$

where δ is any complex number with re $\delta > 0$. We would then have

$$\tilde{R}(s) = \frac{P(s)}{s+\delta}.$$

But it seems simpler to take $\delta = 1$.

Now define

$$\tilde{q}(t) = p(t) - (s_0+1)\, \tilde{r}(t). \tag{32}$$

It follows from (31) that

$$\tilde{Q}(s) = \frac{s-s_0}{s+1} P(s). \tag{33}$$

The extra factor now has a simple zero at $s = s_0$, as required, but is also regular in the whole of the re $s \geqslant 0$.

We note that

$$\begin{aligned} \tilde{r}_1(t) &= \beta \int_0^t e^{-u}\, du + \int_0^t dv \int_0^v e^{-v+u} p(u) \\ &= \beta - \beta e^{-t} + \int_0^t p(u)\, du \int_u^t e^{-v+u}\, dv \\ &= \beta - \beta e^{-t} + \int_0^t p(u)\, (1 - e^{-t+u})\, du \\ &= p_1(t) - \tilde{r}(t). \end{aligned} \tag{34}$$

Hence from (32) and (34)

$$\tilde{q}_1(t) = -s_0 p_1(t) + (s_0+1)\, \tilde{r}(t). \tag{35}$$

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LEMMA 3. *If $\phi(t)$ is Lebesgue integrable in any relevant finite interval, and if*

$$\phi(t) = o(p_1(t)),$$

then

$$\int_0^t e^{-t+u}\phi(u)\,du = o(p_1(t)).$$

Proof. We apply (9) and (13), with $c = \frac{1}{2}$.

Given any $\epsilon > 0$, choose t_0 so that

$$|\phi(t)| < \epsilon|p_1(t)| \quad (t > t_0).$$

We may suppose that $t_0 \geqslant T(\frac{1}{2})$ (with the notation used in (10), (11)). Having fixed t_0, we have

$$\begin{aligned}\int_0^{t_0} e^{-t+u}\phi(u)\,du &= \text{const. } e^{-t}\\ &= o(p_1(t)) \quad \text{(by (13))}.\end{aligned}$$

But, using (10) (with t, u interchanged), we have

$$\begin{aligned}\left|\int_{t_0}^t e^{-t+u}\,\phi(u)\,du\right| &\leqslant \epsilon\int_{t_0}^t e^{-t+u}\,|p_1(u)|\,du\\ &\leqslant \epsilon|p_1(t)|\int_{t_0}^t e^{\frac{1}{2}(-t+u)}\,du\\ &\leqslant 2\epsilon|p_1(t)|.\end{aligned}$$

Hence the result.

LEMMA 4. *Assume that* (9) *holds. Let s_0 be a given complex number with* re $s_0 > 0$. *Suppose that $P(s)$ has a zero of order n at $s = s_0$. Define $\tilde{q}(t)$ by* (32).

Then

(*a*) $\tilde{q}(t) = o(\tilde{q}_1(t))$,

(*b*) $f(t) \in o(N, \tilde{q}, \beta)$ *if and only if*

$$f(t) = Ct^n e^{s_0 t} + g(t), \tag{36}$$

where $g(t) \in o(N, p, \beta)$ and C is a constant.

Proof. Applying Lemma 3, with $\phi(t) = p(t)$, and (13), we deduce that

$$\tilde{r}(t) = o(p_1(t)). \tag{37}$$

Hence, by (32),

$$\tilde{q}(t) = o(p_1(t)). \tag{38}$$

Also, by (35) and (37),

$$\tilde{q}_1(t) = p_1(t)\,(-s_0 + o(1)).$$

Thus, since $s_0 \neq o$, $o(\tilde{q}_1(t))$ is the same as $o(p_1(t))$. This result will be used several times. In particular, (*a*) now follows at once from (38). To prove (*b*), suppose first that (36) holds. Write

$$\phi(t) = \beta g(t) + \int_0^t p(t-u)\,g(u)\,du.$$

Thus we are given that

$$\phi(t) = o(p_1(t)). \tag{39}$$

Since

$$\phi(t) = \beta g(t) + (p * g)_t \quad \text{and} \quad \tilde{r}(t) = \beta e^{-t} + (e^{-t} * p)_t,$$

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it follows from the associativity of convolutions that

$$\begin{aligned}\tilde{r} * g &= \beta(e^{-t} * g) + (e^{-t} * p) * g \\ &= e^{-t} * (\beta g + p * g) \\ &= e^{-t} * \phi \\ &= o(p_1(t)) \quad \text{(by Lemma 3).}\end{aligned}$$

Hence, by (32) and (39),

$$\beta g(t) + \int_0^t \tilde{q}(t-u)\, g(u)\, du = o(p_1(t)) = o(\tilde{q}_1(t)),$$

so that $g(t) \in o(N, \tilde{q}, \beta)$.

It follows from Lemma 2 that

$$Ct^n e^{s_0 t} \in o(N, \tilde{q}, \beta)$$

and thus

$$f(t) \in o(N, \tilde{q}, \beta).$$

The converse implication depends on the result

$$p(t) = \tilde{q}(t) + (s_0 + 1) \int_0^t e^{s_0(t-u)}\, \tilde{q}(u)\, du + \beta(s_0 + 1)\, e^{s_0 t}. \tag{40}$$

This may be obtained by considering Laplace transforms, since, by (33),

$$P(s) = \left(1 + \frac{s_0 + 1}{s - s_0}\right) \tilde{Q}(s);$$

but it may also easily be verified directly.

Now suppose that

$$f(t) \in o(N, \tilde{q}, \beta).$$

Write

$$\psi(t) = \beta f(t) + \int_0^t \tilde{q}(t-u) f(u)\, du$$

thus we are given that

$$\psi(t) = o(\tilde{q}_1(t)) = o(p_1(t)). \tag{41}$$

By (40) and the associativity of convolutions,

$$\begin{aligned}\beta f(t) + \int_0^t p(t-u) f(u)\, du &= \psi(t) + (s_0 + 1) \int_0^t e^{s_0(t-u)}\, \psi(u)\, du \\ &= \psi(t) + (s_0 + 1)\, e^{s_0 t} \int_0^\infty e^{-s_0 u}\, \psi(u)\, du - (s_0 + 1) \int_t^\infty e^{s_0(t-u)}\, \psi(u)\, du.\end{aligned}$$

Write

$$A = (s_0 + 1) \int_0^\infty e^{-s_0 u}\, \psi(u)\, du;$$

so that A is a complex constant. Using (41) and applying Lemma 1 with $k = 0$, we deduce that

$$\beta f(t) + \int_0^t p(t-u) f(u)\, du = A\, e^{s_0 t} + o(p_1(t)).$$

It therefore follows from the corollary to Lemma 2 that

$$f(t) - \frac{A}{P^{(n)}(s_0)}\, t^n\, e^{s_0 t} \in o(N, p, \beta).$$

(Since $P(s)$ has a zero of order n, $P^{(n)}(s_0) \neq 0$.)

Thus (36) is satisfied, with $C = A/P^{(n)}(s_0)$, and the proof is completed.

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LEMMA 5. *If*
$$f(u) = \alpha + \int_{-\infty}^{\infty} e^{-iut} a(t)\,dt \quad (\alpha \neq 0)$$
with $\int_{-\infty}^{\infty} |a(t)|\,dt < \infty$ *and if for real* u, $f(u) \neq 0$, *then*
$$\frac{1}{f(u)} = \frac{1}{\alpha} + \int_{-\infty}^{\infty} e^{-iut} b(t)\,dt,$$
with
$$\int_{-\infty}^{\infty} |b(t)|\,dt < \infty.$$

The proof of this lemma is essentially given in Lemma 4 of Wiener and Pitt (10).

LEMMA 6. *If*
$$R(s) = \alpha + \int_0^{\infty} e^{-st} r(t)\,dt \quad \text{where} \quad \int_0^{\infty} |r(t)|\,dt < \infty \quad (\alpha \neq 0),$$
and if, for
$$\operatorname{re} s \geqslant 0, \quad R(s) \neq 0,$$
then
$$\frac{1}{R(s)} = \frac{1}{\alpha} + \int_0^{\infty} e^{-st} b(t)\,dt, \quad \text{where} \quad \int_0^{\infty} |b(t)|\,dt < \infty.$$

Proof. We have, for $\gamma \geqslant 0$,
$$R(\gamma + iu) = \alpha + \int_0^{\infty} e^{-\gamma t - iut} r(t)\,dt.$$
So for fixed $\gamma \geqslant 0$ the conditions of Lemma 5 are satisfied, with
$$a(t) = r(t)\,e^{-\gamma t} \quad (t > 0)$$
$$= 0 \quad (t < 0).$$
Hence, there is a function $b_\gamma(t)$ such that
$$\frac{1}{R(\gamma + iu)} = \frac{1}{\alpha} + \int_{-\infty}^{\infty} e^{-iut} b_\gamma(t)\,dt, \tag{42}$$
with
$$\int_{-\infty}^{\infty} |b_\gamma(t)|\,dt < \infty.$$

It follows from equation (42) and with the aid of a known theorem (Titchmarsh (8), Theorem 14) that for a given $\gamma \geqslant 0$ we have, for almost all t,
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{1}{R(\gamma + iu)} - \frac{1}{\alpha} \right\} e^{iut}\,du = b_\gamma(t) \quad (C, 1),$$
where the $(C, 1)$ summability of the integral to $b_\gamma(t)$ is taken as meaning that
$$\frac{1}{2\pi} \int_{-T}^{T} \left\{ 1 - \frac{|u|}{T} \right\} \left\{ \frac{1}{R(\gamma + iu)} - \frac{1}{\alpha} \right\} e^{iut}\,du \to b_\gamma(t) \quad \text{as} \quad T \to \infty. \tag{43}$$
Hence, for given $\gamma \geqslant 0$, putting $\gamma + iu = s$ in (43), we get
$$b_\gamma(t)\,e^{\gamma t} = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left\{ \frac{1}{R(s)} - \frac{1}{\alpha} \right\} e^{st}\,ds \tag{44}$$
for almost all t, with the integral taken in the $(C, 1)$ sense (this being given the obvious meaning).

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Now for fixed γ_1, γ_2 ($\gamma_1 \geqslant 0, \gamma_2 \geqslant 0$)

$$\int \left(\frac{1}{R(s)} - \frac{1}{\alpha}\right) e^{st}\, ds = 0 \tag{45}$$

by Cauchy's theorem, where the integral is taken round a rectangle with vertices at $\gamma_1 \pm iT, \gamma_2 + iT$.

By a theorem of Doetsch ((1), p. 51), $R(\gamma + iT) \to \alpha$; $R(\gamma - iT) \to \alpha$ as $T \to \infty$, uniformly for $\gamma_1 \leqslant \gamma \leqslant \gamma_2$. Hence the contribution to (45) of the horizontal portions tends to zero as $T \to \infty$, so that for any given t

$$\int_{\gamma_2 - iT}^{\gamma_2 + iT} \left(\frac{1}{R(s)} - \frac{1}{\alpha}\right) e^{st}\, ds = \int_{\gamma_1 - iT}^{\gamma_1 + iT} \left(\frac{1}{R(s)} - \frac{1}{\alpha}\right) e^{st}\, ds + o(1). \tag{46}$$

By the regularity of the $(C, 1)$ method for integrals, it follows that if the integral (44) is summable $(C, 1)$ for $\gamma = \gamma_1$ and any given t, then it is summable to the same sum for $\gamma = \gamma_2$ and that t.

Now write $b_0(t) = b(t)$. We have seen that $b(t)$ is given by (44) (taken in the $(C, 1)$ sense, with $\gamma = 0$) for almost all t, hence, by (44) and (46), it follows that for any given γ,

$$b_\gamma(t) = b(t)\, e^{-\gamma t}$$

for almost all t, so that we may replace $b_\gamma(t)$ by $b(t)\, e^{-\gamma t}$ in (42).

Now suppose that $t < 0$. Then if Γ denotes the semicircle $|s| = T$, $\operatorname{re} s \geqslant 0$ (taken in a counter clockwise sense) we have, again by Cauchy's theorem,

$$\int_\Gamma \left(\frac{1}{R(s)} - \frac{1}{\alpha}\right) e^{st}\, ds = \int_{-iT}^{iT} \left(\frac{1}{R(s)} - \frac{1}{\alpha}\right) e^{st}\, ds.$$

But $\{1/R(s) - (1/\alpha)\}$ is bounded in the half plane $\operatorname{re} s \geqslant 0$. Hence, since $t < 0$, it follows by Jordan's lemma that

$$\int_\Gamma \left(\frac{1}{R(s)} - \frac{1}{\alpha}\right) e^{st}\, ds \to 0 \quad \text{as} \quad T \to \infty.$$

Hence

$$\int_{-i\infty}^{i\infty} \left(\frac{1}{R(s)} - \frac{1}{\alpha}\right) e^{st}\, ds = 0 \tag{47}$$

where the integral is taken as a principal value, i.e. as

$$\lim_{T \to \infty} \int_{-iT}^{iT} \left(\frac{1}{R(s)} - \frac{1}{\alpha}\right) e^{st}\, ds = 0.$$

Hence, again by the regularity of the $(C, 1)$ method (47) holds also when the integral is taken in the $(C, 1)$ sense; in other words,

$$b(t) = 0 \quad \text{for} \quad t < 0.$$

Thus we have, from (42),

$$\frac{1}{R(s)} = \frac{1}{\alpha} + \int_0^\infty e^{-st}\, b(t)\, dt,$$

with

$$\int_0^\infty |b(t)|\, dt < \infty.$$

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5. Now define

$$q(t) = \alpha p(t) + \beta r(t) + \int_0^t p(t-u)\, r(u)\, du \quad (\alpha \neq 0), \tag{48}$$

with

$$\left.\begin{aligned} R(s) &= \alpha + \int_0^\infty e^{-st} r(t)\, dt \\ \int_0^\infty |r(t)|\, dt &< \infty. \end{aligned}\right\} \tag{49}$$

It may be observed here that $R(s)$ has only a finite number of zeros in $\operatorname{re} s \geqslant 0$. Since

$$\left| \int_0^\infty e^{-st} r(t)\, dt \right| < \tfrac{1}{2}|\alpha|$$

outside a bounded rectangle $0 \leqslant \operatorname{re} s \leqslant A$, $-G \leqslant \operatorname{im} s \leqslant G$, and $R(s)$ can only have a finite number of zeros in the rectangle. Write, for $\operatorname{re} s > 0$,

$$P(s) = \beta + \int_0^\infty e^{-st} p(t)\, dt, \tag{50}$$

$$Q(s) = \alpha\beta + \int_0^\infty e^{-st} q(t)\, dt. \tag{51}$$

By (14), the integral in (50) is absolutely convergent. Therefore, by (48) and (49),

$$Q(s) = P(s)\, R(s). \tag{52}$$

We further assume that

$$\int_0^t |p(u)|\, du = O(p_1(t)). \tag{53}$$

LEMMA 7. *Assume that* (9) *and* (53) *hold. Define* $q(t)$, $R(s)$ *and* $Q(s)$ *as in* (48), (49) *and* (51). *Suppose that* $R(s)$ *has no zeros in* $\operatorname{re} s \geqslant 0$ *and* $r(t) \to 0$ *as* $t \to \infty$. *Then*

$$(a)\ \ q(t) = o(q_1(t)),$$

$$(b)\ \ o(N, p, \beta) = o(N, q, \alpha\beta).$$

Proof. We have

$$q(t) = \alpha p(t) + \beta r(t) + \int_0^t p(u)\, r(t-u)\, du.$$

It is enough to consider the case for $t > t_0$.

Given any $\epsilon > 0$, there is a t_0 such that $|p(t)| < \epsilon |p_1(t)|$ for $t > t_0$. Supposing that $t > t_0$, we have

$$\begin{aligned} \left| \int_{t_0}^t p(u)\, r(t-u)\, du \right| &\leqslant \epsilon \int_{t_0}^t |p_1(u)|\, |r(t-u)|\, du \\ &\leqslant \epsilon |p_1(t)| \int_{t_0}^t |r(t-u)|\, e^{-\alpha(t-u)} du \quad \text{(by (10))} \\ &< \epsilon |p_1(t)| \int_0^\infty |r(w)| e^{-cw}\, dw \\ &= \epsilon |p_1(t)| \,.\, \text{const.} \end{aligned}$$

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Therefore $$q(t) = o(p_1(t)). \tag{54}$$

(If $p(t)$ is not equivalent to 0, it follows from (53) that, for sufficiently large t,

$$|p_1(t)| \geqslant c > 0 \quad (c \text{ constant}). \tag{54*}$$

If $p(t)$ is equivalent to 0, then $p_1(t) = \beta$; thus, by (6), $\beta \neq 0$ so that (54*) again holds. In either case (54) holds, since $r(t) = o(1)$. (Also,

$$\begin{aligned} q_1(t) &= \alpha\beta + \int_0^t q(u)\,du \\ &= \alpha\beta + \alpha(p_1(t) - \beta) + \beta\rho(t) + \int_0^t p(t-u)\,\rho(u)\,du, \end{aligned}$$

where $$\rho(u) = \int_0^u r(v)\,dv.$$

Write $\int_0^\infty r(t)\,dt = \rho$, so that as $u \to \infty$, $\rho(u) = \rho + o(1)$. Since $\rho(u) \to \rho$ and (N, p, β) is regular, we get

$$q_1(t) = (\alpha + \rho + o(1))\,p_1(t) = (R(0) + o(1))\,p_1(t). \tag{55}$$

Since $R(0) \neq 0$, (a) follows. To prove (b), let

$$\psi(t) = \beta f(t) + \int_0^t p(t-u) f(u)\,du.$$

Suppose that $$\psi(t) = o(p_1(t)).$$

Now $$\alpha\beta f(t) + \int_0^t q(t-u) f(u)\,du = \alpha\psi(t) + \int_0^t r(t-u)\,\psi(u)\,du.$$

Since $$\psi(t) = o(p_1(t)), \quad \int_0^\infty |r(t)|\,dt < \infty \quad \text{and} \quad r(t) \to 0,$$

then $$\int_0^t r(t-u)\,\psi(u)\,du = o(p_1(t)).$$

Therefore $$\alpha\beta f(t) + \int_0^t q(t-u) f(u)\,du = o(p_1(t))$$

and hence $$\alpha\beta f(t) + \int_0^t q(t-u) f(u)\,du = o(q_1(t)),$$

i.e. $$o(N, p, \beta) \subseteq o(N, q, \alpha\beta).$$

It can be easily verified that with the given conditions

$$\int_0^t |q(u)|\,du = O(q_1(t)).$$

Again, $P(s) = Q(s)/R(s)$ where $1/R(s)$ satisfies the similar conditions to those of $R(s)$ (by Lemma 6). We thus get

$$o(N, q, \alpha\beta) \subseteq o(N, p, \beta).$$

Thus (b) follows.

6. *Main theorem.*

THEOREM 1. *Assume that* (9) *and* (53) *hold. Define* $q(t)$, $R(s)$, $Q(s)$ *as in* (48), (49) *and* (51). *Suppose that* $r(t) \to 0$ *as* $t \to \infty$. *Assume also that no zeros of* $R(s)$ *occur on* re $s = 0$,

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and let the zeros of $R(s)$ *in* $\operatorname{re} s > 0$ *occur at* $s = s_1, s_2, s_3, \ldots, s_k$ *(all different), with multiplicities* $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k$, *and suppose that the zeros of* $R(s)$ *in* $\operatorname{re} s > 0$ *are zeros of* $P(s)$ *with non-negative multiplicities* $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. *Then* $f(t) \in o(N, q, \alpha\beta)$ *if and only if*

$$f(t) = g(t) + \sum_{i=1}^{k} e^{s_i t} \sum_{j=\lambda_i}^{\lambda_i+\gamma_i-1} C_{ij} t^j \tag{56}$$

where $g(t) \in o(N, p, \beta)$, $C_{ij} = const.$

Proof. Consider the functions

$$P_v(s) = \left(\frac{s-s_1}{s+1}\right)^v P(s), \quad R_v(s) = \left(\frac{s-s_1}{s+1}\right)^{-v} R(s) \quad (v = 1, 2, \ldots, \gamma_1),$$

$$P_{\gamma_1+v}(s) = \left(\frac{s-s_2}{s+s}\right)^v P_{\gamma_1}(s), \quad R_{\lambda_1+v}(s) = \left(\frac{s-s_2}{s+1}\right)^{-v} R_{\gamma_1}(s) \quad (v = 1, 2, \ldots, \gamma_2),$$

.....

$$P^*(s) = \prod_{i=1}^{k} \left(\frac{s-s_i}{s+1}\right)^{\gamma_i} P(s), \quad R^*(s) = \prod_{i=1}^{k} \left(\frac{s-s_2}{s+1}\right)^{-\gamma_i} R(s),$$

so that $$Q(s) = R(s)\,P(s) = R_1(s)\,P_1(s) = \ldots = R^*(s)\,P^*(s).$$

We will first prove that $R^*(s)$ satisfies conditions similar to those of $R(s)$. It is enough to prove the case for one factor and the general result will follow by successive applications.

Take $$R^*(s) = \frac{s+1}{s-s_0} R(s).$$

Writing $\hat{R}(s) = R(s) - \alpha$, this gives

$$\hat{R}^*(s) + \alpha = (\hat{R}(s) + \alpha)\left(1 + \frac{1+s_0}{s-s_0}\right); \tag{57}$$

i.e. $$\hat{R}^*(s) = \frac{1+s_0}{s-s_0}\alpha + \left(1 + \frac{1+s_0}{s-s_0}\right)\hat{R}(s),$$

so that $$r^*(t) = (1+s_0)\,\alpha e^{s_0 t} + r(t) + (1+s_0)\int_0^t e^{s_0(t-u)} r(u)\,du. \tag{58}$$

In general, if we defined $r^*(t)$ by (58) with $\int_0^\infty |r(t)|\,dt < \infty$ its Laplace transform would be defined only for $\operatorname{re} s > \operatorname{re} s_0$, and so (57) would hold in that region. But if $R(s_0)$ vanishes, this means, by (49), that

$$\alpha + \int_0^\infty e^{-s_0 u} r(u)\,du = 0,$$

so that (58) can be written as

$$r^*(t) = r(t) - (1+s_0)\int_t^\infty e^{s_0(t-u)} r(u)\,du. \tag{59}$$

It follows from (59) that if

$$\int_0^\infty |r(t)|\,dt < \infty \quad \text{and} \quad r(t) \to 0,$$

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then $r^*(t)$ has the same properties. Hence its Laplace transform is defined for $\operatorname{re} s \geqslant 0$, so that by analytic continuation, (57) will hold everywhere in $\operatorname{re} s \geqslant 0$. Applying Lemma 4, with $Q(s) = P_1(s)$, we have $f(t) \in o(N, p_1, \beta)$ if and only if $f_1(t) = f(t) - C_1 e^{s_1 t} t^{\lambda_1} \in o(N, p, \beta)$. Next, by Lemma 4, with $P_2(s)$, $P_1(s)$, $R_1(s)$ in place of $Q(s)$, $P(s)$, $R(s)$, we have $f(t) \in o(N, p_2, \beta)$ if and only if $f(t) - C_2 e^{s_1 t} t^{\lambda_1+1} \in o(N, p_1, \beta)$ which holds, by the previous case, if and only if

$$f_2(t) = f(t) - C_2 e^{s_1 t} t^{\lambda_1+1} - C_1 e^{s_1 t} t^{\lambda_1} \in o(N, p, \beta).$$

Continuing in this way we obtain $f(t) \in o(N, p^*, \beta)$ if and only if

$$f(t) - \sum_{i=1}^{k} e^{s_i t} \sum_{j=\lambda_i}^{\lambda_i+\gamma_i-1} C_{ij} t^j \in o(N, p, \beta),$$

and finally, by Lemma 7, with P^*, Q for P, Q

$$f(t) \in o(N, q, \alpha\beta) \quad \text{if and only if} \quad f(t) \in o(N, p^*, \beta).$$

Thus the theorem is proved.

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