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MINIMIZING DIRICHLET EIGENVALUES ON CUBOIDS OF UNIT MEASURE

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Abstract. We consider the minimization of Dirichlet eigenvalues λ_k, $k \in \mathbb{N}$, of the Laplacian on cuboids of unit measure in $\mathbb{R}^3$. We prove that any sequence of optimal cuboids in $\mathbb{R}^3$ converges to a cube of unit measure in the sense of Hausdorff as $k \to \infty$. We also obtain an upper bound for that rate of convergence.

§1. *Introduction.* The eigenvalues of the Laplacian have been the object of intensive study over the last century. Of particular interest are related shape optimization problems. For $k \in \mathbb{N}$, the goal is to optimize the kth eigenvalue of the Laplacian with boundary conditions over a collection of open sets in $\mathbb{R}^m$. This collection satisfies geometric constraints, such as fixed Lebesgue measure or fixed perimeter.

For an open set $\Omega \subset \mathbb{R}^m$, $m \geqslant 2$, of finite Lebesgue measure $|\Omega|$, we let $\lambda_k(\Omega)$, $k \in \mathbb{N}$, denote the Dirichlet eigenvalues of the Laplacian on Ω which are strictly positive, arranged in non-decreasing order and counted with multiplicity: that is,

$$\lambda_1(\Omega) \leqslant \lambda_2(\Omega) \leqslant \lambda_3(\Omega) \leqslant \cdots \leqslant \lambda_k(\Omega) \leqslant \cdots .$$

This sequence accumulates at $+\infty$.

We consider the minimization problem given by

$$\lambda_k^*(m) := \inf\{\lambda_k(\Omega) : \Omega \text{ open in } \mathbb{R}^m, |\Omega| = c\}.$$

It was shown by Faber and Krahn that among all open sets in $\mathbb{R}^m$ of measure c, the ball of measure c minimizes the first Dirichlet eigenvalue (see [15]). Krahn and Szego proved that, among all open sets in $\mathbb{R}^m$ of measure c, the second Dirichlet eigenvalue is minimized by the union of two disjoint balls of measure $c/2$ each (see [15]). For $k \geqslant 3$, the existence of an open set of prescribed measure which minimizes the kth Dirichlet eigenvalue remains unresolved to date. However, in the class of quasi-open sets of prescribed measure, it was shown by Bucur in [7] that a minimizer does exist and that such a minimizer is bounded and has finite perimeter. Independently, Mazzoleni and Pratelli, in [17], proved the existence of a minimizer in the collection of quasi-open sets. For any lower semi-continuous, increasing function of the first k Dirichlet eigenvalues,

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they proved the existence of a minimizer which is bounded in terms of k and m independently of the function. It was shown in [5] that, for $k \leqslant m+1$, any bounded minimizer of $\lambda_k(\Omega)$ has at most $\min\{7, k\}$ components.

No optimal domains are known for λ_k with $k \geqslant 3$. In particular, the conjecture that if $m = 2$, then $\lambda_3(\Omega)$ is bounded from below by the third eigenvalue of the disc with the same measure as Ω is open. There are no obvious candidates for minimizers of λ_k with $k \geqslant 5$ in any dimension $m \geqslant 2$. Even for $m = 2$, minimizers need not be discs or disjoint unions of discs (see [21]). Furthermore, it was shown in [6] that, for $k \geqslant 5$, $\lambda_k(\Omega)$ cannot be minimized by a disc or a disjoint union of discs. The numerical investigation [1] suggests that for some values of k the minimizers may not have any symmetries.

Pólya's conjecture for Dirichlet eigenvalues asserts that for all bounded, open sets $\Omega \subset \mathbb{R}^m$, $\lambda_k(\Omega) \geqslant 4\pi^2(\omega_m|\Omega|)^{-2/m}k^{2/m}$, where ω_m denotes the measure of a ball in $\mathbb{R}^m$ of radius 1. It was shown in [11] that Pólya's conjecture is equivalent to $\lambda_k^*(m)$ being asymptotically equal to $4\pi^2(\omega_m c)^{-2/m}k^{2/m}$ as $k \to \infty$.

It is also interesting to consider the optimization of the eigenvalues of the Laplacian subject to other geometric constraints, such as fixed perimeter. For the Dirichlet eigenvalues, existence of a minimizer in the class of open sets in $\mathbb{R}^m$ of finite Lebesgue measure and prescribed perimeter was shown in [12]. Moreover, it was shown there that any minimizer is bounded and connected, and regularity results for the boundary were also obtained. Bucur and Freitas, in [9], showed that any sequence of minimizers of λ_k in $\mathbb{R}^2$ with perimeter ℓ converges in the sense of Hausdorff to the disc of perimeter ℓ as $k \to \infty$. They also showed that if the collection of admissible sets is restricted to the collection of n-sided, convex, planar polygons of perimeter ℓ, then any sequence of minimizers converges to the regular n-sided polygon of perimeter ℓ as $k \to \infty$. For $m \geqslant 2$, other constraints were considered in [4], including perimeter and moment of inertia, subject to an additional convexity constraint. Further results for the Dirichlet eigenvalues were obtained in [3], [8], [5], [9] and [4]. Some of the results of [3] follow directly from those in [4], while the results of [12] supersede those of [8].

Recently, Antunes and Freitas considered the problem of minimizing λ_k over all planar rectangles of unit measure (see [2]). In [2, Theorem 2.1], they showed that any sequence of minimizing rectangles for the Dirichlet eigenvalues converges to the unit square in the sense of Hausdorff as $k \to \infty$.

In Theorem 1.1 below, we obtain the corresponding three-dimensional result for the Dirichlet eigenvalues of the Laplacian on cuboids in $\mathbb{R}^3$ of unit measure. In addition, we obtain an estimate for the rate of convergence. Let R_{a_1,a_2,a_3} denote a cuboid in $\mathbb{R}^3$ of side lengths a_1, a_2, a_3 such that $a_1a_2a_3 = 1$ and $a_1 \leqslant a_2 \leqslant a_3$: that is,

$$R_{a_1,a_2,a_3} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < a_1, \\ 0 < x_2 < a_2, 0 < x_3 < (a_1a_2)^{-1}, a_1 \leqslant a_2 \leqslant a_3\}. \quad (1.1)$$

We prove the following theorem.

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THEOREM 1.1. (i) *Let $k \in \mathbb{N}$. The variational problem*

$$\lambda_k^* := \inf\{\lambda_k(R_{a_1,a_2,a_3}) : a_1 \leqslant a_2 \leqslant a_3\}$$

has an optimal cuboid $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ with side lengths $a_{1,k}^* \leqslant a_{2,k}^* \leqslant a_{3,k}^*$ such that $a_{1,k}^* a_{2,k}^* a_{3,k}^* = 1$.

(ii)

$$a_{3,k}^* \leqslant 1 + O(k^{-(2-\beta)/6}), \quad k \to \infty, \tag{1.2}$$

where β is an exponent of the remainder in

$$\#\{(i_1, i_2, i_3) \in \mathbb{Z}^3 : i_1^2 + i_2^2 + i_3^2 \leqslant R^2\} - \frac{4\pi}{3} R^3 = O(R^\beta), \quad R \to \infty.$$

Furthermore, any sequence of optimal cuboids $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ converges to the unit cube in $\mathbb{R}^3$ in the sense of Hausdorff as $k \to \infty$.

The best estimate known to date is that, for any $\epsilon > 0$, $\beta = \frac{21}{16} + \epsilon$ (see [**14**]). Hence (1.2) holds for $\beta = \frac{21}{16} + \epsilon, \epsilon > 0$. The conjecture for the optimal remainder is $\beta = 1 + \epsilon, \epsilon > 0$ (see [**10**]).

A heuristic explanation for this asymptotic shape result is the following (see also [**2**]). For any cuboid R in $\mathbb{R}^3$ with measure $|R|$ and perimeter $\mathrm{Per}(R)$,

$$\lambda_k(R) = \left(\frac{6\pi^2 k}{|R|}\right)^{2/3} + \frac{(3\pi^5)^{1/3}\mathrm{Per}(R)k^{1/3}}{2^{5/3}|R|^{4/3}} + o(k^{1/3}), \quad k \to \infty. \tag{1.3}$$

So, if $|R| = 1$, then (1.3) suggests that the cuboid that minimizes $\lambda_k(R)$, $k \to \infty$, is the one with minimal perimeter, that is, the unit cube.

The Dirichlet eigenvalues of the Laplacian on a cuboid R_{a_1,a_2,a_3} (as in (1.1)) are given by

$$\frac{\pi^2 i_1^2}{a_1^2} + \frac{\pi^2 i_2^2}{a_2^2} + \frac{\pi^2 i_3^2}{a_3^2}, \quad i_1, i_2, i_3 \in \mathbb{N}. \tag{1.4}$$

By listing these in non-decreasing order including multiplicities, the kth Dirichlet eigenvalue on R_{a_1,a_2,a_3}, $\lambda_k(R_{a_1,a_2,a_3})$, is the kth item of this list. In Table 1, we list the optimal cuboids for the first few Dirichlet eigenvalues.

Let $\lambda \in \mathbb{R}$, let $\lambda \geqslant 0$ and let $a_1, a_2, a_3 \in \mathbb{R}$ be such that $a_1 a_2 a_3 = 1$ and $a_1 \leqslant a_2 \leqslant a_3$. With (1.4) in mind, we define

$$E(\lambda) := \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} \leqslant \frac{\lambda}{\pi^2}\right\}. \tag{1.5}$$

The ellipsoid $E(\lambda)$ has semi-axes

$$r_1 = \frac{a_1 \lambda^{1/2}}{\pi}, \qquad r_2 = \frac{a_2 \lambda^{1/2}}{\pi}, \qquad r_3 = \frac{a_3 \lambda^{1/2}}{\pi},$$

and $|E(\lambda)| = (4/3\pi^2)\lambda^{3/2}$.

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Table 1: The first eight minimized Dirichlet eigenvalues, the side lengths of the cuboids that realize them, and the corresponding minimizing modes.

k	λ_k^*	$a_{1,k}^*, a_{2,k}^*, a_{3,k}^*$	Minimizing modes
1	$3\pi^2$	$1, 1, 1$	$(1, 1, 1)$
2	$3 \cdot 2^{2/3}\pi^2$	$2^{-1/3}, 2^{-1/3}, 2^{2/3}$	$(1, 1, 2)$
3	$3 \cdot 2^{-2/3}5^{2/3}\pi^2$	$(\frac{2}{5})^{1/3}, (\frac{5}{2})^{1/6}, (\frac{5}{2})^{1/6}$	$(1, 2, 1)$
4	$6\pi^2$	$1, 1, 1$	$(2, 1, 1)$
5	$3^{5/3}\pi^2$	$3^{-1/3}, 3^{-1/3}, 3^{2/3}$	$(1, 1, 3)$
6	$3 \cdot 2^{4/3}\pi^2$	$2^{-2/3}, 2^{1/3}, 2^{1/3}$ or $2^{-2/3}, 2^{-2/3}, 2^{4/3}$	$(1, 2, 2)$ or $(1, 1, 4)$
7	$3 \cdot 5^{2/3}\pi^2$	$(\frac{5}{8})^{1/6}, (\frac{5}{8})^{1/6}, 2 \cdot 5^{-1/3}$ or $5^{-1/3}, 5^{-1/3}, 5^{2/3}$	$(2, 1, 2)$ or $(1, 1, 5)$
8	$9\pi^2$	$1, 1, 1$	$(2, 2, 1)$

By (1.4) and (1.5), we see that the Dirichlet eigenvalues $\lambda_1(R_{a_1,a_2,a_3}), \dots, \lambda_k(R_{a_1,a_2,a_3})$ (counted with multiplicities) correspond to the integer lattice points that are inside or on the ellipsoid $E(\lambda_k)$ in the first octant (excluding the coordinate planes). Thus, in order to minimize λ_k among all cuboids given by (1.1), we wish to determine the three-dimensional ellipsoid $E(\lambda)$ of minimal measure which encloses k integer lattice points in the first octant (excluding the coordinate planes).

For $n \in \mathbb{N}$, $n \geqslant 2$, estimates for the number of integer lattice points which are inside or on an n-dimensional ellipsoid have been widely studied from a number-theoretical viewpoint. However, in order to use these estimates, it is crucial that the corresponding cuboids are bounded as $k \to \infty$. As in the two-dimensional case, this is the most difficult part of the proof.

This paper is organized as follows. In §2, we prove Theorem 1.1(i). In §3, we obtain bounds for lattice point sums which are key ingredients in the proofs of the lemmas in §4. In that section, we follow the strategy of [**2**] and prove that the side lengths of a sequence of minimal cuboids $(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*})_k$ are bounded uniformly in k. This is achieved by first obtaining an upper bound for the counting function $N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j(R_{a_1,a_2,a_3}) \leqslant \lambda\}$ for arbitrary cuboids. Using the maximality of $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ and comparing with the unit cube gives the required uniform bound. Finally, in §5, we use known estimates for the number of integer lattice points that are inside and on an ellipsoid to conclude the proof of Theorem 1.1(ii).

§2. *Proof of Theorem 1.1(i).*

Proof. Fix $k \in \mathbb{N}$. Suppose that $\{R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}\}_{\ell\in\mathbb{N}}$ is a minimizing sequence for λ_k such that $a_{3,k}^{(\ell)} \to \infty$ as $\ell \to \infty$. In order to preserve the measure constraint, $a_{1,k}^{(\ell)} \to 0$ as $\ell \to \infty$. So

$$\lambda_k(R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}) > \frac{\pi^2}{(a_{1,k}^{(\ell)})^2} \to \infty \quad \text{as } \ell \to \infty.$$

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However, for the unit cube in $\mathbb{R}^3$, $\lambda_k \leqslant 3\pi^2 k^2 < +\infty$. This contradicts the assumption that $\{R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}\}_{\ell\in\mathbb{N}}$ is a minimizing sequence for λ_k. So any minimizing sequence $\{R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}\}_{\ell\in\mathbb{N}}$ for λ_k is such that $a_{1,k}^{(\ell)}, a_{2,k}^{(\ell)}, a_{3,k}^{(\ell)}$ are bounded as $\ell \to \infty$. Hence, for each $i \in \{1, 2, 3\}$, there exists a convergent subsequence, again denoted by $a_{i,k}^{(\ell)}$, such that $a_{i,k}^{(\ell)} \to a_{i,k}^*$ for some $a_{i,k}^* \in (0, \infty)$. Since $(a_1, a_2, a_3) \mapsto \lambda_k(R_{a_1,a_2,a_3})$ is continuous, $\lambda_k(R_{a_{1,k}^{(\ell)},a_{2,k}^{(\ell)},a_{3,k}^{(\ell)}}) \to \lambda_k(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*})$ as $\ell \to \infty$. Hence $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$ is an optimal cuboid for λ_k. □

It is not difficult to see that the above argument can also be used to prove the existence of an optimal cuboid for λ_k in $\mathbb{R}^m$ with $m \geqslant 4$.

§3. *Key lemmas to prove boundedness of an optimal cuboid.* The following lemmas are crucial in the proofs that follow in §4.

LEMMA 3.1. *Let $y \geqslant 0$, $a \geqslant 0$. For $n \in \{1, 2\}$,*

$$\sum_{i=1}^{\lfloor y^{1/2}/a\rfloor} (y - a^2 i^2)^{n/2} \leqslant \frac{\sqrt{\pi}}{2a} \frac{\Gamma((n+2)/2)}{\Gamma((n+3)/2)} y^{(n+1)/2} - \frac{1}{2} y^{n/2} + \frac{(2an)^{n/2}}{(n+2)^{(n+2)/2}} y^{n/4}. \tag{3.1}$$

Proof. We know that

$$\sum_{i=1}^{\lfloor y^{1/2}/a\rfloor} (y - a^2 i^2)^{n/2} = a^n \sum_{i=1}^{\lfloor y^{1/2}/a\rfloor} \left(\left(\frac{y^{1/2}}{a} \right)^2 - i^2 \right)^{n/2}. \tag{3.2}$$

Let $R = y^{1/2}/a$ and consider $\sum_{i=1}^{\lfloor R\rfloor} g(i)$, where

$$g(i) = (R^2 - i^2)^{n/2}. \tag{3.3}$$

Then, for $0 \leqslant i \leqslant R$,

$$g'(i) = -ni(R^2 - i^2)^{(n-2)/2} \leqslant 0,$$
$$g''(i) = n(R^2 - i^2)^{(n-4)/2}((n-1)i^2 - R^2) \leqslant 0.$$

So $i \mapsto g(i)$ is decreasing on $[0, R]$ and, since $n = 1$ or $n = 2$, g is also concave on $[0, R]$. We note that since g is decreasing, $\sum_{i=1}^{\lfloor R\rfloor} g(i)$ is the total area of the rectangles of width 1 and height $g(i)$, $i \in \{1, \ldots, \lfloor R\rfloor\}$ which are inscribed in the curve $g(x)$ for $0 \leqslant x \leqslant R$. Due to the concavity of g on $(0, R)$, we can bound $\sum_{i=1}^{\lfloor R\rfloor} g(i)$ from above by the area under g minus the area of the inscribed triangles that sit on top of the aforementioned rectangles: that is,

$$\sum_{i=1}^{\lfloor R\rfloor} g(i) \leqslant \int_0^R g(i)\, di - \frac{1}{2} \sum_{i=1}^{\lfloor R\rfloor} (g(i-1) - g(i)) - \frac{1}{2}(R - \lfloor R\rfloor) g(\lfloor R\rfloor). \tag{3.4}$$

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We know that

$$\int_0^R g(i)\,di = R^{n+1}\int_0^1 (1-t^2)^{n/2}\,dt$$
$$= \frac{R^{n+1}}{2}\int_0^1 (1-s)^{n/2}\frac{1}{\sqrt{s}}\,ds = \frac{\sqrt{\pi}}{2}\frac{\Gamma((n+2)/2)}{\Gamma((n+3)/2)}R^{n+1}, \quad (3.5)$$

where we have used [**13**, 3.191.3, 8.384.1].

We also know that

$$-\frac{1}{2}\sum_{i=1}^{\lfloor R\rfloor}(g(i-1)-g(i)) - \frac{1}{2}(R-\lfloor R\rfloor)g(\lfloor R\rfloor)$$
$$= -\frac{1}{2}R^n + \frac{1}{2}(1+\lfloor R\rfloor - R)(R^2-\lfloor R\rfloor^2)^{n/2}$$
$$= -\frac{1}{2}R^n + \frac{1}{2}(1+\lfloor R\rfloor - R)(R+\lfloor R\rfloor)^{n/2}(R-\lfloor R\rfloor)^{n/2}$$
$$\leqslant -\frac{1}{2}R^n + \frac{1}{2}(2R)^{n/2}\max_{0\leqslant\beta<1}(1-\beta)\beta^{n/2}$$
$$= -\frac{1}{2}R^n + \frac{(2n)^{n/2}}{(n+2)^{(n+2)/2}}R^{n/2}. \quad (3.6)$$

Combining (3.2), (3.4), (3.5) and (3.6) gives (3.1). □

Applying the previous lemma with $n=1$, $y=(a_2^2/\pi^2)\lambda$ and $a=a_2/a_1$, we recover the result of [**2**, Theorem 3.1]. Since g (as in (3.3)) is decreasing on $[0, y^{1/2}/a]$, the following lemma holds for all $n\in\mathbb{N}$.

LEMMA 3.2. *Let $y\geqslant 0$, $a\geqslant 0$. For $n\in\mathbb{N}$,*

$$\sum_{i=1}^{\lfloor y^{1/2}/a\rfloor}(y-a^2i^2)^{n/2} \leqslant \int_0^{y^{1/2}/a}(y-a^2i^2)^{n/2}\,di = \frac{\sqrt{\pi}}{2a}\frac{\Gamma((n+2)/2)}{\Gamma((n+3)/2)}y^{(n+1)/2}.$$

§4. *Uniform boundedness of an optimal cuboid.* With $E(\lambda)$ as defined in (1.5), we define the counting function

$$N(\lambda) := \#\{j\in\mathbb{N} : \lambda_j(R_{a_1,a_2,a_3})\leqslant\lambda\} = \#\{(i_1,i_2,i_3)\in\mathbb{N}^3\cap E(\lambda)\}.$$

We now use the results of §3 to obtain an upper bound for $N(\lambda)$.

LEMMA 4.1. *For $\lambda\geqslant 0$ and $a_1\leqslant a_2\leqslant a_3$, $E(\lambda)$ and $N(\lambda)$ as above,*

$$N(\lambda)\leqslant\frac{\lambda^{3/2}}{6\pi^2} - \frac{\lambda}{8\pi a_1} + \frac{\lambda^{1/2}}{16a_1^2}. \quad (4.1)$$

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Proof. For $(i_1, i_2, i_3) \in \mathbb{N}^3 \cap E(\lambda)$,

$$i_3 \leqslant \left\lfloor \left(\frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2}i_1^2 - \frac{a_3^2}{a_2^2}i_2^2 \right)_+^{1/2} \right\rfloor,$$

where "+" denotes the positive part. Hence

$$N(\lambda) \leqslant \sum_{i_1 \in \mathbb{N}} \sum_{i_2 \in \mathbb{N}} \left\lfloor \left(\frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2}i_1^2 - \frac{a_3^2}{a_2^2}i_2^2 \right)_+^{1/2} \right\rfloor \tag{4.2}$$

$$\leqslant \sum_{i_1=1}^{\lfloor a_1(\lambda^{1/2}/\pi)\rfloor} \sum_{i_2=1}^{\lfloor a_2(\lambda/\pi^2 - i_1^2/a_1^2)^{1/2}\rfloor} \left(\frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2}i_1^2 - \frac{a_3^2}{a_2^2}i_2^2 \right)^{1/2}. \tag{4.3}$$

Applying Lemma 3.2 with $y = (a_3^2/\pi^2)\lambda - (a_3^2/a_1^2)i_1^2$, $a = a_3/a_2$, $n = 1$ to (4.3), gives

$$N(\lambda) \leqslant \sum_{i_1=1}^{\lfloor a_1(\lambda^{1/2}/\pi)\rfloor} \frac{\pi a_2}{4a_3}\left(\frac{a_3^2}{\pi^2}\lambda - \frac{a_3^2}{a_1^2}i_1^2 \right)$$

$$= \sum_{i_1=1}^{\lfloor a_1(\lambda^{1/2}/\pi)\rfloor} \frac{\pi a_2 a_3}{4}\left(\frac{\lambda}{\pi^2} - \frac{i_1^2}{a_1^2} \right). \tag{4.4}$$

Applying Lemma 3.1 with $y = \lambda/\pi^2$, $a = 1/a_1$, $n = 2$, we obtain that

$$\frac{\pi a_2 a_3}{4} \sum_{i_1=1}^{\lfloor a_1(\lambda^{1/2}/\pi)\rfloor} \left(\frac{\lambda}{\pi^2} - \frac{i_1^2}{a_1^2} \right) \leqslant \frac{\pi a_2 a_3}{4}\left(\frac{2a_1}{3\pi^3}\lambda^{3/2} - \frac{1}{2\pi^2}\lambda + \frac{1}{4\pi a_1}\lambda^{1/2} \right)$$

$$= \frac{\lambda^{3/2}}{6\pi^2} - \frac{\lambda}{8\pi a_1} + \frac{\lambda^{1/2}}{16a_1^2}. \tag{4.5}$$

By (4.4) and (4.5), (4.1) follows. □

We now prove that the side lengths $a_{1,k}^*, a_{2,k}^*, a_{3,k}^*$ of an optimal cuboid $R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*}$ in $\mathbb{R}^3$ are uniformly bounded.

LEMMA 4.2. *For all $k \in \mathbb{N}$,*

$$a_{3,k}^* \leqslant 319.$$

Proof. Since (4.1) holds for all $\lambda \geqslant 0$ and all cuboids, it holds for $\lambda = \lambda_k^*$ and an optimal cuboid $R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*}$, so

$$k \leqslant N(\lambda_k^*) \leqslant \frac{(\lambda_k^*)^{3/2}}{6\pi^2} - \frac{\lambda_k^*}{8\pi a_{1,k}^*} + \frac{(\lambda_k^*)^{1/2}}{16(a_{1,k}^*)^2},$$

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and, by rearranging,

$$\frac{(\lambda_k^*)^{3/2} - 6\pi^2 k}{6\pi^2 \lambda_k^*} \geqslant \frac{1}{8\pi a_{1,k}^*} - \frac{(\lambda_k^*)^{-1/2}}{16(a_{1,k}^*)^2}. \tag{4.6}$$

The left-hand side of (4.6) is an increasing function of λ_k^*, so it is bounded from above by $(\nu_k^{3/2} - 6\pi^2 k)/6\pi^2 \nu_k$, where ν_k is the kth Dirichlet eigenvalue of the Laplacian on the unit cube in $\mathbb{R}^3$. We obtain a lower bound for the right-hand side of (4.6) by using the fact that

$$\lambda_k^* \geqslant \lambda_1(R_{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*}) \geqslant \frac{\pi^2}{(a_{1,k}^*)^2}$$

implies that

$$-\frac{(\lambda_k^*)^{-1/2}}{16(a_{1,k}^*)^2} \geqslant -\frac{1}{16\pi a_{1,k}^*}. \tag{4.7}$$

Hence, by (4.7),

$$\frac{\nu_k^{3/2} - 6\pi^2 k}{6\pi^2 \nu_k} \geqslant \frac{1}{16\pi a_{1,k}^*},$$

which implies that

$$a_{1,k}^* \geqslant \frac{1}{16\pi} \frac{6\pi^2 \nu_k}{\nu_k^{3/2} - 6\pi^2 k}. \tag{4.8}$$

We now obtain a uniform lower bound for $a_{1,k}^*$. By an estimate of Gauss,

$$\begin{aligned} N(\nu_k) &= \#\left\{(i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 \leqslant \frac{\nu_k}{\pi^2}\right\} \\ &\geqslant \frac{\omega_3}{8}\left(\frac{\nu_k^{1/2}}{\pi} - 3^{1/2}\right)_+^3 \geqslant \frac{\nu_k^{3/2}}{6\pi^2} - \frac{3^{1/2}\nu_k}{2\pi}. \end{aligned}$$

Let Θ_k denote the multiplicity of ν_k. Then $N(\nu_k) \leqslant k + \Theta_k - 1$. In addition, $\Theta_k = \#\{(i_1, i_2, i_3) \in \mathbb{N}^3 : i_1^2 + i_2^2 + i_3^2 = \nu_k/\pi^2\}$ is the number of integer lattice points in the first octant that lie on the sphere in $\mathbb{R}^3$ which is centred at $(0, 0, 0)$ and has radius $\nu_k^{1/2}/\pi$. By projection onto the plane $i_3 = 0$, each of these lattice points corresponds to an integer lattice point which lies inside or on the circle $\{(i_1, i_2) \in \mathbb{Z}^2 : i_1^2 + i_2^2 = \nu_k/\pi^2\}$ in the first quadrant. The number of integer lattice points that lie inside or on this circle is bounded from above by $\nu_k/4\pi$, that is, the area inscribed by the circle in the first quadrant. Thus

$$\nu_k^{3/2} \leqslant 6\pi^2 k + 3\pi \nu_k\left(\tfrac{1}{2} + 3^{1/2}\right). \tag{4.9}$$

Hence, by (4.8) and (4.9),

$$a_{1,k}^* \geqslant \left(8\left(\tfrac{1}{2} + 3^{1/2}\right)\right)^{-1}. \tag{4.10}$$

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Using that $a^*_{1,k} \leqslant a^*_{2,k} \leqslant a^*_{3,k}$, $a^*_{1,k}a^*_{2,k}a^*_{3,k} = 1$ and (4.10), we deduce that

$$a^*_{3,k} \leqslant \frac{1}{(a^*_{1,k})^2} \leqslant 64\left(\frac{1}{2} + 3^{1/2}\right)^2 \leqslant 319. \qquad \square$$

The main obstructions to proving a corresponding result to Theorem 1.1(ii) in higher dimensions $m \geqslant 4$ are the following. Firstly, for $m \geqslant 4$, the corresponding upper bound for $N(\lambda)$ to (4.2) involves lattice point sums $\sum_{i=1}^{\lfloor R \rfloor} g(i)$ with $g(i)$, R as in (3.3) and $n \geqslant 3$. For $n \geqslant 3$, $y^{1/2}/(a\sqrt{n-1})$ is an inflection point of g in $(0, y^{1/2}/a)$ and so g is not concave on $(0, y^{1/2}/a)$. Thus, the above approach cannot be used to obtain an upper bound for the left-hand side of (3.1) when $n \geqslant 3$. Secondly, the higher-dimensional equivalent of (4.1) will contain more terms in the right-hand side. The leading term in that right-hand side is the Weyl term. However, the lower order terms are bounds which are uniform in a_1, for example. Their usefulness depends on the numerical coefficients which show up. These, in turn, depend on lower-dimensional lattice point sums.

§5. *Proof of Theorem 1.1(ii).* The minimizers $R_{a^*_{1,k},a^*_{2,k},a^*_{3,k}}$ of λ_k need not be unique. From this point onwards, we consider an arbitrary subsequence of minimizers denoted by $(R_{a^*_{1,k},a^*_{2,k},a^*_{3,k}})_k$.

For $E(\lambda)$ as defined in (1.5), we introduce the following notation.

$$\begin{aligned} T(\lambda) &= \#\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \cap E(\lambda)\}, \\ T_{x_1}(\lambda) &= \#\{(0, x_2, x_3) \in (\{0\} \times \mathbb{Z}^2) \cap E(\lambda)\}, \\ T^+_{x_1}(\lambda) &= \#\{(0, x_2, x_3) \in (\{0\} \times \mathbb{N}^2) \cap E(\lambda)\}. \end{aligned}$$

$T(\lambda)$ is the total number of integer lattice points that are inside or on the three-dimensional ellipsoid $E(\lambda)$. Similarly, $T_{x_1}(\lambda)$ is the number of integer lattice points that are inside or on the two-dimensional ellipse which is centred at $(0, 0)$ and has semi-axes $a_2\lambda^{1/2}/\pi$, $a_3\lambda^{1/2}/\pi$. The number of these lattice points that lie in the first quadrant (excluding the axes) is denoted by $T^+_{x_1}(\lambda)$. Then $T_{x_2}(\lambda)$, $T^+_{x_2}(\lambda)$ etc. are defined similarly. Thus

$$\begin{aligned} T(\lambda) = 8N(\lambda) &+ 4T^+_{x_1}(\lambda) + 4T^+_{x_2}(\lambda) + 4T^+_{x_3}(\lambda) \\ &+ 2\left\lfloor \frac{a_1\lambda^{1/2}}{\pi} \right\rfloor + 2\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor + 2\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor + 1, \end{aligned}$$

which implies that

$$\begin{aligned} N(\lambda) = \frac{1}{8}T(\lambda) &- \frac{1}{2}T^+_{x_1}(\lambda) - \frac{1}{2}T^+_{x_2}(\lambda) - \frac{1}{2}T^+_{x_3}(\lambda) \\ &- \frac{1}{4}\left\lfloor \frac{a_1\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{4}\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{4}\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{8}. \end{aligned}$$

In addition,

$$T_{x_1}(\lambda) = 4T^+_{x_1}(\lambda) + 2\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor + 2\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor + 1,$$

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which implies that

$$T_{x_1}^{+}(\lambda) = \frac{1}{4}T_{x_1}(\lambda) - \frac{1}{2}\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{2}\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor - \frac{1}{4},$$

and, similarly for $T_{x_2}^{+}(\lambda)$, $T_{x_3}^{+}(\lambda)$. Thus

$$\begin{aligned} N(\lambda) = {} & \frac{1}{8}T(\lambda) - \frac{1}{8}T_{x_1}(\lambda) - \frac{1}{8}T_{x_2}(\lambda) - \frac{1}{8}T_{x_3}(\lambda) \\ & + \frac{1}{4}\left\lfloor \frac{a_1\lambda^{1/2}}{\pi} \right\rfloor + \frac{1}{4}\left\lfloor \frac{a_2\lambda^{1/2}}{\pi} \right\rfloor + \frac{1}{4}\left\lfloor \frac{a_3\lambda^{1/2}}{\pi} \right\rfloor + \frac{1}{4}. \end{aligned} \tag{5.1}$$

Below we use this expression for $N(\lambda)$ in order to prove Theorem 1.1(ii).

Proof of Theorem 1.1(ii). By setting $\lambda = \lambda_k^*$ in (5.1) and considering an optimal cuboid $R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*}$,

$$\begin{aligned} k \leqslant N(\lambda_k^*) = {} & \frac{1}{8}T(\lambda_k^*) - \frac{1}{8}T_{x_1}(\lambda_k^*) - \frac{1}{8}T_{x_2}(\lambda_k^*) - \frac{1}{8}T_{x_3}(\lambda_k^*) \\ & + \frac{1}{4}\left\lfloor \frac{a_{1,k}^*(\lambda_k^*)^{1/2}}{\pi} \right\rfloor + \frac{1}{4}\left\lfloor \frac{a_{2,k}^*(\lambda_k^*)^{1/2}}{\pi} \right\rfloor + \frac{1}{4}\left\lfloor \frac{a_{3,k}^*(\lambda_k^*)^{1/2}}{\pi} \right\rfloor + \frac{1}{4}. \end{aligned} \tag{5.2}$$

By Lemma 4.2, the $\{a_{1,k}^*, a_{2,k}^*, a_{3,k}^*\}$ are uniformly bounded, so it is possible to make use of known estimates for the number of integer lattice points that are inside or on a three-dimensional ellipsoid or a two-dimensional ellipse. In particular, there exists $C < \infty$ such that, for all $\lambda \geqslant 0$,

$$\frac{4}{3\pi^2}\lambda^{3/2} - C\lambda^{\beta/2} \leqslant T(\lambda) \leqslant \frac{4}{3\pi^2}\lambda^{3/2} + C\lambda^{\beta/2} + 1, \tag{5.3}$$

where β is as defined in the Introduction. Similarly, there exists $D < \infty$ such that, for all $\lambda \geqslant 0$,

$$\frac{a_2a_3}{\pi}\lambda - D\lambda^{\theta/2} \leqslant T_{x_1}(\lambda) \leqslant \frac{a_2a_3}{\pi}\lambda + D\lambda^{\theta/2} + 1, \tag{5.4}$$

where θ is an exponent of the remainder in Gauss' circle problem

$$\#\{(i_1, i_2) \in \mathbb{Z}^2 : i_1^2 + i_2^2 \leqslant R^2\} - \pi R^2 = O(R^\theta),\ R \to \infty.$$

The best estimate known to date is $\theta > \frac{131}{208}$ (see the Introduction in [**16**]). Hence the formula above holds for $\theta = \frac{131}{208} + \epsilon$ for any $\epsilon > 0$. The corresponding inequalities to (5.4) also hold for $T_{x_2}(\lambda)$, $T_{x_3}(\lambda)$. Using these inequalities and (5.2), we obtain an upper bound for $N(\lambda_k^*)$,

$$\begin{aligned} k \leqslant N(\lambda_k^*) \leqslant {} & \frac{(\lambda_k^*)^{3/2}}{6\pi^2} - \frac{1}{8\pi}\left(\frac{1}{a_{1,k}^*} + \frac{1}{a_{2,k}^*} + \frac{1}{a_{3,k}^*}\right)\lambda_k^* + \frac{C}{8}(\lambda_k^*)^{\beta/2} \\ & + \frac{1}{4\pi}(a_{1,k}^* + a_{2,k}^* + a_{3,k}^*)(\lambda_k^*)^{1/2} + \frac{3D}{8}(\lambda_k^*)^{\theta/2} + \frac{3}{8}. \end{aligned} \tag{5.5}$$

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Rearranging (5.5) gives

$$\frac{1}{a_{1,k}^*}+\frac{1}{a_{2,k}^*}+\frac{1}{a_{3,k}^*} \leqslant 8\pi\left(\frac{(\lambda_k^*)^{3/2}-6\pi^2 k}{6\pi^2\lambda_k^*}\right)+\pi C(\lambda_k^*)^{-(2-\beta)/2} + 2(a_{1,k}^*+a_{2,k}^*+a_{3,k}^*)(\lambda_k^*)^{-1/2}+3\pi D(\lambda_k^*)^{-(2-\theta)/2} + 3\pi(\lambda_k^*)^{-1}.$$

Since $((\lambda_k^*)^{3/2}-6\pi^2 k)/6\pi^2\lambda_k^*$ is an increasing function of λ_k^*, we can replace λ_k^* by ν_k, where ν_k is the kth Dirichlet eigenvalue of the Laplacian on the unit cube in $\mathbb{R}^3$. Thus, by Pólya's inequality $\lambda_k^* \geqslant (6\pi^2 k)^{2/3}$ (see [19, 20]),

$$\begin{aligned}
&\frac{1}{a_{1,k}^*}+\frac{1}{a_{2,k}^*}+\frac{1}{a_{3,k}^*}\\
&\quad\leqslant 8\pi\left(\frac{\nu_k^{3/2}-6\pi^2 k}{6\pi^2\nu_k}\right)+\pi C(\lambda_k^*)^{-(2-\beta)/2}+3\pi(\lambda_k^*)^{-1}\\
&\qquad+2(a_{1,k}^*+a_{2,k}^*+a_{3,k}^*)(\lambda_k^*)^{-1/2}+3\pi D(\lambda_k^*)^{-(2-\theta)/2}\\
&\quad\leqslant 8\pi\left(\frac{\nu_k^{3/2}-6\pi^2 k}{6\pi^2\nu_k}\right)+\pi C(6\pi^2)^{-(2-\beta)/3}k^{-(2-\beta)/3}+3\pi(6\pi^2)^{-2/3}k^{-2/3}\\
&\qquad+2(a_{1,k}^*+a_{2,k}^*+a_{3,k}^*)(6\pi^2)^{-1/3}k^{-1/3}+3\pi D(6\pi^2)^{-(2-\theta)/3}k^{-(2-\theta)/3}\\
&\quad= 8\pi\left(\frac{\nu_k^{3/2}-6\pi^2 k}{6\pi^2\nu_k}\right)+O(k^{-(2-\beta)/3}).
\end{aligned} \tag{5.6}$$

To obtain an upper bound for $(\nu_k^{3/2}-6\pi^2 k)/6\pi^2\nu_k$ we proceed as follows. By (5.1), with $\lambda=\nu_k$,

$$N(\nu_k)=\frac{1}{8}T(\nu_k)-\frac{3}{8}T_{x_1}(\nu_k)+\frac{3}{4}\left\lfloor\frac{\nu_k^{1/2}}{\pi}\right\rfloor+\frac{1}{4}. \tag{5.7}$$

Since $a_1=a_2=a_3=1$, by (5.3) and (5.4),

$$\frac{4}{3\pi^2}\nu_k^{3/2}-C\nu_k^{\beta/2}\leqslant T(\nu_k) \tag{5.8}$$

and

$$T_{x_1}(\nu_k)\leqslant\frac{\nu_k}{\pi}+D\nu_k^{\theta/2}+1, \tag{5.9}$$

where β and θ are as in (5.3), (5.4). Again, let Θ_k denote the multiplicity of ν_k. Thus, by (5.7), (5.8) and (5.9), we obtain a lower bound for $N(\nu_k)$,

$$k+\Theta_k-1\geqslant N(\nu_k)\geqslant\frac{\nu_k^{3/2}}{6\pi^2}-\frac{C}{8}\nu_k^{\beta/2}-\frac{3}{8\pi}\nu_k-\frac{3D}{8}\nu_k^{\theta/2}+\frac{3}{4\pi}\nu_k^{1/2}-\frac{7}{8},$$

which implies that

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$$\begin{aligned}\frac{\nu_k^{3/2}-6\pi^2 k}{6\pi^2\nu_k} &\leqslant \frac{3}{8\pi}+\frac{C}{8}\nu_k^{-(2-\beta)/2}\\ &\quad+\frac{3D}{8}\nu_k^{-(2-\theta)/2}-\frac{3}{4\pi}\nu_k^{-1/2}+\Theta_k\nu_k^{-1}-\frac{1}{8}\nu_k^{-1}\\ &\leqslant \frac{3}{8\pi}+\frac{C}{8}\nu_k^{-(2-\beta)/2}+\frac{3D}{8}\nu_k^{-(2-\theta)/2}+\Theta_k\nu_k^{-1}\\ &\leqslant \frac{3}{8\pi}+\frac{C}{8}(6\pi^2)^{-(2-\beta)/3}k^{-(2-\beta)/3}\\ &\quad+\frac{3D}{8}(6\pi^2)^{-(2-\theta)/3}k^{-(2-\theta)/3}+\Theta_k\nu_k^{-1},\end{aligned}$$

by Pólya's inequality.

We know that $\Theta_k = \#\{(i_1,i_2,i_3)\in\mathbb{N}^3 : i_1^2+i_2^2+i_3^2 = \nu_k/\pi^2\}$ is the number of integer lattice points in the first octant that lie on the sphere in $\mathbb{R}^3$ which is centred at $(0,0,0)$ and has radius $\nu_k^{1/2}/\pi$. It is well known that $\#\{(x_1,x_2,x_3)\in\mathbb{Z}^3 : x_1^2+x_2^2+x_3^2 = d\} = O(d^{1/2+o(1)})$.

The following routine proof was communicated by T. Wooley. Let $n = d-x_3^2$. Now $|x_3|\leqslant d^{1/2}$, so, for $x_3\in[-d^{1/2},d^{1/2}]\cap\mathbb{Z}$, there are at most $2d^{1/2}+1$ possible values of n. If $n=0$, then $x_1^2+x_2^2=0$ has one solution $(0,0)\in\mathbb{Z}^2$. Suppose that $n\neq 0$. Let $R(n)$ denote the number of pairs $(x_1,x_2)\in\mathbb{Z}^2$ such that $x_1^2+x_2^2=n$. Then

$$\#\{(x_1,x_2,x_3)\in\mathbb{Z}^3 : x_1^2+x_2^2+x_3^2=d\}\leqslant 1+\sum_{|z|\leqslant d^{1/2}}R(d-z^2).$$

By [18, Corollary 3.23],

$$R(n) = 4\sum_{d|n,d>0,d\text{ odd}}\left(\frac{-1}{d}\right),$$

where the sum is taken over all positive, odd divisors of n and $\left(\frac{-1}{d}\right)$ is the quadratic residue symbol. Thus $R(n)\leqslant 4D(n)$, where $D(n)$ denotes the number of positive divisors of n. By [18, Theorem 8.31], for every $\epsilon>0$, there exists n_ϵ such that, for $n>n_\epsilon$,

$$D(n) < n^{(1+\epsilon)\log 2/\log\log n},$$

which implies that $D(n) = O(n^\epsilon)$. Therefore

$$\begin{aligned}&\#\{(x_1,x_2,x_3)\in\mathbb{Z}^3 : x_1^2+x_2^2+x_3^2=d\}\\ &\quad\leqslant 1+O\left(\sum_{|z|\leqslant d^{1/2}}(d-z^2)^\epsilon\right)\leqslant 1+O(d^{1/2+\epsilon}).\end{aligned}$$

So $\Theta_k = O(\nu_k^{1/2+o(1)})$ and $\Theta_k\nu_k^{-1} = O(\nu_k^{-1/2+o(1)}) = O(k^{-1/3+o(1)})$. Thus

$$\frac{\nu_k^{3/2}-6\pi^2 k}{6\pi^2\nu_k}\leqslant\frac{3}{8\pi}+O(k^{-(2-\beta)/3}). \tag{5.10}$$

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So, by (5.6) and (5.10), we deduce that

$$\frac{1}{a_{1,k}^*}+\frac{1}{a_{2,k}^*}+\frac{1}{a_{3,k}^*}\leqslant 3+O(k^{-(2-\beta)/3}),\quad k\to\infty. \tag{5.11}$$

Furthermore, by the arithmetic mean–geometric mean inequality applied to $1/a_{1,k}^*+1/a_{2,k}^*$ and by (5.11),

$$2(a_{3,k}^*)^{1/2}+\frac{1}{a_{3,k}^*}\leqslant 3+O(k^{-(2-\beta)/3}),\quad k\to\infty.$$

Let $a_{3,k}^*=1+\delta_k$, where $\delta_k>0$. Then

$$2(1+\delta_k)^{3/2}+1\leqslant 3+3\delta_k+O(k^{-(2-\beta)/3}),\quad k\to\infty.$$

Since $a_{3,k}^*\leqslant 319$, $\delta_k\leqslant 399$. Hence $(1+\delta_k)^{3/2}\geqslant 1+\frac{3}{2}\delta_k+\frac{3}{160}\delta_k^2$ for $0<\delta_k\leqslant 399$ and we deduce that $\delta_k\leqslant O(k^{-(2-\beta)/6})$, $k\to\infty$. As this estimate is independent of the subsequence $(R_{a_{1,k}^*,a_{2,k}^*,a_{3,k}^*})_k$, we arrive at the conclusion of Theorem 1.1(ii). □

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