Canad. Math. Bull. 2026, pp. 1–15
http://dx.doi.org/10.4153/S0008439526102276
© The Author(s), 2026. Published by Cambridge University Press on behalf of Canadian Mathematical Society

GCD-closed sets and divisibility among power LCM matrices

Guangyan Zhu

Abstract. Let a, b, and n be positive integers and let $S = \{x_1, \cdots, x_n\}$ be a set of n distinct positive integers. We denote by $[S^a]$ the $n \times n$ matrix having the ath power of the least common multiple of x_i and x_j as its (i, j)-entry. For $x \in S$, let $G_S(x) = \{d \in S : d < x, d|x \text{ and } (d|y|x, y \in S) \Rightarrow y \in \{d, x\}\}$. In this article, we show that $[S^a]$ divides $[S^b]$ in the ring of $n \times n$ matrices over the integers if $a|b$ and S is gcd closed (i.e., $\gcd(x_i, x_j) \in S$ for all integers i and j with $1 \le i, j \le n$) and satisfies the condition $\mathcal{G}$ (That is, for any $x \in S$, either $G_S(x)$ contains at most one element, or $G_S(x)$ contains at least two elements and satisfies that $\operatorname{lcm}(y_1, y_2) = x$ and $\gcd(y_1, y_2) \in G_S(y_1) \cap G_S(y_2)$ for any $\{y_1, y_2\} \subseteq G_S(x)$). This confirms a conjecture of Hong proposed in (2026, *Bull. Aust. Math. Soc.*, 113, 231–243).

1 Introduction

Let a, b, and n be positive integers. For any integers x and y, we denote by (x, y) the greatest common divisor of integers x and y and by $[x, y]$ their least common multiple. Let $\mathbb{Z}$ denote the ring of integers and let $|T|$ stand for the cardinality of a finite set T of integers. Let f be an arithmetic function and let $S = \{x_1, \ldots, x_n\}$. Let $(f(S))$ and $(f[S])$ denote the $n \times n$ matrices whose (i, j)-entries are $f((x_i, x_j))$ and $f([x_i, x_j])$, respectively. Define $\langle n \rangle := \{1, \ldots, n\}$. For any real number e, let ξ_e be the arithmetic function defined for any positive integer x by $\xi_e(x) = x^e$. The $n \times n$ matrices $(\xi_a(x_i, x_j)) = ((x_i, x_j)^a)$ (abbreviated by (S^a)) and $(\xi_a[x_i, x_j]) = ([x_i, x_j]^a)$ (abbreviated by $[S^a]$) are called the *ath power GCD matrix* on S and the *ath power LCM matrix* on S, respectively. The set S is said to be *factor closed* (written as FC for short) if $(x \in S, d > 0, d|x) \Rightarrow d \in S$. We say that S is *gcd closed* if S contains (x_i, x_j) for all integers i and j with $1 \le i, j \le n$. Obviously, any FC set contains 1 and is gcd closed but the converse is not true. For instance, if the set S is FC, then for any integer $x > 1$, the set $xS := \{xy | y \in S\}$ is gcd closed but not FC since $1 \notin xS$. One hundred and fifty years ago, Smith [26] proved that

$$\det([x_i, x_j]) = \prod_{k=1}^{n} \varphi(x_k)\pi(x_k) \tag{1.1}$$

Received by the editors June 10, 2026 UTC; revised July 3, 2026 UTC; accepted July 4, 2026 UTC.
Published online on Cambridge Core July 9, 2026.
The research was supported in part by the Startup Research Fund of Hubei Minzu University for Doctoral Scholars (Grant No. BS25008)
AMS subject classification: **11C20**, 11A05, 15B36.
Keywords: Divisibility, power GCD matrix, power LCM matrix, greatest-type divisor, gcd-closed set.

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

if S is FC, where φ is the Euler's totient function and π is the multiplicative function defined for the prime power p^r by $\pi(p^r) = -p$. After that, many generalizations of Smith's determinant (1.1) and related results were published (see, e.g., [1–25] and [27–35]). In particular, an elegant result was gotten by Hong et al. [10] stating that for any integer $n \geq 2$, one has

$$\det([i,j])_{2\leq i,j\leq n} = \Big(\prod_{k=1}^{n}\varphi(k)\pi(k)\Big)\sum_{\substack{t=1\\ t \text{ is square free}}}^{n}\frac{t\mu(t)}{\varphi(t)},$$

where μ is the M öbius function and an integer $x \geq 1$ is called *square free* if x is not divisible by the square of any prime number.

Hong [12] introduced the concept of greatest-type divisor when he solved the famous Bourque–Ligh conjecture [4]. For any integer $x \in S$, if $(y < x, y|z|x$ and $y, z \in S) \Rightarrow z \in \{y, x\}$, then y is called a *greatest-type divisor* of x. Let $G_S(x) := \{y \in S : y$ is a greatest-type divisor of x in $S\}$. Let $M_n(\mathbb{Z})$ stand for the ring of $n \times n$ matrices over the integers. In 1992, Bourque and Ligh [4] proved that (S) divides $[S]$ in the ring $M_n(\mathbb{Z})$ if S is FC. Namely, there exists a $B \in M_n(\mathbb{Z})$ such that $[S] = B(S)$ or $[S] = (S)B$. In 2002, Hong [14] showed that such a factorization is not true when S is gcd closed and $\max_{x\in S}\{|G_S(x)|\} = 2$.

On the other hand, Hong [17] initiated the investigation of divisibility among power LCM matrices. It was proved in [17] that $[S^a]|[S^b]$ if $a|b$ and S is a divisor chain (i.e., $x_{\sigma(1)}|\ldots|x_{\sigma(n)}$ for a permutation σ of $\langle n\rangle$). Evidently, a divisor chain is gcd closed but not conversely. Tan and Li [27] generalized Hong's result [17] by proving that $[S^a]|[S^b]$ holds in the ring $M_{|S|}(\mathbb{Z})$ if $a|b$ and S consists of finitely many coprime divisor chains with $1 \in S$, and that such divisibility relation is not true if $a \nmid b$. Zhu and Li [33] confirmed the conjecture of Hong raised in [17] stating that if $a|b$ and S is a gcd-closed set with $\max_{x\in S}\{|G_S(x)|\} = 1$, then $[S^a] \mid [S^b]$ in the ring $M_n(\mathbb{Z})$. In 2026, Hong [18] established the same divisibility result when S is FC. As in [18], for any set S of positive integers and for any $x \in S$ with $|G_S(x)| \geq 2$, we say that the two distinct greatest-type divisors y_1 and y_2 of x in S *satisfy the condition* $\mathcal{G}$ if $[y_1, y_2] = x$ and $(y_1, y_2) \in G_S(y_1) \cap G_S(y_2)$. We say that x *satisfies the condition* $\mathcal{G}$ if any two distinct greatest-type divisors of x in S satisfy the condition $\mathcal{G}$. Moreover, we say that a set S of positive integers *satisfies the condition* $\mathcal{G}$ if any element x in S satisfies that either $|G_S(x)| \leq 1$, or $|G_S(x)| \geq 2$ and x satisfies the condition $\mathcal{G}$. In [18], Hong showed that any FC set is a gcd-closed set satisfying the condition $\mathcal{G}$. Furthermore, Hong raised the following conjecture in Section 3 of [18].

Conjecture 1.1 [18, Conjecture 3.4] *Let a and b be positive integers with $a|b$ and let S be a gcd-closed set satisfying the condition $\mathcal{G}$. Then $[S^a] \mid [S^b]$ in the ring $M_{|S|}(\mathbb{Z})$.*

For the case $\max_{x\in S}\{|G_S(x)|\} = 1$, by the theorem of Zhu and Li [33], we know that this conjecture is true. For the case $\max_{x\in S}\{|G_S(x)|\} = 2$, by [28], one knows that Conjecture 1.1 holds. Furthermore, Zhu et al. [34] confirmed Conjecture 1.1 for the case $\max_{x\in S}\{|G_S(x)|\} = 3$. But Conjecture 1.1 is still kept open when S is the general gcd-closed set satisfying the condition $\mathcal{G}$.

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

In this article, our main goal is to study the divisibility among power LCM matrices. We introduce a new method to investigate Conjecture 1.1. The main result of this article can be stated as follows.

Theorem 1.2 *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and let a and b be positive integers such that $a|b$. Then the ath power LCM matrix $[S^a]$ divides the bth power LCM matrix $[S^b]$ in the ring $M_{|S|}(\mathbb{Z})$.*

This article is organized as follows. After introducing some necessary preliminary results, we proceed to the core of our argument. In fact, we first recall and show in Section 2 some results related to the arithmetic properties of gcd-closed sets that satisfy the condition $\mathcal{G}$ and then present a formula for the inverse of power LCM matrices. Finally, in Section 3, by proving a key lemma, we present the proof of Theorem 1.2.

The divisibility of matrices under consideration is invariant under relabeling the elements of S. Specifically, for any permutation σ on $\langle n\rangle$, we have $[S^a] \mid [S^b]$ if and only if $[S^a_\sigma] \mid [S^b_\sigma]$. Therefore, without loss of generality, we may assume throughout the rest of this article that the set $S = \{x_1, \ldots, x_n\}$ is ordered such that $x_1 < \cdots < x_n$.

2 Preliminary lemmas

Throughout this section, let $x_m \in S$ with $G_S(x_m) = \{x_{m_1}, \ldots, x_{m_s}\}$, $2 \le m \le n = |S|$ and $1 \le s \le m-1$. Let $1 \le k \le s$ and for any integers $i_1, \ldots, i_k$ with $1 \le i_1 < \cdots < i_k \le s$, we define

$$x_{m_{i_1 \ldots i_k}} := (x_{m_{i_1}}, \ldots, x_{m_{i_k}}).$$

Then $x_{m_{1\ldots s}} = (x_{m_1}, \ldots, x_{m_s})$. In what follows, we recall and show several results that are needed in the proof of our main result. For any $j \in \langle s\rangle$, define $P_j := \frac{x_m}{x_{m_j}}$. We begin with a known result which is due to Zhu and Yu [35] and is about the arithmetic property of the gcd-closed set S which satisfies the condition $\mathcal{G}$.

Lemma 2.1 [35] *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and $x_m \in S$ with $G_S(x_m) = \{x_{m_1}, \ldots, x_{m_s}\}$. Let k be an integer with $1 \le k \le s$. Then for any integers $i_1, \ldots, i_k$ with $1 \le i_1 < \cdots < i_k \le s$, we have*

$$x_{m_{i_1 \ldots i_k}} P_{i_1} \ldots P_{i_k} = x_m.$$

Lemma 2.2 [32] *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and $x_l, x_m \in S$ with $G_S(x_m) = \{x_{m_1}, \ldots, x_{m_s}\}$. Let $1 \le k \le s-1$. Assume that $(x_l, x_m) \mid x_{m_{1\ldots k}}$ and $(x_l, x_m) \nmid x_{m_j}$ for all integers j with $k+1 \le j \le s$. Let $1 \le i_1 < \cdots < i_h \le k$ with $1 \le h \le k$ and $k+1 \le j_1 < \cdots < j_t \le s$ with $1 \le t \le s-k$. Then each of the following is true:*

(i). $(x_l, x_{m_{i_1 \ldots i_h}}) = (x_l, x_m)$.

(ii). $[x_l, x_{m_{j_1 \ldots j_t}}] = [x_l, x_m]$.

(iii). $[x_l, x_{m_{i_1 \ldots i_h j_1 \ldots j_t}}] = [x_l, x_{m_{i_1 \ldots i_h}}]$.

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

Lemma 2.3 [6, Theorem 3] *If S is gcd-closed set of n positive integers and $(f(S))$ is nonsingular, then*

$$((f(S))^{-1})_{ij} := \sum_{\substack{x_i|x_k \\ x_j|x_k}} \frac{c_{ik}c_{jk}}{\alpha_f(x_k)}$$

with

$$\alpha_f(x_k) := \sum_{\substack{d|x_k \\ d\nmid x_t, x_t<x_k}} (f*\mu)(d) \tag{2.1}$$

and

$$c_{ij} := \sum_{\substack{dx_i|x_j \\ dx_i\nmid x_t, x_t<x_j}} \mu(d). \tag{2.2}$$

Lemma 2.4 [16] *Let S be a gcd-closed set and let $\alpha_f(x_k)$ be given as in (2.1). Then*

$$\alpha_f(x_k) = \sum_{J\subseteq G_S(x_k)} (-1)^{|J|} f\big(\gcd(J\cup\{x_k\})\big).$$

Lemma 2.5 [6] *If S is gcd closed, then*

$$\det(f(S)) = \prod_{k=1}^{n} \alpha_f(x_k)$$

with $\alpha_f(x_k)$ being given as in (2.1).

Lemma 2.6 [35] *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and let $x_m \in S$ be such that $\{x_{m_1},\dots,x_{m_s}\} \subseteq G_S(x_m)$. For any real number e, define*

$$\beta_{\xi_e}(s,x_m) := x_m^e + \sum_{t=1}^{s}(-1)^t \sum_{1\le i_1<\cdots<i_t\le s} x_{m_{i_1\dots i_t}}^e.$$

Then

$$\beta_{\xi_e}(s,x_m) = x_{m_{1\dots s}}^e \prod_{j=1}^{s}\left(\left(\frac{x_m}{x_{m_j}}\right)^e - 1\right).$$

For any real number e, the reciprocal arithmetic function $\frac{1}{\xi_e}$ is defined for any positive integer x by $\frac{1}{\xi_e}(x) := \frac{1}{x^e}$. In fact, one has $\frac{1}{\xi_e} = \xi_{-e}$.

Lemma 2.7 *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ and $x_m \in S$ with $G_S(x_m) = \{x_{m_1},\dots,x_{m_s}\}$. Then*

$$\alpha_{\frac{1}{\xi_a}}(x_m) = x_{m_{1\dots s}}^{-a} \prod_{j=1}^{s}\left(\left(\frac{x_m}{x_{m_j}}\right)^{-a} - 1\right).$$

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

Proof Setting $f = \frac{1}{\xi_a}$ in Lemma 2.4, by Lemmas 2.4 and 2.6, we get that

$$\alpha_{\frac{1}{\xi_a}}(x_m) = \beta_{\xi_{-a}}(s, x_m) = x_{m_{1\ldots s}}^{-a} \prod_{j=1}^{s} \left(\left(\frac{x_m}{x_{m_j}} \right)^{-a} - 1 \right)$$

as required. So Lemma 2.7 is proved. ∎

For any real number e and for any gcd-closed set S satisfying the condition $\mathcal{G}$, Zhu and Yu [35] proved that the eth power LCM matrix $[S^e]$ is nonsingular when $e \neq 0$ which confirmed another conjecture of Hong raised in [18]. In the following, we present the explicit formulas for the inverses of the eth power GCD matrix (S^e) and the eth power LCM matrix $[S^e]$.

Lemma 2.8 *Let S be a gcd-closed set of n positive integers and let S satisfy the condition $\mathcal{G}$. Then for any nonzero real number e, the eth power GCD matrix (S^e) and the eth power LCM matrix $[S^e]$ are both nonsingular, and for any integers i and j with $1 \leq i, j \leq n$, we have*

$$((S^e)^{-1})_{ij} := \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik} c_{jk}}{\alpha_{\xi_e}(x_k)}$$

and

$$([S^e]^{-1})_{ij} := \frac{1}{x_i^e x_j^e} \sum_{\substack{x_i | x_k \\ x_j | x_k}} \frac{c_{ik} c_{jk}}{\alpha_{\frac{1}{\xi_e}}(x_k)},$$

where c_{ik} and c_{jk} are defined as in (2.2).

Proof Since

$$[x_i, x_j]^e = \frac{x_i^e x_j^e}{(x_i, x_j)^e},$$

we have

$$\begin{aligned} [S^e] &= \operatorname{diag}(x_1^e, \ldots, x_n^e) \cdot \left(\frac{1}{\xi_e}(x_i, x_j) \right) \cdot \operatorname{diag}(x_1^e, \ldots, x_n^e) \\ &=: \operatorname{diag}(x_1^e, \ldots, x_n^e) \cdot B \cdot \operatorname{diag}(x_1^e, \ldots, x_n^e), \end{aligned} \tag{2.3}$$

where $B := \left(\frac{1}{\xi_e}(x_i, x_j) \right) = (\xi_{-e}(S))$ is the $(-e)$th power GCD matrix defined on S. We take the determinants on both sides of (2.3) and obtain that

$$\det[S^e] = \det B \cdot \prod_{k=1}^{n} x_k^{2e}. \tag{2.4}$$

Since the two diagonal matrices in (2.3) are nonsingular, the nonsingularity of the eth power LCM matrix $[S^e]$ is equivalent to that of B. Thus, we first prove that B is nonsingular.

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

By Lemma 2.5, the determinant of B is equal to

$$\det B = \prod_{k=1}^{n} \alpha_{\frac{1}{\xi_e}}(x_k). \tag{2.5}$$

Hence, it suffices to show that each factor $\alpha_{\frac{1}{\xi_e}}(x_k)$ is nonzero. First, from Lemma 2.4, we have

$$\alpha_{\frac{1}{\xi_e}}(x_1) = x_1^{-e} \neq 0. \tag{2.6}$$

Next, for any $2 \leq m \leq n = |S|$, let $G_S(x_m) = \{x_{m_1}, \ldots, x_{m_s}\}$ with $s = |G_S(x_m)| \geq 1$. For each $x_{m_j} \in G_S(x_m)$ $(1 \leq j \leq s)$, since $-e \neq 0$ and $x_m \neq x_{m_j}$, we have

$$\left(\frac{x_m}{x_{m_j}}\right)^{-e} - 1 \neq 0.$$

Applying Lemma 2.7 then yields that

$$\alpha_{\frac{1}{\xi_e}}(x_m) = x_{m_{1\ldots s}}^{-e} \prod_{j=1}^{s} \left(\left(\frac{x_m}{x_{m_j}}\right)^{-e} - 1\right) \neq 0. \tag{2.7}$$

Combining (2.5)–(2.7), we conclude that $\det B \neq 0$, so B is nonsingular. Replacing $-e$ by e, this gives that the eth power GCD matrix (S^e) is nonsingular since $e \neq 0$. Moreover, together with (2.4), this implies that $[S^e]$ is nonsingular.

Furthermore, we apply Lemma 2.3 to the function $f = \xi_e$ and get the formula for $(S^e)^{-1}$. Letting $f = \frac{1}{\xi_e}$ in Lemma 2.3 gives the entrywise formula for B^{-1} as follows:

$$\left(B^{-1}\right)_{ij} = \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{c_{ik}c_{jk}}{\alpha_{\frac{1}{\xi_e}}(x_k)}. \tag{2.8}$$

Finally, combining (2.3) and (2.8) yields the desired result immediately.

This completes the proof of Lemma 2.8. ∎

Lemma 2.9 [32] *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ with $x_m \in S$. For any integer r with $1 \leq r \leq m$, if c_{rm} is defined as in (2.2) and $G_S(x_m) = \{x_{m_1}, \ldots, x_{m_s}\}$, then*

$$c_{rm} = \begin{cases} 1, & \text{if } r = m, \\ (-1)^k, & \text{if } r = m_{i_1 \ldots i_k}\ (1 \leq i_1 < \cdots < i_k \leq s,\ 1 \leq k \leq s), \\ 0, & \text{otherwise.} \end{cases}$$

3 A key lemma and proof of Theorem 1.2

In this section, with the aid of the lemmas presented in Section 2, we prove Theorem 1.2. For any integers l and m with $1 \leq l, m \leq |S|$, we define the two-variable function H as follows:

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

$$H(l,m) := \frac{1}{\alpha_{\frac{1}{\xi_a}}(x_m)} \sum_{x_r | x_m} \frac{c_{rm}[x_l, x_r]^b}{x_r^a}. \tag{3.1}$$

First of all, we prove a key lemma that plays a crucial role in the proof of Theorem 1.2.

Lemma 3.1 *Let S be a gcd-closed set satisfying the condition $\mathcal{G}$ with $x_l, x_m \in S$, and let $G_S(x_m) = \{x_{m_1}, \ldots, x_{m_s}\}$ with $s \geq 1$. Let a and b be positive integers. Then*

$$H(l,m) = \begin{cases} (-1)^s \Big(\dfrac{x_l}{(x_l,x_m)}\Big)^b x_m^a x_{m_{1\ldots s}}^{b-a} \displaystyle\prod_{j=1}^{s} \frac{\left(\frac{x_m}{x_{m_j}}\right)^{b-a} - 1}{\left(\frac{x_m}{x_{m_j}}\right)^{a} - 1}, & \text{if } (x_l,x_m) \mid x_{m_{1\ldots s}}, \\ x_l^b, \quad \text{if } x_m \mid x_l, \\ (-1)^k \Big(\dfrac{x_l}{(x_l,x_m)}\Big)^b x_m^a x_{m_{1\ldots k}}^{b-a} \displaystyle\prod_{j=1}^{k} \frac{\left(\frac{x_m}{x_{m_j}}\right)^{b-a} - 1}{\left(\frac{x_m}{x_{m_j}}\right)^{a} - 1}, & \text{if } (x_l,x_m) \mid x_{m_{1\ldots k}} \\ \text{for some } 1 \leq k \leq s-1 \text{ and } (x_l,x_m) \nmid x_{m_j} \text{ for all } k+1 \leq j \leq s. \end{cases}$$

Proof By (3.1) and Lemma 2.9, we have

$$\begin{aligned} \alpha_{\frac{1}{\xi_a}}(x_m) H(l,m) &= \sum_{x_r | x_m} \frac{c_{rm}[x_l, x_r]^b}{x_r^a} \\ &= \frac{[x_l, x_m]^b}{x_m^a} + \sum_{t=1}^{s} (-1)^t \sum_{1 \leq j_1 < \cdots < j_t \leq s} \frac{[x_l, x_{m_{j_1 \ldots j_t}}]^b}{x_{m_{j_1 \ldots j_t}}^a}. \end{aligned} \tag{3.2}$$

Since $(x_l,x_m) | x_m$, it follows that either $(x_l,x_m) | x_{m_{1\ldots s}}$, or $(x_l,x_m) = x_m$, or there exist an integer k with $1 \leq k \leq s-1$ and k integers $i_1, \ldots, i_k$ with $1 \leq i_1 < \cdots < i_k \leq s$ such that $(x_l,x_m) | x_{m_{i_1 \ldots i_k}}$ and $(x_l,x_m) \nmid x_{m_j}$ for all $j \in \langle s \rangle \backslash \{i_1, \ldots, i_k\}$. So we only need to consider the following three cases.

Case 1. $(x_l,x_m) | x_{m_{1\ldots s}}$. For $1 \leq j_1 < \cdots < j_t \leq s$ with $1 \leq t \leq s$, since $x_{m_{1\ldots s}} | x_{m_{j_1 \ldots j_t}} | x_m$, we have $(x_l,x_m) | x_{m_{j_1 \ldots j_t}} | x_m$ and

$$(x_l, x_{m_{j_1 \ldots j_t}}) = (x_l, (x_{m_{j_1 \ldots j_t}}, x_m)) = ((x_l,x_m), x_{m_{j_1 \ldots j_t}}) = (x_l,x_m).$$

Thus, by (3.2) and Lemma 2.6, we have

$$\begin{aligned} \alpha_{\frac{1}{\xi_a}}(x_m) H(l,m) &= \frac{x_l^b x_m^b}{x_m^a (x_l,x_m)^b} + \sum_{t=1}^{s} (-1)^t \sum_{1 \leq j_1 < \cdots < j_t \leq s} \frac{x_l^b x_{m_{j_1 \ldots j_t}}^b}{x_{m_{j_1 \ldots j_t}}^a (x_l, x_{m_{j_1 \ldots j_t}})^b} \\ &= \frac{x_l^b x_m^{b-a}}{(x_l,x_m)^b} + \sum_{t=1}^{s} (-1)^t \sum_{1 \leq j_1 < \cdots < j_t \leq s} \frac{x_l^b x_{m_{j_1 \ldots j_t}}^b}{x_{m_{j_1 \ldots j_t}}^a (x_l,x_m)^b} \\ &= \frac{x_l^b x_m^{b-a}}{(x_l,x_m)^b} + \frac{x_l^b}{(x_l,x_m)^b} \sum_{t=1}^{s} (-1)^t \sum_{1 \leq j_1 < \cdots < j_t \leq s} \frac{x_{m_{j_1 \ldots j_t}}^b}{x_{m_{j_1 \ldots j_t}}^a} \end{aligned}$$

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

$$= \frac{x_l^b}{(x_l, x_m)^b}\Big(x_m^{b-a} + \sum_{t=1}^{s}(-1)^t \sum_{1\le j_1<\cdots<j_t\le s} x_{m_{j_1\cdots j_t}}^{b-a}\Big)$$
$$= \frac{x_l^b}{(x_l, x_m)^b}\beta_{\xi_{b-a}}(s, x_m)$$
$$= \frac{x_l^b}{(x_l, x_m)^b} x_{m_{1\cdots s}}^{b-a} \prod_{j=1}^{s}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a} - 1\Big).$$

Then it follows from Lemmas 2.1 and 2.7 that

$$H(l,m) = \Big(\frac{x_l}{(x_l, x_m)}\Big)^b \frac{x_{m_{1\cdots s}}^{b-a} \prod_{j=1}^{s}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a} - 1\Big)}{x_{m_{1\cdots s}}^{-a} \prod_{j=1}^{s}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{-a} - 1\Big)}$$
$$= \Big(\frac{x_l}{(x_l, x_m)}\Big)^b x_{m_{1\cdots s}}^b (-1)^s \prod_{j=1}^{s}\Big(\frac{x_m}{x_{m_j}}\Big)^a \cdot \prod_{j=1}^{s}\frac{\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a} - 1}{\Big(\frac{x_m}{x_{m_j}}\Big)^a - 1}$$
$$= \Big(\frac{x_l}{(x_l, x_m)}\Big)^b x_{m_{1\cdots s}}^b (-1)^s \Big(\frac{x_m}{x_{m_{1\cdots s}}}\Big)^a \prod_{j=1}^{s}\frac{\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a} - 1}{\Big(\frac{x_m}{x_{m_j}}\Big)^a - 1}$$
$$= \Big(\frac{x_l}{(x_l, x_m)}\Big)^b x_{m_{1\cdots s}}^{b-a} x_m^a (-1)^s \prod_{j=1}^{s}\frac{\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a} - 1}{\Big(\frac{x_m}{x_{m_j}}\Big)^a - 1}$$
$$= (-1)^s \Big(\frac{x_l}{(x_l, x_m)}\Big)^b x_m^a x_{m_{1\cdots s}}^{b-a} \prod_{j=1}^{s}\frac{\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a} - 1}{\Big(\frac{x_m}{x_{m_j}}\Big)^a - 1}$$

as expected.

Case 2. $(x_l, x_m) = x_m$. Then $x_m | x_l$. Clearly, $x_{m_{j_1\cdots j_t}} | x_m | x_l$ for $1 \le j_1 < \cdots < j_t \le s$. This implies that

$$[x_l, x_m] = [x_l, x_{m_{j_1 j_2 \cdots j_t}}] = x_l.$$

Therefore, by (3.2) and Lemmas 2.6 and 2.7, we have

$$\alpha_{\frac{1}{\xi_a}}(x_m)H(l,m) = \frac{x_l^b}{x_m^a} + \sum_{t=1}^{s}(-1)^t \sum_{1\le j_1<\cdots<j_t\le s} \frac{[x_l, x_{m_{j_1\cdots j_t}}]^b}{x_{m_{j_1\cdots j_t}}^a}$$
$$= \frac{x_l^b}{x_m^a} + \sum_{t=1}^{s}(-1)^t \sum_{1\le j_1<\cdots<j_t\le s} \frac{x_l^b}{x_{m_{j_1\cdots j_t}}^a}$$
$$= x_l^b\Big(x_m^{-a} + \sum_{t=1}^{s}(-1)^t \sum_{1\le j_1<\cdots<j_t\le s} x_{m_{j_1\cdots j_t}}^{-a}\Big)$$
$$= x_l^b \beta_{\xi_{-a}}(s, x_m) = x_l^b \alpha_{\frac{1}{\xi_a}}(x_m),$$

and so $H(l,m) = x_l^b$ as desired.

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

Case 3. There is an integer k with $1 \le k \le s-1$ and k integers $i_1, \dots, i_k$ with $1 \le i_1 < \cdots < i_k \le s-1$ such that $(x_l, x_m) | x_{m_{i_1 \dots i_k}}$ and $(x_l, x_m) \nmid x_{m_j}$ for all $j \in \langle s \rangle \backslash \{i_1, \dots, i_k\}$. WLOG, one may let $(x_l, x_m) \mid x_{m_{1\dots k}}$ and $(x_l, x_m) \nmid x_{m_j}$ for all $k+1 \le j \le s$. By (3.2), we get that

$$\begin{aligned}
\alpha_{\frac{1}{\xi_a}}(x_m) H(l,m) &= \frac{[x_l, x_m]^b}{x_m^a} + \sum_{h=1}^{k} (-1)^h \sum_{1 \le i_1 < \cdots < i_h \le k} \frac{[x_l, x_{m_{i_1 \dots i_h}}]^b}{x_{m_{i_1 \dots i_h}}^a} \\
&\quad + \sum_{t=1}^{s-k} (-1)^t \sum_{k+1 \le j_1 < \cdots < j_t \le s} \frac{[x_l, x_{m_{j_1 \dots j_t}}]^b}{x_{m_{j_1 \dots j_t}}^a} \\
&\quad + \sum_{h=1}^{k} \sum_{t=1}^{s-k} (-1)^{h+t} \sum_{\substack{1 \le i_1 < \cdots < i_h \le k \\ k+1 \le j_1 < \cdots < j_t \le s}} \frac{[x_l, x_{m_{i_1 \dots i_h j_1 \dots j_t}}]^b}{x_{m_{i_1 \dots i_h j_1 \dots j_t}}^a} \\
&= E_1 + E_2,
\end{aligned}$$

where

$$E_1 := \frac{[x_l, x_m]^b}{x_m^a} + \sum_{h=1}^{k} (-1)^h \sum_{1 \le i_1 < \cdots < i_h \le k} \frac{[x_l, x_{m_{i_1 \dots i_h}}]^b}{x_{m_{i_1 \dots i_h}}^a}$$

and

$$\begin{aligned}
E_2 := &\sum_{t=1}^{s-k} (-1)^t \sum_{k+1 \le j_1 < \cdots < j_t \le s} \frac{[x_l, x_{m_{j_1 \dots j_t}}]^b}{x_{m_{j_1 \dots j_t}}^a} \\
&+ \sum_{h=1}^{k} \sum_{t=1}^{s-k} (-1)^{h+t} \sum_{\substack{1 \le i_1 < \cdots < i_h \le k \\ k+1 \le j_1 < \cdots < j_t \le s}} \frac{[x_l, x_{m_{i_1 \dots i_h j_1 \dots j_t}}]^b}{x_{m_{i_1 \dots i_h j_1 \dots j_t}}^a}.
\end{aligned}$$

First of all, we compute E_1. By Lemmas 2.2(i) and 2.6, we obtain that

$$\begin{aligned}
E_1 &= \frac{[x_l, x_m]^b}{x_m^a} + \sum_{h=1}^{k} (-1)^h \sum_{1 \le i_1 < \cdots < i_h \le k} \frac{[x_l, x_{m_{i_1 \dots i_h}}]^b}{x_{m_{i_1 \dots i_h}}^a} \\
&= \frac{x_l^b x_m^b}{x_m^a (x_l, x_m)^b} + \sum_{h=1}^{k} (-1)^h \sum_{1 \le i_1 < \cdots < i_h \le k} \frac{x_l^b x_{m_{i_1 \dots i_h}}^b}{x_{m_{i_1 \dots i_h}}^a (x_l, x_{m_{i_1 \dots i_h}})^b} \\
&= \frac{x_l^b x_m^{b-a}}{(x_l, x_m)^b} + \sum_{h=1}^{k} (-1)^h \sum_{1 \le i_1 < \cdots < i_h \le k} \frac{x_l^b x_{m_{i_1 \dots i_h}}^{b-a}}{(x_l, x_m)^b} \\
&= \left(\frac{x_l}{(x_l, x_m)} \right)^b \left(x_m^{b-a} + \sum_{h=1}^{k} (-1)^h \sum_{1 \le i_1 < \cdots < i_h \le k} x_{m_{i_1 \dots i_h}}^{b-a} \right) \\
&= \left(\frac{x_l}{(x_l, x_m)} \right)^b \beta_{\xi_{b-a}}(k, x_m) \\
&= \left(\frac{x_l}{(x_l, x_m)} \right)^b x_{m_{1\dots k}}^{b-a} \prod_{j=1}^{k} \left(\left(\frac{x_m}{x_{m_j}} \right)^{b-a} - 1 \right).
\end{aligned}$$

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

Subsequently, we calculate E_2. From Lemmas 2.1, 2.2(i)–(iii), and 2.6, we can deduce that

$$
\begin{aligned}
E_2 &= \sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s} \frac{[x_l, x_{m_{j_1\cdots j_t}}]^b}{x^a_{m_{j_1\cdots j_t}}} \\
&\quad + \sum_{h=1}^{k}\sum_{t=1}^{s-k}(-1)^{h+t} \sum_{\substack{1\le i_1<\cdots<i_h\le k\\ k+1\le j_1<\cdots<j_t\le s}} \frac{[x_l, x_{m_{i_1\cdots i_h j_1\cdots j_t}}]^b}{x^a_{m_{i_1\cdots i_h j_1\cdots j_t}}} \\
&= \sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s} \frac{[x_l, x_m]^b}{x^a_{m_{j_1\cdots j_t}}} \\
&\quad + \sum_{h=1}^{k}\sum_{t=1}^{s-k}(-1)^{h+t} \sum_{\substack{1\le i_1<\cdots<i_h\le k\\ k+1\le j_1<\cdots<j_t\le s}} \frac{[x_l, x_{m_{i_1\cdots i_h}}]^b}{x^a_{m_{i_1\cdots i_h j_1\cdots j_t}}} \\
&= \sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s} \frac{x_l^b x_m^b}{(x_l, x_m)^b x^a_{m_{j_1\cdots j_t}}} \\
&\quad + \sum_{h=1}^{k}\sum_{t=1}^{s-k}(-1)^{h+t} \sum_{\substack{1\le i_1<\cdots<i_h\le k\\ k+1\le j_1<\cdots<j_t\le s}} \frac{x_l^b x^b_{m_{i_1\cdots i_h}}}{(x_l, x_{m_{i_1\cdots i_h}})^b x^a_{m_{i_1\cdots i_h j_1\cdots j_t}}} \\
&= \sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s} \frac{x_l^b x_m^b}{(x_l, x_m)^b x^a_{m_{j_1\cdots j_t}}} \\
&\quad + \sum_{h=1}^{k}\sum_{t=1}^{s-k}(-1)^{h+t} \sum_{\substack{1\le i_1<\cdots<i_h\le k\\ k+1\le j_1<\cdots<j_t\le s}} \frac{x_l^b x^b_{m_{i_1\cdots i_h}}}{(x_l, x_m)^b x^a_{m_{i_1\cdots i_h j_1\cdots j_t}}} \\
&= \Big(\frac{x_l}{(x_l, x_m)}\Big)^b \Big(\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s} \frac{x_m^b}{\frac{x_m^a}{(P_{j_1}\cdots P_{j_t})^a}} \\
&\quad + \sum_{h=1}^{k}\sum_{t=1}^{s-k}(-1)^{h+t} \sum_{\substack{1\le i_1<\cdots<i_h\le k\\ k+1\le j_1<\cdots<j_t\le s}} \frac{\frac{x_m^b}{(P_{i_1}\cdots P_{i_h})^b}}{\frac{x_m^a}{(P_{i_1}\cdots P_{i_h}P_{j_1}\cdots P_{j_t})^a}} \Big) \\
&= \Big(\frac{x_l}{(x_l, x_m)}\Big)^b \Big(\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s} x_m^{b-a}(P_{j_1}\cdots P_{j_t})^a \\
&\quad + \sum_{h=1}^{k}\sum_{t=1}^{s-k}(-1)^{h+t} \sum_{\substack{1\le i_1<\cdots<i_h\le k\\ k+1\le j_1<\cdots<j_t\le s}} \frac{x_m^{b-a}}{(P_{i_1}\cdots P_{i_h})^{b-a}} \cdot (P_{j_1}\cdots P_{j_t})^a \Big) \\
&= \Big(\frac{x_l}{(x_l, x_m)}\Big)^b \Big(\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s} (P_{j_1}\cdots P_{j_t})^a \Big(x_m^{b-a} \\
&\quad + \sum_{h=1}^{k}(-1)^h \sum_{1\le i_1<\cdots<i_h\le k} \frac{x_m^{b-a}}{(P_{i_1}\cdots P_{i_h})^{b-a}} \Big) \Big)
\end{aligned}
$$

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b \Big(\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s}(P_{j_1}\dots P_{j_t})^a \Big(x_m^{b-a} + \sum_{h=1}^{k}(-1)^h \sum_{1\le i_1<\cdots<i_h\le k} x_{m_{i_1\dots i_h}}^{b-a}\Big)\Big)$$

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b \Big(\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s}(P_{j_1}\dots P_{j_t})^a \beta_{\xi_{b-a}}(k,x_m)\Big)$$

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b \Big(\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s}(P_{j_1}\dots P_{j_t})^a x_{m_{1\dots k}}^{b-a} \times \prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big)\Big)$$

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_{m_{1\dots k}}^{b-a}\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big)\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s}(P_{j_1}\dots P_{j_t})^a$$

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_{m_{1\dots k}}^{b-a}\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big)\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s}\frac{x_m^a}{\frac{x_m^a}{(P_{j_1}\dots P_{j_t})^a}}$$

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_{m_{1\dots k}}^{b-a}\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big)\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s}\frac{x_m^a}{x_{m_{j_1\dots j_t}}^a}$$

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_{m_{1\dots k}}^{b-a}\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big) \times \Big(x_m^a\Big(x_m^{-a}+\sum_{t=1}^{s-k}(-1)^t \sum_{k+1\le j_1<\cdots<j_t\le s}^{s-k} x_{m_{j_1\dots j_t}}^{-a}\Big)-1\Big)$$

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_{m_{1\dots k}}^{b-a}\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big)(x_m^a\beta_{\xi_{-a}}(s-k,x_m)-1)$$

$$= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_{m_{1\dots k}}^{b-a}\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big)\Big(x_m^a\Big(x_{m_{(k+1)\dots s}}^{-a}\prod_{j=k+1}^{s}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{-a}-1\Big)-1\Big).$$

Thus,

$$E_1+E_2 = \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_{m_{1\dots k}}^{b-a}x_m^a x_{m_{(k+1)\dots s}}^{-a}\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big)\prod_{j=k+1}^{s}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{-a}-1\Big),$$

and so by Lemmas 2.1 and 2.7, we have

$$H(l,m) = \frac{E_1+E_2}{\alpha_{\frac{1}{\xi_a}}(x_m)} = \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_m^a x_{m_{1\dots k}}^{b-a}x_{m_{(k+1)\dots s}}^{-a}x_{m_{1\dots s}}^{a}\frac{\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{b-a}-1\Big)}{\prod_{j=1}^{k}\Big(\Big(\frac{x_m}{x_{m_j}}\Big)^{-a}-1\Big)}$$

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

$$
\begin{aligned}
&= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_m^a x_{m_{1\dots k}}^{b-a} x_{m_{(k+1)\dots s}}^{-a} x_{m_{1\dots s}}^a \\
&\quad \times (-1)^k \Big(\prod_{j=1}^k \frac{x_m}{x_{m_j}}\Big)^a \frac{\prod_{j=1}^k \Big(\big(\frac{x_m}{x_{m_j}}\big)^{b-a}-1\Big)}{\prod_{j=1}^k \Big(\big(\frac{x_m}{x_{m_j}}\big)^{a}-1\Big)} \\
&= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_m^a x_{m_{1\dots k}}^{b-a} x_{m_{(k+1)\dots s}}^{-a} x_{m_{1\dots s}}^a \\
&\quad \times (-1)^k (P_1\dots P_k)^a \prod_{j=1}^k \frac{\big(\frac{x_m}{x_{m_j}}\big)^{b-a}-1}{\big(\frac{x_m}{x_{m_j}}\big)^{a}-1} \\
&= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_m^a x_{m_{1\dots k}}^{b-a} \frac{(P_{k+1}\dots P_s)^a}{x_m^a} x_{m_{1\dots s}}^a \\
&\quad \times (-1)^k (P_1\dots P_k)^a \prod_{j=1}^k \frac{\big(\frac{x_m}{x_{m_j}}\big)^{b-a}-1}{\big(\frac{x_m}{x_{m_j}}\big)^{a}-1} \\
&= \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_m^a x_{m_{1\dots k}}^{b-a} (-1)^k \prod_{j=1}^k \frac{\big(\frac{x_m}{x_{m_j}}\big)^{b-a}-1}{\big(\frac{x_m}{x_{m_j}}\big)^{a}-1} \\
&= (-1)^k \Big(\frac{x_l}{(x_l,x_m)}\Big)^b x_m^a x_{m_{1\dots k}}^{b-a} \prod_{j=1}^k \frac{\big(\frac{x_m}{x_{m_j}}\big)^{b-a}-1}{\big(\frac{x_m}{x_{m_j}}\big)^{a}-1}
\end{aligned}
$$

as asserted.

This finishes the proof of Lemma 3.1. ∎

Finally, we conclude this article by giving the proof of Theorem 1.2.

Proof of Theorem 1.2 Let S be a gcd-closed set which satisfies the condition $\mathcal{G}$ and let a and b be positive integers with $a \mid b$. For any integers i and j with $1 \le i, j \le |S|$, by Lemma 2.8, we have

$$
\begin{aligned}
\big([S^b][S^a]^{-1}\big)_{ij} &= \sum_{r=1}^{|S|} [x_i,x_r]^b \frac{1}{x_r^a x_j^a} \sum_{\substack{x_r \mid x_t \\ x_j \mid x_t}} \frac{c_{rt}c_{jt}}{\alpha_{\frac{1}{\xi_a}}(x_t)} \\
&= \sum_{x_j \mid x_t} c_{jt} \frac{1}{x_j^a \alpha_{\frac{1}{\xi_a}}(x_t)} \sum_{x_r \mid x_t} \frac{c_{rt}[x_i,x_r]^b}{x_r^a} \\
&= \sum_{x_j \mid x_t} c_{jt} \frac{H(i,t)}{x_j^a}
\end{aligned}
$$

with $H(i,t)$ being defined as in (3.1).

In what follows, we prove that

$$
\big([S^b][S^a]^{-1}\big)_{ij} \in \mathbb{Z}
$$

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

for all integers i and j with $1 \le i, j \le |S|$. By the definition of c_{jt} as in (2.2), we know that $c_{jt} \in \mathbb{Z}$. So we need only to show that for any positive integer t with $x_j \mid x_t$, we have

$$\frac{H(i,t)}{x_j^a} \in \mathbb{Z}.$$

This will be shown in what follows.

For any given integers i and t with $1 \le i, t \le |S|$ and $x_j | x_t$, we consider the following two cases.

Case 1. $t = 1$. Then $x_t = x_1 = \gcd(S)$. At this moment, we must have $x_j = x_1$. By Lemmas 2.4 and 2.9, we obtain that

$$\frac{H(i,t)}{x_j^a} = \frac{H(i,1)}{x_1^a} = \frac{1}{x_1^a \alpha_{\frac{1}{\xi_a}}(x_1)} \sum_{x_r | x_1} \frac{c_{r1}[x_i, x_r]^b}{x_r^a} = \frac{[x_i, x_1]^b}{x_1^a} = \frac{x_i^b}{x_1^a} = x_i^{b-a}\Big(\frac{x_i}{x_1}\Big)^a \in \mathbb{Z}$$

as one expects.

Case 2. $t \ge 2$. Since S is gcd closed, one has $x_1 \mid x_t$ and so $|G_S(x_t)| \ge 1$. Write $G_S(x_t) := \{x_{t_1}, \ldots, x_{t_s}\}$ with $s \ge 1$. Since $a \mid b$ and S satisfies the condition $\mathcal{G}$ as well as $x_j \mid x_t$, by Lemma 3.1, we have

$$\frac{H(i,t)}{x_j^a} = \begin{cases} (-1)^s \Big(\dfrac{x_i}{(x_i, x_t)}\Big)^b x_{t_{1\ldots s}}^{b-a} \Big(\dfrac{x_t}{x_j}\Big)^a \displaystyle\prod_{j=1}^{s} \frac{\left(\frac{x_t}{x_{t_j}}\right)^{b-a} - 1}{\left(\frac{x_t}{x_{t_j}}\right)^a - 1} \in \mathbb{Z}, & \text{if } (x_i, x_t) \mid x_{t_{1\ldots s}}, \\ x_i^{b-a}\Big(\dfrac{x_i}{x_j}\Big)^a \in \mathbb{Z}, & \text{if } x_t \mid x_i, \\ (-1)^k \Big(\dfrac{x_i}{(x_i, x_t)}\Big)^b \Big(\dfrac{x_t}{x_j}\Big)^a x_{t_{1\ldots k}}^{b-a} \displaystyle\prod_{j=1}^{k} \frac{\left(\frac{x_t}{x_{t_j}}\right)^{b-a} - 1}{\left(\frac{x_t}{x_{t_j}}\right)^a - 1} \in \mathbb{Z}, & \text{if } (x_i, x_t) \mid x_{t_{1\ldots k}} \\ \text{for some } 1 \le k \le s-1 \text{ and } (x_i, x_t) \nmid x_{t_j} \text{ for all } k+1 \le j \le s. \end{cases}$$

It then follows that

$$\frac{H(i,t)}{x_j^a} \in \mathbb{Z}$$

as required.

This concludes the proof of Theorem 1.2. ∎

Acknowledgements The author is grateful to the anonymous referee for insightful and invaluable suggestions and comments that improved greatly the presentation of the article.

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

References

[1] T. M. Apostol, *Arithmetic properties of generalized Ramanujan sums*. Pacific J. Math. **41**(1972), 281–293.
[2] S. Beslin and S. Ligh, *Another generalization of Smith's determinant*. Bull. Aust. Math. Soc. **40**(1989), 413–415.
[3] R. Bhatia and J. A. Dias da Silva, *Infinite divisibility of GCD matrices*. Amer. Math. Monthly **115**(2008), 551–553.
[4] K. Bourque and S. Ligh, *On GCD and LCM matrices*. Linear Algebra Appl. **174**(1992), 65–74.
[5] K. Bourque and S. Ligh, *Matrices associated with classes of arithmetical functions*. J. Number Theory **45**(1993), 367–376.
[6] K. Bourque and S. Ligh, *Matrices associated with arithmetical functions*. Linear Multilinear Algebra **34**(1993), 261–267.
[7] K. Bourque and S. Ligh, *Matrices associated with multiplicative functions*. Linear Algebra Appl. **216**(1995), 267–275.
[8] W. D. Feng, S. F. Hong, and J. R. Zhao, *Divisibility properties of power LCM matrices by power GCD matrices on gcd-closed sets*. Discrete Math. **309**(2009), 2627–2639.
[9] T. Hilberdink, *Determinants of multiplicative Toeplitz matrices*. Acta Arith. **125**(2006), 265–284.
[10] S. A. Hong, S. N. Hu, and Z. B. Lin, *On a certain arithmetical determinant*. Acta Math. Hungar. **150**(2016), 372–382.
[11] S. A. Hong and Z. B. Lin, *New results on the value of a certain arithmetical determinant*. Publ. Math. Debrecen **93**(2018), 171–187.
[12] S. F. Hong, *On the Bourque-Ligh conjecture of least common multiple matrices*. J. Algebra **218**(1999), 216–228.
[13] S. F. Hong, *Gcd-closed sets and determinants of matrices associated with arithmetical functions*. Acta Arith. **101**(2002), 321–332.
[14] S. F. Hong, *On the factorization of LCM matrices on gcd-closed sets*. Linear Algebra Appl. **345**(2002), 225–233.
[15] S. F. Hong, *Factorization of matrices associated with classes of arithmetical functions*. Colloq. Math. **98**(2003), 113–123.
[16] S. F. Hong, *Nonsingularity of matrices associated with classes of arithmetical functions*. J. Algebra **281**(2004), 1–14.
[17] S. F. Hong, *Divisibility properties of power GCD matrices and power LCM matrices*. Linear Algebra Appl. **428**(2008), 1001–1008.
[18] S. F. Hong, *Divisibility among power GCD matrices and power LCM matrices*. Bull. Aust. Math. Soc. **113**(2026), 231–243.
[19] S. F. Hong and K. S. Enoch Lee, *Asymptotic behavior of eigenvalues of reciprocal power LCM matrices*. Glasg. Math. J. **50**(2008), 163–174.
[20] S. F. Hong and R. Loewy, *Asymptotic behavior of eigenvalues of greatest common divisor matrices*. Glasg. Math. J. **46**(2004), 551–569.
[21] S. F. Hong and R. Loewy, *Asymptotic behavior of the smallest eigenvalue of matrices associated with completely even functions* $(\mathrm{mod}\ r)$. Int. J. Number Theory **7**(2011), 1681–1704.
[22] S. F. Hong, J. Zhao, and Y. Z. Yin, *Divisibility properties of Smith matrices*. Acta Arith. **132**(2008), 161–175.
[23] M. Li and Q. R. Tan, *Divisibility of matrices associated with multiplicative functions*. Discret. Math. **311**(2011), 2276–2282.
[24] Z. B. Lin and S. A. Hong, *More on a certain arithmetical determinant*. Bull. Aust. Math. Soc. **97**(2018), 15–25.
[25] P. J. McCarthy, *A generalization of Smith's determinant*. Canad. Math. Bull. **29**(1986), 109–113.
[26] H. J. S. Smith, *On the value of a certain arithmetical determinant*. Proc. Lond. Math. Soc. **7**(1875–1876), 208–212.
[27] Q. R. Tan and M. Li, *Divisibility among power GCD matrices and among power LCM matrices on finitely many coprime divisor chains*. Linear Algebra Appl. **438**(2013), 1454–1466.
[28] J. X. Wan and G. Y. Zhu, *Divisibility among power GCD and power LCM matrices on certain gcd-closed sets*. Linear Multilinear Algebra **74**(2026), 144–165.
[29] Y. Yamasaki, *Arithmetical properties of multiple Ramanujan sums*. Ramanujan J. **21**(2010), 241–261.
[30] G. Y. Zhu, *On the divisibility among power GCD and power LCM matrices on gcd-closed sets*. Int. J. Number Theory **18**(2022), 1397–1408.
[31] G. Y. Zhu, *On a certain determinant for a U.F.D*. Colloq. Math. **171**(2023), 49–59.

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.

[32] G. Y. Zhu, *Gcd-closed sets and divisibility among power GCD matrices and power LCM matrices*. Period. Math. Hungar. in press.
[33] G. Y. Zhu and M. Li, *On the divisibility among power LCM matrices on gcd-closed sets*. Bull. Aust. Math. Soc. **107**(2023), 31–39.
[34] G. Y. Zhu, Y. Y. Luo, and J. X. Wan, *Factorization of power GCD matrices and power LCM matrices on certain gcd-closed sets*. Int. J. Number Theory. https://doi.org/10.1142/S1793042126501204
[35] G. Y. Zhu and J. M. Yu, *Proof of a conjecture of Hong on nonsingularity of power LCM matrices on gcd-closed sets*. Proc. Amer. Math. Soc. **154**(2026), 1911–1925.

School of Mathematics and Statistics, Hubei Minzu University, Enshi City, China
e-mail: 2009043@hbmzu.edu.cn

Downloaded from https://www.cambridge.org/core. 17 Nov 2026 at 07:17:28, subject to the Cambridge Core terms of use.