CONSTRAINED OPEN MAPPING THEOREMS AND APPLICATIONS

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ABSTRACT

Some constrained open mapping theorems are obtained via Ekeland's variational principle. The constraint need only be a closed subset when the mapping is assumed to be Lipschitz, or a closed convex cone when the mapping is assumed to be closed. Generalizations of some previous results of Welsh and others are obtained. Apart from the presence of a constraint and a different method, the differentiability assumptions made are weaker. As applications, two results on the constrained controllability of nonlinear systems are given.

1. *Introduction*

It is well known that the open mapping theorem for nonlinear mappings plays an important role in both theory and applications. In particular, it is an important tool in the study of local controllability of nonlinear systems (see Chukwu and Lenhart [**3**], Klamka [**6**]) and differential inclusions (see Frankowska [**5**]). Most of the known results on open mappings need differentiability assumptions for the mappings, usually Gâteaux or Fréchet differentiability. It is useful to weaken these assumptions. Some authors use the set-valued derivative instead of the traditional one (see [**5**] and Penot [**8**] and references therein). Recently, a concept of weak Gâteaux inverse derivative was introduced by Welsh [**12**] and used to obtain an open mapping theorem by applying Brézis and Browder's ordering principle.

For the purpose of applications, constrained open mapping theorems are most important (for example, see [**2**, **3**, **6**]).

In this paper, we will relax Welsh's notion of weak inverse derivative to one that our results suggest is more useful. We use Ekeland's variational principle in place of Brézis and Browder's theorem, which simplifies the arguments, to generalize Welsh's result to the case with constraint under weaker conditions. The constraint is only a closed subset or a closed convex cone in some cases. Our results also generalize the corresponding ones in [**2**] and in Ray and Cramer [**9**] by weakening the differentiability requirement and attaching a perturbation. One of our theorems, which replaces differentiability by a suitable approximation in the assumptions, generalizes an open mapping theorem in Deimling [**4**] by weakening some conditions and including a constraint. These generalizations of the cited results of [**2**] and [**4**] cannot easily be deduced from Welsh's result even in the unconstrained case. As applications, our results are used to study the constrained controllability of some nonlinear systems. Two conclusions are obtained which allow exact locally or globally constrained controllability of nonlinear systems if some associated linear systems are constrained controllable. The first conclusion improves the theorem in [**6**] by removing the nonempty interior demand for the constraint. In the second one, which generalizes the corresponding result in [**7**], we do not impose differentiability assumptions.

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2. *Preliminaries*

We recall some notions and fix some notation.

Let X be a Banach space, $D \subset X$ and $\alpha \geqslant 0$. The closure and interior of D are denoted by $\bar{D}$ and $\operatorname{int}(D)$ respectively, the open unit ball in X is $B_X = \{x \in X : \|x\| < 1\}$, the unit sphere is $S_X = \{x \in X : \|x\| = 1\}$. If $x_0 \in X$, we let $B_X(x_0, \alpha) = x_0 + \alpha B_X$. If $x \in D$, the tangent cone $T_D(x)$ of D at x is defined by

$$T_D(x) = \{v \in X : \liminf_{h \to 0} h^{-1} d(x + hv, D) = 0\}.$$

It is known that $v \in T_D(x)$ if and only if there exist $v_n \in X$ and positive real numbers $h_n \to 0$ such that $v_n \to v$, $x + h_n v_n \in D$.

The following notions can be found in [**2**].

Let X, Y be two Banach spaces, $F : \operatorname{Dom}(F) \subset X \longrightarrow 2^Y \setminus \varnothing$ be a set-valued mapping. F is said to be Lipschitz with constant l in $U \subset \operatorname{Dom}(F)$ if

$$\text{for all } x_1, x_2 \in U, \quad F(x_1) \subset F(x_2) + l\|x_1 - x_2\| B_Y;$$

F is said to be pseudo-Lipschitz around $(x, y) \in \operatorname{Graph}(F)$ (the graph of F) if there exist $l > 0$ and neighbourhoods U of x and V of y such that

$$\text{for all } x_1, x_2 \in U, \quad F(x_1) \cap V \subset F(x_2) + l\|x_1 - x_2\| B_Y.$$

Our new concept is defined as follows.

DEFINITION 2.1. Let X, Y be normed spaces, $f : D \subset X \longrightarrow Y$, $x_0 \in X$, $\gamma \geqslant 0$. We say that f possesses a γ-Gâteaux inverse derivative $\Gamma(x_0) : Y \longrightarrow X$ at $x_0 \in D$ if for every $y \in Y$ and $\lambda > 0$ with $x_0 + \lambda \Gamma(x_0) y \in D$, there exists $w_y \in \bar{B}_Y$ such that

$$f(x_0 + \lambda \Gamma(x_0) y) = f(x_0) + \lambda y + \lambda \gamma \|y\| w_y + o(\lambda). \tag{2.1}$$

Briefly, $\Gamma(x_0)$ is said to be the γ-G inverse derivative of f at x_0 or f is γ-G inverse differentiable at x_0.

REMARK 2.2. If $\gamma \geqslant 1$, (2.1) is satisfied for every function by taking $\Gamma(x_0) = 0$, so we are usually interested in the case $\gamma < 1$.

REMARK 2.3. Whether w_y depends on λ is not important for the use in the next section. We will suppose that w_y is independent of λ except in Theorem 3.16.

If $\gamma = 0$, or $w_y \equiv 0$ in the definition, this coincides with the notion 'weak Gâteaux inverse derivative' defined in [**12**] and the notion 'directional contractor' defined in [**1**] if, in addition, $\Gamma(x_0)$ is linear and bounded. Obviously, if f has a weak Gâteaux inverse derivative, then it possesses γ-G inverse derivatives for all $\gamma \geqslant 0$, but the converse is not true. The following proposition shows that a Lipschitz perturbation of a γ-G inverse differentiable map is γ'-G inverse differentiable, which property is not shared by a weak Gâteaux inverse differentiable map.

PROPOSITION 2.4. *Let X, Y be normed spaces. Suppose that $f : D \subset X \longrightarrow Y$ has a γ-G inverse derivative $\Gamma(x)$ at $x \in X$ such that*

$$\|\Gamma(x) y\| \leqslant M\|y\| \quad \text{with } M > 0, \quad \text{for all } y \in B_Y. \tag{2.2}$$

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Suppose that $g: D \longrightarrow Y$ *is Lipschitz of constant* k *in* D *(with* $kM+\gamma<1$*). Then* $f+g$ *possesses a* $(\gamma+kM)$-G *inverse derivative* $\Gamma_1(x)$ *at* x *and* $\|\Gamma_1(x)y\| \leqslant M\|y\|$ *for all* $y \in Y$.

Proof. Let $h>1$ be given and let

$$\Gamma_1(x)y = h\|y\|\,\Gamma(x)\left(\frac{y}{h\|y\|}\right), \quad \text{for all } y \in Y.$$

Then, for each $y \in Y$ and each $\lambda>0$ with $x+\lambda\Gamma_1(x)y \in D$, we have

$$f(x+\lambda\Gamma_1(x)y) = f(x+\lambda h\|y\|\,\Gamma(x)(y/h\|y\|)) = f(x)+\lambda y+\lambda\gamma\|y\|\,w_y+o(\lambda).$$

The Lipschitz property of g and (2.2) imply that

$$\|g(x+\lambda\Gamma_1(x)y)-g(x)\| \leqslant \lambda kM\|y\|.$$

That is, there exists $v_y \in \overline{B_Y}$ such that

$$g(x+\lambda\Gamma_1(x)y)-g(x) = \lambda kM\|y\|\,v_y.$$

Let

$$u_y = \frac{\gamma}{\gamma+kM}w_y+\frac{kM}{\gamma+kM}v_y.$$

Then, $u_y \in \overline{B_Y}$ and

$$f(x+\lambda\Gamma_1(x)y)+g(x+\lambda\Gamma_1(x)y) = f(x)+g(x)+\lambda y+\lambda(\gamma+kM)\|y\|\,u_y+o(\lambda).$$

Obviously $\|\Gamma_1(x)y\| \leqslant M\|y\|$ for all $y \in Y$. □

It is easy to show that if f is Fréchet differentiable at x and $Df(x)$ is surjective, then f has a bounded 0-G inverse derivative. Hence, Proposition 2.4 shows that if L is continuous linear and onto and $f-L$ is Lipschitz, then f has a γ-G inverse derivative with $\gamma<1$ provided that the Lipschitz constant is sufficiently small. In Corollary 3.10 and Theorem 3.16, we will construct γ-G inverse derivatives with a given $\gamma<1$ for a given function, which illustrates the usefulness of this concept.

3. *Constrained open mapping theorems*

In this section, we use the γ-G inverse derivative to derive some constrained open mapping theorems. First, we consider the case when the mapping f is Lipschitz and the constraint is only a closed subset; secondly we consider the case when f is only closed and the constraint is a closed convex cone.

In the following, we always suppose that X, Y are Banach spaces. For the convenience of the reader, we state the version of Ekeland's variational principle that we shall use [**2**, Theorem 3.3.1].

LEMMA 3.1 (Ekeland's variational principle). *Suppose that* $V: Z \longmapsto \mathbb{R} \cup \{+\infty\}$ *is a lower semicontinuous nontrivial function defined on a complete metric space* (Z,d) *and bounded from below. Let* $\bar{x} \in \mathrm{Dom}(V)$ *and* $\varepsilon>0$ *be fixed. Then there exists* $\hat{x} \in Z$ *such that*

$$V(\hat{x}) < V(x)+\varepsilon d(x,\hat{x}), \quad \text{for each } x \neq \hat{x},$$
$$V(\hat{x})+\varepsilon d(\bar{x},\hat{x}) \leqslant V(\bar{x}).$$

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Our first result is as follows.

THEOREM 3.2. *Let $K \subset X$ be a closed subset, $x_0 \in K$, $\delta > 0$ and $0 \leqslant \gamma < 1$. Suppose that Ω is a neighbourhood of K, $f: \Omega \longrightarrow Y$ is continuous and, restricted to $\Omega \cap B_X(x_0, \delta)$, is Lipschitz with constant l. If f possesses a γ-G inverse derivative $\Gamma(x)$ for each $x \in K \cap B_X(x_0, \delta)$ and*

(i) $\|\Gamma(x)\, y\| \leqslant M \|y\|$ *with* $M \geqslant 0$, *for all* $y \in B_Y$, $x \in K \cap B_X(x_0, \delta)$;

(ii) $\Gamma(x)\, B_Y \subset T_K(x) + \alpha \Gamma(x)\, B_Y$ *with* $\alpha \in [0, 1)$, *for all* $x \in K \cap B_X(x_0, \delta)$;

(iii) $\alpha M l < 1 - \gamma$;

then $B_Y(f(x_0), \beta) \subset f(B_X(x_0, \delta) \cap K)$ with $\beta = ((1 - \gamma - \alpha M l)/M(1 + \alpha))\, \delta$ and the set-valued mapping

$$f_K^{-1}(y) = \{x \in K \cap B_X(x_0, \delta) : f(x) = y\}$$

is pseudo-Lipschitz around $(f(x_0), x_0)$.

Proof. Let $\rho \in (0, \beta)$ and $\varepsilon \in (\rho/\delta, \beta/\delta)$. Then

$$(1 + \alpha)\, \varepsilon M + \alpha M l + \gamma < 1 \quad \text{and} \quad \rho < \varepsilon \delta.$$

Choose $\eta > 0$, $\eta_1 > 0$ so small that $\rho < \varepsilon(\delta - \eta) - \eta_1$, and

$$x \in B_X(x_0, \eta) \quad \text{implies} \quad \|f(x) - f(x_0)\| < \eta_1.$$

For each pair $y, y_1 \in B_Y(f(x_0), \rho)$, let $\bar{x} \in f_K^{-1}(y_1) \cap B_X(x_0, \eta)$ (if the intersection is empty, we let $\bar{x} = x_0$), and consider the function

$$V(x) = \|y - f(x)\|$$

on the metric space $(K, \|\cdot\|)$.

By Ekeland's variational principle, there exists $\hat{x} \in K$ such that

$$\|y - f(\hat{x})\| \leqslant \|y - f(x)\| + \varepsilon \|x - \hat{x}\|, \quad \text{for all } x \in K, \tag{3.1}$$

$$\|\hat{x} - \bar{x}\| \leqslant \frac{1}{\varepsilon}(\|y - f(\bar{x})\| - \|y - f(\hat{x})\|) \leqslant \frac{\|y - f(\bar{x})\|}{\varepsilon}.$$

Since

$$\|y - f(\bar{x})\| \leqslant \|y - f(x_0)\| + \|f(x_0) - f(\bar{x})\| \leqslant \rho + \eta_1,$$

we have

$$\|\hat{x} - x_0\| \leqslant \|\hat{x} - \bar{x}\| + \|\bar{x} - x_0\| \leqslant \frac{\rho + \eta_1}{\varepsilon} + \eta < \delta, \tag{3.2}$$

that is, $\hat{x} \in K \cap B_X(x_0, \delta)$.

We will prove that $y = f(\hat{x})$.

If $y \neq f(\hat{x})$, let $z = \Gamma(\hat{x})((y - f(\hat{x}))/h\|y - f(\hat{x})\|)$ with $h > 1$ and $\varepsilon\alpha M + \varepsilon M + \alpha M l + \gamma < 1/h$. Then $(y - f(\hat{x}))/h\|y - f(\hat{x})\| \in B_Y$, and, by assumption (ii), there is $\hat{y} \in B_Y$ such that $z \in T_K(\hat{x}) + \alpha \Gamma(\hat{x})\, \hat{y}$. Therefore there exist $v_n \in Y$, $\lambda_n \to 0$, such that

$$v_n \longrightarrow z - \alpha \Gamma(\hat{x})\, \hat{y}, \quad \hat{x} + \lambda_n v_n \in K.$$

We may suppose that λ_n is so small that $\hat{x} + \lambda_n z \in \Omega$. Substituting $x = \hat{x} + \lambda_n v_n$ into (3.1), we obtain

$$\begin{aligned} \|y - f(\hat{x})\| &\leqslant \|y - f(\hat{x} + \lambda_n v_n)\| + \varepsilon \lambda_n \|v_n\| \\ &\leqslant \|y - f(\hat{x} + \lambda_n z)\| + \|f(\hat{x} + \lambda_n z) - f(\hat{x} + \lambda_n v_n)\| + \varepsilon \lambda_n \|v_n\| \\ &\leqslant \|y - f(\hat{x} + \lambda_n z)\| + l \lambda_n \|z - v_n\| + \varepsilon \lambda_n \|v_n\|. \end{aligned} \tag{3.3}$$

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By the definition of γ-G inverse derivative, there exists $w \in B_Y$ such that

$$f(\hat{x}+\lambda_n z) = f(\hat{x}) + \lambda_n \frac{y-f(\hat{x})}{h\|y-f(\hat{x})\|} + \lambda_n \gamma w/h + o(\lambda_n),$$

so that (if n is sufficiently large)

$$\begin{aligned}\|y-f(\hat{x})\| \leqslant \left(1-\frac{\lambda_n}{h\|y-f(\hat{x})\|}\right)\|y-f(\hat{x})\| + \lambda_n \gamma + \|o(\lambda_n)\| \\ + l\lambda_n\|z-v_n\| + \varepsilon\lambda_n\|v_n\|,\end{aligned}$$

and therefore

$$1 \leqslant h\left(\frac{\|o(\lambda_n)\|}{\lambda_n} + l\|z-v_n\| + \varepsilon\|v_n\| + \gamma\right).$$

Letting $n \to \infty$, we have

$$\begin{aligned}1 &\leqslant h(l\|\alpha\Gamma(\hat{x})\,\hat{y}\| + \gamma + \varepsilon\|z-\alpha\Gamma(\hat{x})\,\hat{y}\|) \\ &\leqslant h(\alpha l M\|\hat{y}\| + \gamma + \varepsilon\|z\| + \varepsilon\alpha M\|\hat{y}\|) \\ &\leqslant h[\alpha l M + \varepsilon(1+\alpha)M + \gamma] < 1.\end{aligned}$$

This is a contradiction. Therefore $y = f(\hat{x})$ and, by (3.2), $\hat{x} \in f_K^{-1}(y) \cap B_X(x_0, \delta)$. Thus

$$f(B_X(x_0,\delta) \cap K) \supset B_Y(f(x_0), \rho).$$

Since $\rho < \beta$ is arbitrary, it can be replaced by β in the above inclusion, so we obtain the first conclusion.

Next, we note that

$$d(\bar{x}, f_K^{-1}(y)) \leqslant d(\bar{x}, \hat{x}) \leqslant \frac{1}{\varepsilon}\|y-f(\bar{x})\|.$$

Since $\bar{x} \in f_K^{-1}(y_1) \cap B_X(x_0, \eta)$ is arbitrary and $y_1 = f(\bar{x})$, we have

$$f_K^{-1}(y_1) \cap B_X(x_0,\eta) \subset f_K^{-1}(y) + \frac{1}{\varepsilon}\|y-y_1\|\,\bar{B}_X,$$

which means that $f_K^{-1}(y)$ is pseudo-Lipschitz around $(f(x_0), x_0)$.

This completes the proof. □

REMARK 3.3. Although the condition $\alpha < 1$ is not used in the proof, our assumptions imply that α must be less than 1. In fact, from Definition 2.1 and our assumptions, we have

$$\|y + \|y\|\gamma w_y + o(\lambda)/\lambda\| = \|f(x+\lambda\Gamma(x)\,y) - f(x)\|/\lambda \leqslant lM\|y\|$$

for each $x \in B_X(x_0,\delta)$ and y and $\lambda > 0$ with $x + \lambda\Gamma(x)\,y \in B_X(x_0,\delta)$. Here, $w_y \in \bar{B}_Y$ is as in Definition 2.1. By letting $\lambda \to 0$, we obtain

$$1 - \gamma \leqslant \|(y/\|y\|) + \gamma w_y\| \leqslant lM.$$

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Hence assumption (iii) implies that $\alpha < (1-\gamma)/(lM) \leqslant 1$.

THEOREM 3.4. *Suppose that condition* (ii) *in Theorem* 3.2 *is replaced by*

(ii)′ $\Gamma(x) S_Y \subset T_K(x) + \alpha\Gamma(x) \bar{B}_Y$, $x \in K \cap B_X(x_0, \delta)$;

then the same conclusion holds.

Proof. The proof is similar to that of Theorem 3.2. We need only replace h by 1 and take $\hat{y} \in \bar{B}_Y$. □

REMARK 3.5. In Theorems 3.2 and 3.4, if $K = X$, then assumption (ii) is satisfied automatically with $\alpha = 0$ and there is no need to impose the Lipschitz assumption on f, because we can directly substitute $x = \hat{x} + \lambda_n z$, instead of $\hat{x} + \lambda_n v_n$, into (3.1) to obtain (3.3) without the second term. In fact, in this case, f only needs to be closed as we will show in Theorem 3.8. Thus Theorems 3.2 and 3.4 are generalizations of the main result of [**12**] which is the case $K = X$ and $\gamma = 0$. We also remark that, in [**12**], assumption (i) is supposed to be satisfied for all $y \in Y$.

If K is convex, we have a better conclusion.

COROLLARY 3.6. *Suppose that conditions* (i), (ii) *and* (iii) *of Theorem* 3.2 *and the following condition hold:*

(iv) *K is convex.*

Then $B_Y(f(x_0), ((1-\gamma)/M)\,\delta) \subset f(B_X(x_0, \delta) \cap K)$ *and also* $f_K^{-1}(y)$ *is pseudo-Lipschitz around* $(f(x_0), x_0)$.

Proof. Since K is convex, $T_K(x)$ is a closed convex cone for each $x \in K$, so for each pair of constants $c_1 > 0, c_2 > 0$, we have

$$c_1 T_K(x) + c_2 T_K(x) \subset T_K(x).$$

By condition (ii) for each $x \in B_X(x_0, \delta)$

$$\begin{aligned}\Gamma(x) B_Y &\subset T_K(x) + \alpha\Gamma(x) B_Y \subset T_K(x) + \alpha(T_K(x) + \alpha\Gamma(x) B_Y) \\ &\subset T_K(x) + \alpha^2\Gamma(x) B_Y \subset \ldots \subset T_K(x) + \alpha^n\Gamma(x) B_Y.\end{aligned}$$

By condition (i), $\Gamma(x) B_Y$ is bounded, so

$$\Gamma(x) B_Y \subset \overline{T_K(x)} = T_K(x).$$

Therefore, condition (ii) holds also with $\alpha = 0$ and so Theorem 3.1 applies with $\alpha = 0$ to complete the proof. □

REMARK 3.7. This corollary means that if the constraint is convex, then the Lipschitz constant can be arbitrarily large.

Now, we give a result where we only suppose that f is closed, that is, f has closed graph, but another condition on the inverse derivative is imposed and the constraint should be a closed convex cone.

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THEOREM 3.8. *Let $K \subset X$ be a closed convex cone, $x_0 \in K, \delta > 0, \gamma \in [0,1)$. Suppose that $f: K \longrightarrow Y$ is closed and possesses a γ-G inverse derivative $\Gamma(x)$ with $\gamma < 1$ for every $x \in K$. If there exists $M > 0$ such that*

$$\|\Gamma(x)y\| \leqslant M\|y\|, \quad \text{for all } y \in B_Y, x \in K \cap B_X(x_0, \delta), \tag{3.4}$$

$$\Gamma(x)B_Y \subset K, \quad \text{for all } x \in K \cap B_X(x_0, \delta), \tag{3.5}$$

then

$$f(B_X(x_0,\delta) \cap K) \supset B_Y\left(f(x_0), \frac{1-\gamma}{M}\delta\right)$$

and $f_K^{-1}(y)$ is pseudo-Lipschitz around $(f(x_0), x_0)$. Therefore, if $U \subset X$ is an open subset and there exist $M = M(x_0) > 0, \delta = \delta(x_0)$ for each $x_0 \in U$ which satisfy (3.4) *and* (3.5)*, then $f(U \cap K)$ is an open set in Y.*

Proof. We define a new metric d on K by

$$d(x_1, x_2) = \max\{\|x_1 - x_2\|, \theta\|f(x_1) - f(x_2)\|\} \quad \text{for } x_1, x_2 \in K,$$

where θ is positive and such that $\theta(1+\gamma) < M$. Since f is closed, the space K endowed with the metric d is complete and is denoted by K_d in the following.

Let $\beta = ((1-\gamma)/M)\delta, \rho \in (0, \beta), \varepsilon \in (\rho/\delta, \beta/\delta)$ and $y \in B_Y(f(x_0), \rho)$. Then $1 > \gamma + \varepsilon M$. Obviously, $f: K_d \longrightarrow Y$ and $\|\cdot\|: K_d \longrightarrow [0, \infty)$ are continuous. Applying Ekeland's variational principle to the function $V(x) := \|y - f(x)\|$ in the space K_d, we obtain $\hat{x} \in K_d$ such that

$$\|y - f(\hat{x})\| \leqslant \|y - f(x)\| + \varepsilon d(x, \hat{x}), \quad \text{for all } x \in K_d, \tag{3.6}$$

$$d(\hat{x}, x_0) \leqslant \frac{1}{\varepsilon}(\|y - f(x_0)\| - \|y - f(\hat{x})\|) \leqslant \frac{\rho}{\varepsilon} < \delta. \tag{3.7}$$

If $y \neq f(\hat{x})$, let $h > 1$, $z = \Gamma(\hat{x})((y - f(\hat{x}))/h\|y - f(\hat{x})\|)$. Then, by (3.5), $z \in K$. Since K is a convex cone, $\hat{x} + \lambda z \in K$ for each real number λ. Replacing x in (3.6) by $\hat{x} + \lambda z$, we obtain

$$\begin{aligned}\|y - f(\hat{x})\| &\leqslant \|y - f(\hat{x} + \lambda z)\| + \varepsilon d(\hat{x} + \lambda z, \hat{x}) \\ &\leqslant \|y - f(\hat{x} + \lambda z)\| + \varepsilon \max\{\lambda\|z\|, \theta\|f(\hat{x} + \lambda z) - f(\hat{x})\|\}.\end{aligned}$$

By the definition of γ-G inverse derivative, there exists $w \in B_Y$ such that

$$f(\hat{x} + \lambda z) = f(\hat{x}) + \lambda \frac{y - f(\hat{x})}{h\|y - f(\hat{x})\|} + o(\lambda) + \lambda\gamma w.$$

Suppose that $\lambda \in (0, h\|y - f(\hat{x})\|)$. Hence, we have

$$\begin{aligned}\|y - f(\hat{x})\| \leqslant \|y - f(\hat{x})\| - \frac{\lambda}{h} + \|o(\lambda)\| + \lambda\gamma \\ + \varepsilon\lambda \max\left\{\frac{M}{h}, \frac{\theta}{h} + \frac{\theta\|o(\lambda)\|}{\lambda} + \theta\gamma\right\},\end{aligned}$$

that is

$$\frac{1}{h} \leqslant \frac{\|o(\lambda)\|}{\lambda} + \gamma + \varepsilon \max\left\{\frac{M}{h}, \frac{\theta}{h} + \frac{\theta\|o(\lambda)\|}{\lambda} + \theta\gamma\right\}.$$

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By letting $h \to 1$, $\lambda \to 0$, we have

$$1 \leqslant \gamma + \varepsilon \max\{M, \theta(1+\gamma)\} = \gamma + \varepsilon M < 1.$$

This is a contradiction. Therefore, $y = f(\hat{x})$, and by (3.7), $\hat{x} \in f_K^{-1}(y) \cap B_X(x_0, \delta)$, that is,

$$f(B_X(x_0, \delta) \cap K) \supset B_Y(f(x_0), \rho).$$

Letting $\rho \to \beta$, we obtain

$$f(B_X(x_0, \delta) \cap K) \supset B_Y(f(x_0), \beta).$$

To prove that f_K^{-1} is pseudo-Lipschitz, we choose $\rho \in (0, \beta/2)$, $\varepsilon \in (2\rho/\delta, \beta/\delta)$ and $\eta > 0$ small enough such that $2\rho < \varepsilon(\delta - \eta)$. Let $y, y_1 \in B_Y(f(x_0), \rho)$, and let $\bar{x} \in f_K^{-1}(y_1) \cap B_X(x_0, \eta)$ (if this intersection is empty, we let $\bar{x} = x_0$). By Ekeland's variational principle, there exists $\hat{x} \in K_d$ satisfying (3.6) and

$$d(\hat{x}, \bar{x}) \leqslant \frac{1}{\varepsilon}(\|y - f(\bar{x})\|) \leqslant \frac{\|y - y_1\|}{\varepsilon} = \frac{2\rho}{\varepsilon}.$$

Thus

$$\|\hat{x} - x_0\| \leqslant d(\hat{x}, x_0) \leqslant d(\bar{x}, x_0) + d(\hat{x}, \bar{x}) \leqslant \frac{2\rho}{\varepsilon} + \eta < \delta$$

which implies that $y = f(\hat{x})$ as shown above. That is, $\hat{x} \in B_X(x_0, \delta) \cap f_K^{-1}(y)$. Therefore, using the same method as in Theorem 3.1, we see that $f_K^{-1}(y)$ is pseudo-Lipschitz around $(f(x_0), x_0)$. This completes the proof. □

REMARK 3.9. If $K = X$, $\gamma = 0$ and $x_0 = 0, f(x_0) = 0$ in Theorem 3.8, we obtain the result of [**12**]. Moreover, in [**12**], the pseudo-Lipschitz property of f^{-1} is not asserted. Thus Theorem 3.8 generalizes the results in [**12**] and, therefore, the results in [**9**], and our proof seems to be simpler.

To give some other results, we denote by

$$X_K = \begin{cases} K & \text{if } K \text{ is a closed convex cone,} \\ \Omega & \text{if } K \text{ is a closed subset;} \end{cases}$$

$$E_K(x) = \begin{cases} K & \text{if } K \text{ is a closed convex cone,} \\ T_K(x) & \text{if } K \text{ is a closed subset.} \end{cases}$$

Here Ω is a neighbourhood of K.

COROLLARY 3.10. *Let $K \subset X$ be a closed subset or a closed convex cone, $x_0 \in X_K$, $\delta > 0$, let $g: X_K \longrightarrow Y$ be continuous and, restricted to $X_K \cap B_X(x_0, \delta)$, Lipschitz of constant k. Suppose that $f: X_K \longrightarrow Y$ is Gâteaux differentiable in $X_K \cap B_X(x_0, \delta)$ with the Gâteaux derivative $Df(x)$ and there exist $c > 0, \gamma \geqslant 0$ with $\gamma + kc < 1$ such that*

$$B_Y \subset Df(x)(E_K(x) \cap cB_X) + \gamma B_Y, \quad \text{for all } x \in B_X(x_0, \delta) \cap K. \tag{3.8}$$

If f is closed when K is a closed convex cone, or f is continuous and, restricted to $B_X(x_0, \delta)$, Lipschitz when K is only a closed subset, then

$$B_Y\left(f(x_0) + g(x_0), \frac{1 - \gamma - kc}{c}\delta\right) \subset (f+g)(B_X(x_0, \delta) \cap K)$$

and $(f+g)_K^{-1}(y)$ is pseudo-Lipschitz around $(f(x_0) + g(x_0), x_0)$.

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Proof. Let $x \in B_X(x_0, \delta) \cap K$, $0 \neq y \in Y$ be arbitrary. For $h \in (1, 1/(\gamma + kc))$, by (3.8), there exist $v_y \in B_X \cap E_K(x), w_y \in B_Y$ such that

$$\frac{y}{h\|y\|} = Df(x)(cv_y) + \gamma w_y,$$

or

$$y = Df(x)(ch\|y\| v_y) + \gamma h\|y\| w_y. \tag{3.9}$$

Define a mapping $\Gamma(x): Y \longrightarrow X$ as follows

$$\Gamma(x)0 = 0, \quad \Gamma(x)y = ch\|y\| v_y, \quad \text{for all } y \in Y$$

where for each $y \in Y$, we fix a point v_y in $B_X \cap E_K(x)$ satisfying (3.9) so that $\Gamma(x)$ is well defined. Obviously,

$$\|\Gamma(x)y\| \leqslant ch\|y\|, \quad \text{for all } y \in Y,$$

and, by the definition of $Df(x)$,

$$\begin{aligned} f(x + \lambda\Gamma(x)y) &= f(x + \lambda ch\|y\| v_y) \\ &= f(x) + \lambda Df(x)(ch\|y\| v_y) + o(\lambda) \\ &= f(x) + \lambda y + \lambda h\gamma\|y\|(-w_y) + o(\lambda). \end{aligned}$$

Hence $\Gamma(x)$ is a bounded $h\gamma$-G inverse derivative of f at x. By Proposition 2.4,

$$\Gamma_1(x)y := h\|y\| \Gamma(x)\left(\frac{y}{h\|y\|}\right)$$

is a $h(\gamma + kc)$-G inverse derivative of $f + g$ at x and $\|\Gamma_1(x)y\| \leqslant hc\|y\|$ for all $y \in Y$.

On the other hand, since $E_K(x)$ is a cone, $v_y \in E_K(x)$ for every $y \in Y$, so $\Gamma(x)y = ch\|y\| v_y \in E_K(x)$ and, therefore, $\Gamma_1(x)y = ch\|y\| v_y \in E_K(x)$. This means that

$$\Gamma_1(x)B_Y \subset E_K(x).$$

By Theorem 3.8 or Theorem 3.2 with $\alpha = 0$ according to whether K is a cone or not, we have

$$B_Y\left(f(x_0) + g(x_0), \frac{1 - h\gamma - hkc}{hc}\delta\right) \subset (f + g)(B_X(x_0, \delta) \cap K)$$

and $(f + g)_K^{-1}(y)$ is pseudo-Lipschitz around $(f(x_0) + g(x_0), x_0)$. Since h in the above inclusion is arbitrary in $(1, 1/(\gamma + kc))$, it can be replaced by 1 which completes the proof. □

REMARK 3.11. This corollary cannot be derived from the result in [**12**] even in the case $K = X$, $g(x) \equiv 0$.

REMARK 3.12. Corollary 3.10 is a generalization of [**2**, Theorem 3.4.3] where f is supposed to be Fréchet differentiable, $g(x) \equiv 0$ and it is only stated that $f(x_0)$ is an interior point of $f(K)$. Corollary 3.10 also generalizes the corresponding results in [**9**] and [**12**] where the authors only consider the case $K = X$, $\gamma = 0$ and $g(x) \equiv 0$.

By a similar method to that used in Corollary 3.6, we can prove easily that if $Df(x)$ is linear and K is convex, then (3.8) is equivalent to

$$B_Y \subset \overline{Df(x)\left(E_K(x) \cap \frac{c}{1 - \gamma}B_X\right)}.$$

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In this case, the conclusions of Corollary 3.10 remain the same.

The following corollary is a direct consequence of Corollary 3.10.

COROLLARY 3.13. *Let X, Y, K, g, f be as above, $U \subset X$ be an open subset. If for each $x_0 \in U$, there exist $\delta = \delta(x_0)$, $c = c(x_0)$, $\gamma = \gamma(x_0)$ satisfying the assumptions of Corollary* 3.10, *then $(f+g)(U \cap K)$ is open in Y.*

COROLLARY 3.14. *Let $c: [0, \infty) \longrightarrow [0, \infty)$ be a function such that*

$$\limsup_{t \to \infty} \frac{t}{\sup_{s \in [0,t]} c(s)} = \infty. \tag{3.10}$$

Suppose that $K \subset X$ is closed; $\gamma \in [0, 1)$, $f: X_K \longrightarrow Y$ is Gâteaux differentiable on K with the derivative $Df(x)$ such that

$$B_Y \subset Df(x)(E_K(x) \cap B_X(0, c(\|x\|))) + \gamma B_Y. \tag{3.11}$$

If f is locally Lipschitz or f is only closed if K is a cone, then $f(K) = Y$.

Proof. We may suppose that $f(0) = 0$ and $c(\cdot)$ is nondecreasing; otherwise, we use $f(x) - f(0)$ instead of f and $\sup_{s \in [0,t]} c(s)$ instead of c. Then from (3.10), it follows that there exist $t_n \to \infty$ such that

$$t_n / c(t_n) \to \infty.$$

Thus for each $y \in Y$, there exists n with $(1-\gamma)\, t_n / c(t_n) > \|y\|$. Also (3.11) implies that

$$B_Y \subset Df(x)(E_K(x) \cap B_X(0, c(t_n))), \quad \text{for each } x \in K \cap t_n B_X.$$

Therefore, from Corollary 3.10, we obtain $y \in f(K \cap t_n B_X)$. This completes the proof. □

REMARK 3.15. Corollary 3.14 is motivated by [**10**, Theorem 3.1] where the authors considered the case when $K = X$, $\gamma = 0$ and $c(\cdot)$ was supposed to be continuous and $\int_0^\infty [1/c(s)]\, ds = \infty$.

Finally, we give a result which is independent of any differentiability in its assumptions and generalizes the open mapping theorem in [**4**, p. 154].

THEOREM 3.16. *Suppose that $K \subset X$ is a closed subset or a closed convex cone, $x_0 \in K$, $\delta > 0$. Let $L: X_K \longrightarrow Y$ be a positively homogeneous operator (that is, $L(kx) = kL(x)$ for $k \geqslant 0$) and suppose there is $c > 0$ such that*

$$B_Y \subset L(cB_X \cap E_K(x)), \quad \text{for each } x \in K \cap B(x_0, \delta). \tag{3.12}$$

Let $f: X_K \longrightarrow Y$ be continuous and Lipschitz restricted to $B_X(x_0, \delta)$ when K is a closed subset or be only a closed function when K is a closed convex cone. Suppose there exists $\alpha \in (0, 1/c)$ such that

$$\|f(x_1) - f(x_2) - L(x_1 - x_2)\| \leqslant \alpha \|x_1 - x_2\| \quad \text{for all } x_1, x_2 \in K \cap B_X(x_0, \delta). \tag{3.13}$$

Then

$$f(B_X(x_0, \delta) \cap K) \supset B_Y(f(x_0), \delta(1 - \alpha c)/c) \tag{3.14}$$

and $f_K^{-1}(y)$ is pseudo-Lipschitz around $(f(x_0), x_0)$.

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Proof. Let $h > 1$ be such that $h(\alpha + k)c < 1$ and let $x \in K \cap B_X(x_0, \delta)$. By (3.12), for each $y \in Y$, there exists $u_y \in E_K(x) \cap B_X$ such that

$$y = hc\|y\| Lu_y. \tag{3.15}$$

Define the mapping $\Gamma(x): Y \longrightarrow X$ by

$$\Gamma(x)y = hc\|y\| u_y \quad \text{for all } y \in Y,$$

where $u_y \in E_K(x) \cap B_X$ satisfies (3.15) and, in the case when there are several such points, we fix one arbitrarily so that $\Gamma(x)$ is well defined. Then

$$\Gamma(x)B_Y \subset E_K(x) \quad \text{and} \quad \|\Gamma(x)y\| \leqslant hc\|y\| \quad \text{for all } y \in Y$$

since $u_y \in E_K(x) \cap B_X$ and $E_K(x)$ is a cone. Moreover, by the homogeneity of L, we have $L\Gamma(x)y = y$ for each $y \in Y$. Thus, by (3.13), when $\lambda > 0$ is small enough, there exists $v_y \in \bar{B}_Y$ such that

$$\begin{aligned} f(x + \lambda\Gamma(x)y) &= f(x) + L(\lambda\Gamma(x)y) + \alpha\|\lambda\Gamma(x)y\| v_y \\ &= f(x) + \lambda y + \lambda\alpha hc\|y\| \|u_y\| v_y. \end{aligned}$$

Obviously, $w_y := \|u_y\| v_y \in B_Y$. Hence, $\Gamma(x)$ is a αhc-G inverse derivative of f at x. Applying Theorem 3.2 or Theorem 3.8 (note that Theorem 3.8 remains true even when the w_y in Definition 2.1 depends on λ), we have

$$f(B_X(x_0, \delta) \cap K) \supset B_Y(f(x_0), \delta(1 - h\alpha c)/(hc)) \tag{3.16}$$

and $f_K^{-1}(y)$ is pseudo-Lipschitz around $(f(x_0), x_0)$. Letting $h \to 1$ in (3.16), we obtain (3.14). This completes the proof. □

REMARK 3.17. Theorem 3.16 cannot easily be deduced from the result in [**12**] even when $K = X$, because it seems that the weak Gâteaux inverse derivative cannot be constructed.

If $K = X$ and L is continuous linear and onto (therefore (3.12) is satisfied for some $c > 0$), we obtain [**4**, Theorem 15.5]. We note that in [**4**], the domain of f is only $B_X(x_0, \delta)$, and f^{-1} is not asserted to be pseudo-Lipschitz. The same situation occurs in the following corollary.

COROLLARY 3.18. *Suppose that $f: X \longrightarrow Y$ is weakly Gâteaux differentiable (for the definition, see* [**4**, p. 154]*, $K \subset X$ is a closed convex cone and $x_0 \in K$ is such that $Df(x_0)(K) = Y$, $\|Df(x) - Df(x_0)\| \leqslant \alpha$ in K for some $\alpha > 0$. Then $B_Y(f(x_0), \beta) \subset f(B_X(x_0, \delta) \cap K)$ for some $\beta, \delta > 0$ provided that α is sufficiently small.*

Proof. Using the same method as in [**4**, Corollary 15.2], we can prove that $L := Df(x_0)$ satisfies (3.13). By [**11**, Proposition 2.4], (3.12) is satisfied for some $c > 0$. Thus the conclusion follows from Theorem 3.16. □

Using a similar method to that used in Corollary 3.14 and by Theorem 3.16, we have the following corollary.

COROLLARY 3.19. *Under the assumptions of Theorem* 3.16*, suppose that* (3.12) *and* (3.13) *hold for all $\delta > 0$, and the positive constant c is independent of δ (therefore* (3.10) *holds). Then $f(K) = Y$.*

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4. *Applications*

In this section, we use Corollary 3.10 and Theorem 3.16 to consider the local and global exact cone constrained controllability of the nonlinear system

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad t \in [t_0, T] \\ x(t_0) = x_0 \end{cases} \tag{4.1}$$

via a suitable associated linear system. Such problems have been studied by Chukwu and Lenhart [**3**] with the closed unit ball as constraint, and Klamka [**6**] with the constraint being a closed convex cone with nonempty interior. Here, we will suppose the constraint is a closed convex cone, and also give the controllable area which was not asserted in [**3**, **6**]. Unconstrained problems can be found in [**7**] where the mathematical tool used was fixed point theorems.

We suppose that both the state space X and the control space U are Banach spaces, $f:[t_0, T] \times X \times U$ is measurable with respect to the first variable, continuous with respect to the last two arguments.

We also suppose that for each $u \in L^\infty(t_0, T; U)$, equation (4.1) admits exactly one solution which is denoted by $x(t, x_0, u)$, and depends continuously on u.

If $K \subset U$ is the constraint, let

$$U_{\text{ad}} = L^\infty(t_0, T; U), \quad K_{\text{ad}} = \{u \in U_{\text{ad}} : u(t) \in K \text{ a.e.}\},$$

(where a.e. stands for 'almost everywhere') and

$$\mathbf{R}_T(K) = \{x(T, x_0, u) : u \in K_{\text{ad}}\}$$

which is called the constrained reachable set of (4.1) in time T from the initial state x_0 with the constraint K.

DEFINITION 4.1. System (4.1) is said to be K-constrained locally exact controllable at $x \in X$ if $x \in \text{int}(\mathbf{R}_T(K))$. If $E \subset \text{int}(\mathbf{R}_T(K))$, we say (4.1) is K-constrained locally exact controllable on E. If $X = \mathbf{R}_T(K)$, we say (4.1) is K-constrained exact controllable.

Let $D_2 f$, $D_3 f$ be the Gâteaux derivatives of f with respect to the second and third variables respectively when these exist, and associate (4.1) with the linear system

$$\begin{cases} z'(t) = D_2 f(t, x(t, x_0, u), u(t))\, z(t) + D_3 f(t, x(t, x_0, u), u(t))\, v(t), \quad t \in [t_0, T] \\ z(t_0) = 0. \end{cases} \tag{4.2}$$

The unique solution of (4.2) (supposed to exist) for given $u, v \in U_{\text{ad}}$ will be denoted by $z(t, x_0, u, v)$.

The following lemma is easy to obtain from the definition of Gâteaux derivative, so we omit its proof.

LEMMA 4.2. *Suppose that X, Y, Z are normed spaces; $f: X \times Y \longrightarrow Z$ possess linear Gâteaux derivative $Df(x, y)$ at (x, y) in each direction. Then f possesses linear partial Gâteaux derivatives $D_1 f(x, y), D_2 f(x, y)$ at (x, y) in each direction, and*

$$Df(x, y)(u, v) = D_1 f(x, y)\, u + D_2 f(x, y)\, v, \quad \text{for all } u \in X, v \in Y.$$

Next, we give a generalization of [**3**, Lemma 3.1] where f was taken to be Fréchet differentiable.

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LEMMA 4.3. *Suppose that $f(t,x,u)$ possesses a linear Gâteaux derivative in each direction with respect to (x,u), and is Fréchet differentiable with respect to x, and $x(\cdot,x_0,u)$ has a linear Gâteaux derivative with respect to u. Then*

$$D_u x(t,x_0,u)v = z(t,x_0,u,v).$$

Proof. By the definition of $x(t,x_0,u)$

$$x(t,x_0,u) = x_0 + \int_0^t f(s,x(s,x_0,u),u(s))\,ds,$$

and so, by Lemma 4.2,

$$\begin{aligned} D_u x(t,x_0,u)v &= \int_0^t D_u f(s,x(s,x_0,u),u(s))v\,ds \\ &= \int_0^t D_2 f(s,x(s,x_0,u),u(s))\,D_u x(s,x_0,u)v\,ds \\ &\quad + \int_0^t D_3 f(s,x(s,x_0,u),u(s))v(s)\,ds, \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}D_u x(t,x_0,u)v &= D_2 f(t,x(t,x_0,u),u(t))\,D_u x(t,x_0,u)v \\ &\quad + D_3 f(t,x(t,x_0,u),u(t))v(t), \end{aligned}$$

and

$$D_u x(0,x_0,u)v = 0.$$

This says that $z(t) = D_u x(t,x_0,u)v$ is the solution $z(t,x_0,u,v)$ of the associated linear system for the given control v. □

THEOREM 4.4. *Suppose that $f(t,x,u)$ has a linear Gâteaux derivative in each direction with respect to (x,u), and is Fréchet derivable with respect to x. Let $K \subset U$ be a closed convex cone, and suppose that the solution map $x(\cdot,x_0,u)$ of* (4.1) *has a linear Gâteaux derivative with respect to u. If there exist $\delta > 0$, $c > 0$ such that for each $u \in B_{U_{\mathrm{ad}}}(0,\delta)$, the associated linear system* (4.2) *is $K \cap cB_U$-constrained exactly controllable on B_X, then the nonlinear system* (4.1) *is K-constrained locally exact controllable on $B_X(x(T,x_0,0),\delta/c)$.*

Proof. By the assumptions and Lemma 4.3, for each $u \in B_{U_{\mathrm{ad}}}(0,\delta)$, we have

$$\begin{aligned} B_X &\subset \{z(T,x_0,u,v) : v \in cB_{U_{\mathrm{ad}}} \cap K_{\mathrm{ad}}\} \\ &= z(T,x_0,u,cB_{U_{\mathrm{ad}}} \cap K_{\mathrm{ad}}) \\ &= D_u x(T,x_0,u)(cB_{U_{\mathrm{ad}}} \cap K_{\mathrm{ad}}). \end{aligned}$$

It is easy to see that K_{ad} is a closed convex cone and, by assumption, the mapping $u \longmapsto x(T,x_0,u)$ is continuous from U_{ad} to X. By Corollary 3.10,

$$B_X(x(T,x_0,0),\delta/c) \subset x(T,x_0,K_{\mathrm{ad}}),$$

that is, (4.1) is K-constrained locally exact controllable on $B_X(x(T,x_0,0),\delta/c)$. □

Now, we consider the system

$$\begin{cases} x'(t) + Ax(t) = f(t,u(t)), & t \in [t_0,T], \\ x(t_0) = 0. \end{cases} \tag{4.3}$$

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Here, $A: X \longrightarrow X$ is a linear operator which generates a strongly continuous linear operator semigroup $\{S(t): t \in [t_0, T]\}$ so that we can suppose $\|S(t)\| \leqslant m$ for each $t \in [t_0, T]$. Associate (4.3) with the linear system

$$\begin{cases} z'(t) + Az(t) = Bu(t), & t \in [t_0, T] \\ z(t_0) = 0. \end{cases} \tag{4.4}$$

Here, $B: U \longrightarrow X$ is a bounded linear operator. We suppose that f is measurable and there exist $k > 0$ such that

$$\|f(t,u) - f(t,v) - (Bu - Bv)\| \leqslant k\|u - v\| \tag{4.5}$$

for all $t \in [t_0, T]$ and $u, v \in U$.

It is well known that for each $u \in U_{\text{ad}}$, (4.3) admits exactly one solution $x(t, 0, u) =: x(t, u)$ which can be expressed as

$$x(t,u) = \int_{t_0}^{t} S(t-s) f(s, u(s))\, ds \quad \text{for all } t \in [t_0, T],\ u \in U_{\text{ad}}.$$

The solution of equation (4.4) for a given $u \in U_{\text{ad}}$ is denoted by $z(t, u)$. It is easy to show that

$$u \longmapsto x(T,u) \quad \text{and} \quad u \longmapsto z(T,u)$$

are well defined operators from U_{ad} to X and $z(T, \cdot)$ is bounded linear. Moreover, for all $u, v \in U_{\text{ad}}$, we have

$$\begin{aligned} &\|x(T,u) - x(T,v) - [z(T,u) - z(T,v)]\| \\ &\qquad \leqslant \int_{t_0}^{T} \|S(T-s)[f(s,u(s)) - f(s,v(s)) - B(u(s) - v(s))]\|\, ds \\ &\qquad \leqslant mk \int_{t_0}^{T} \|u(s) - v(s)\|\, ds \leqslant mk(T - t_0)\, \|u - v\| =: \alpha \|u - v\|. \end{aligned} \tag{4.6}$$

If in place of $U_{\text{ad}} = L^{\infty}(t_0, T; U)$, we take $U_{\text{ad}} = L^2(t_0, T; U)$, then, for all $u, v \in U_{\text{ad}}$, we can prove similarly that the above inequality is still true with a new constant α being

$$\alpha = mk\sqrt{T - t_0}. \tag{4.7}$$

If $z(T, K_{\text{ad}}) = X$, by [**11**, Proposition 2.4], there exists a constant $c > 0$ such that $B_X \subset z(T, cB_{U_{\text{ad}}})$, that is, (3.12) holds for all $\delta > 0$. Hence, by Corollary 3.19, we have the following theorem.

THEOREM 4.5. *If $z(T, K_{\text{ad}}) = X$, that is, system* (4.4) *is K-constrained exactly controllable, and the constant α in* (4.6) (*or* (4.7) *if $U_{\text{ad}} = L^2(t_0, T; U)$*) *is sufficiently small, then $x(T, K_{\text{ad}}) = X$, that is, system* (4.3) *is K-constrained exactly controllable.*

This theorem means that the constrained controllability of a linear system implies the constrained controllability of a perturbed system if the perturbation is Lipschitz with a sufficiently small constant.

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