Proc. Camb. Phil. Soc. (1974), **75**, 345
PCPS 75–35
Printed in Great Britain

'Line-cobordism'

By M. GATE and G. WHISTON
Mathematics Department University of Durham, South Road, Durham

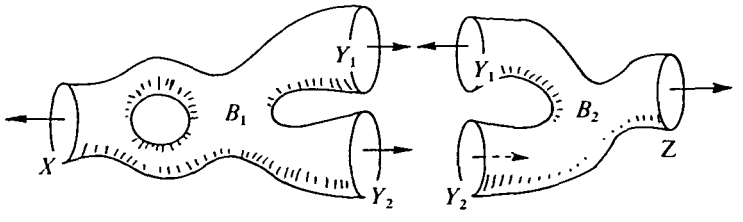
(*Received* 20 *August* 1973)

The following extension problem is of considerable geometrical interest. Given a compact manifold $X = \partial Z$ the boundary of a compact manifold Z, does the normal bundle of X in Z extend to a line bundle on Z? A related question involving foliations is if X is a compact boundary, is X a leaf of a codimension one foliation on any compact manifold bounded by X? Obviously if it were such a leaf, its normal bundle in the bounded manifold would extend. Reinhart posed both these problems in the language of cobordism theories(1),(2) defining a vector cobordism theory to a deal with questions of the first genre and a foliated-cobordism theory to deal with the second. He was able to obtain complete results in his vector-cobordism theory, whilst the foliated-cobordism theory remains an unsolved problem. Our aim in this article is to extend Reinhart's results on vector-cobordism theory to unoriented line bundles. Whereas in vector-cobordism the Euler number is a cobordism invariant, in the cobordism theory that we shall describe, the modulus of the Euler number becomes a cobordism invariant.

Definition 1. *Two compact, manifolds X_1 and X_2 are L-cobordant iff $X_1 \cup X_2 = \partial Z$ where Z is a compact, connected manifold admitting a line bundle θ normal on ∂Z.*

If we ask for the line bundle to be orientable, then if $j_1: X_1 \to Z$ and $j_2: X_2 \to Z$ are the inclusion maps, the line bundles $j_1^*(\theta)$ and $j_2^*(\theta)$ are orientable and their orientations either both agree with the orientations of the normal bundles (inwards) or $j_1^*(\theta)$ is oriented inwards and $j_2^*(\theta)$ is oriented outwards. Note that if we require that $j_1^*(\theta)$ be interior oriented and that $j_2^*(\theta)$ be exterior oriented, L-cobordism reduces to vector-cobordism (V-cobordism) as defined by Reinhart. It is easy to show that L-cobordism is an equivalence relation and that the sum of manifolds induces the structure of a commutative monoid on the set M_L^n of n-dimensional L-cobordism classes.

However, if we restrict θ to be orientable, without any boundary conditions, the new relation is not transitive. The lack of transitivity is illustrated in the example below.



https://doi.org/10.1017/S0305004100048556 Published online by Cambridge University Press

Here, $X \cup Y = \partial B_1$, $Y \cup Z = \partial B_2$ and B_1 carries an orientable line bundle exterior normal on both connected components Y_1 and Y_2 of Y, whilst B_2 carries an oriented line bundle exterior normal on Y_1 and interior oriented on Y_2. The two line bundles do not extend to an oriented line bundle on $B_3 = B_1 \cup_f B_2$ for $B_1 \supset Y \xrightarrow[f]{\text{inc.}} B_2$. One can, however, define an orientable line-cobordism between connected compact manifolds, but the disjoint sum structure has to be discarded.

DEFINITION 2. *Two connected, compact manifolds X_1 and X_2 are called L^0-cobordant iff $X_1 \cup X_2 = \partial Z$ for Z a compact connected manifold admitting an oriented line bundle normal on ∂Z.*

Clearly, Definition 2 reduces to vector cobordism when we ask for exterior orientation on X_1 and interior orientation on X_2. A more restrictive cobordism can be defined in the category of oriented manifolds. We call this SL-cobordism.

DEFINITION 3. *Two compact, oriented manifolds X_1^+ and X_2^+ are called SL-cobordant iff $X_1^+ \cup X_2^- = \partial_0 Z$ for Z a compact, connected oriented manifold carrying a line bundle normal on Z.*

The SL version of L^0-cobordism for oriented manifolds is SL^0-cobordism.

DEFINITION 4. *Two compact, connected, oriented manifolds X_1^+ and X_2^+ are called SL^0-cobordant iff $X_1^+ \cup X_2^- = \partial \circ Z$ for Z a compact connected, oriented manifold admitting an orientable line bundle normal on Z.*

(1) *L-Cobordism.* Suppose that $k^n : M_L^n \to M_0^n$ (the group of unoriented cobordism classes of compact n-dimensional manifolds) is the canonical epimorphism sending L-cobordism classes onto 0-cobordism classes. We define K_L^n as ker (k^n) and obtain the following result.

LEMMA 1. *$K_L^n = 0$ for $n \equiv 1 \bmod (2)$ and for $n \equiv 0 \bmod (2)$, K_L^n is generated by the L-cobordism class of S^n.*

Proof. (i) For $n \equiv 1 \bmod (2)$, $|X|_L \in K_L^n$ implies $|X|_V \in K_V^n$ the kernel of the canonical epimorphism $M_V^n \to M_0^n$. But $K_V^n = 0$ for $n \equiv 1 \bmod (2)$, implying that $X = \partial Z$ for Z a compact manifold admitting a vector field inner normal on X. Hence $|X|_L = 0$ and $K_L^n = 0$.

(ii) In dimensions $\equiv 0 \bmod (2)$, K_V^n is generated by the class of S^n. In fact, if $k_X = \chi/(X)/2$ and $|X|_V \in K_V^n$, $|X|_V = k_X \cdot |S^n|_V$. Now $-|S^n|_V = |S_{2H}^n|_V$, where S_{2H}^n is S^n with two one-handles and clearly $|S^n|_L = |S_{2H}^n|_L$. Hence if

$$|X|_L \in K_L^n, \quad |X|_L = |k_X| \cdot |S^n|_L.$$

LEMMA 2. *S^{2k} is L-cobordant for all $k \geqslant 0$.*

Proof. Let $\overset{0}{D}$ be a small open $2k+1$ disc around the origin O of D^{2k+1} which is equipped with the radial vector field (which has a singularity at the origin) Exercise $\overset{0}{D}$ and identify the antipodal points in the exterior boundary. The resulting manifold ($\mathbb{R}P^{2k+1} \backslash \overset{0}{D}$, the $2k+1$ cross-cap) has boundary S^{2k} and an unorientable line bundle normal to S^{2k}.

https://doi.org/10.1017/S0305004100048556 Published online by Cambridge University Press

An immediate consequence of the above lemmas is

THEOREM 1. *$M_L^n = M_0^n$ in all dimensions. In particular, any unoriented boundary is the boundary of a compact manifold admitting a line bundle normal on the boundary.*

Next, we consider the L-cobordism of oriented compact manifolds.

LEMMA 3. *Suppose that K_{SL}^n is the kernel of the canonical epimorphism $M_{SL}^n \to M_{SO}^n$ from the group of n-dimensional SL-cobordism classes onto the group of n-dimensional oriented cobordism classes. Then for $n \not\equiv 1 \bmod (4)$, $K_{SL}^n = 0$. For $n \equiv 1 \bmod (4)$, K_{SL}^n is the two element cyclic group on $|S^n|_L$.*

Proof. (i) $n \equiv 0 \bmod (2)$. If $|X|_{SL} \in K_{SL}^{2k}$, $|X|_{SV} \in K_{SV}^{2k}$ which is free cyclic on $|S^{2k}|_{SV}$ by Theorem (2) of ref. (1). In fact, $|X|_{SV} = k_X \cdot |S^{2k}|_{SV}$. Thus $|X|_{SL} = |k_X| \cdot |S^{2k}|_{SL}$. In Lemma 1, we showed that S^{2k} is the boundary of the $(2k+1)$-dimensional cross-cap which admits a non-orientable line bundle normal on S^{2k}. Since $\mathbb{R}P^{2k+1}$ happens to be orientable, $|S^{2k}|_{SL} = 0$ which implies that $K_{SL}^{2k} = 0$. (ii) $n \equiv -1 \bmod (4)$. Reinhart's result $K_{SV}^{4k-1} = 0$ implies that $K_{SL}^{4k-1} = 0$. (iii) $n \equiv 1 \bmod (4)$. If $|X|_{SL} \in K^{4k+1}$ and $|X|_{SL} \neq 0$, Reinhart's result on K_{SV}^{4k+1} implies that $|X|_{SL} = |S^{4k+1}|_{SL}$. Now $|S^{4k+1}|_{SL} \neq 0$. To prove this, we first note that if Z has oriented boundary S^{4k+1}, $\chi(Z) \equiv 1 \bmod (2)$. For if $\partial_0 Z_1 = \partial_0 Z_2 = S^{4k+1}$, glue Z_1 and Z_2 along S^{4k+1} to obtain a closed, compact, oriented manifold Z_3 of dimension $4k+2$. Because

$$\chi(Z_3) = \chi(Z_1) + \chi(Z_2) \quad \text{and} \quad \chi(Z_3) \equiv 0 \bmod (2), \quad \chi(Z_1) \equiv \chi(Z_2) \bmod (2).$$

Hence because $S^{4k+1} = \partial_0 D^{4k+2}$ which has Euler number 1, $\partial_0 Z = S^{4k+1}$ implies that $\chi(Z) \equiv 1 \bmod (2)$. We suppose that $|S^{4k+1}|_{SL} = 0$ and obtain a contradiction. If

$$S^{4k+1} = \partial_0 Z$$

for Z a compact, connected, oriented manifold with a line bundle L normal to S^{4k+1}, L must be unorientable because $|S^{4k+1}|_{SV} \neq 0$. Therefore there is a two-fold connected covering space Z^* of Z which carries an orientable line bundle L^* covering L which is normal on $\partial Z^* = (\partial Z)_1 \cup (\partial Z)_2$. L^* is interior oriented on $(\partial Z)_1$ and exterior oriented on $(\partial Z)_2$. Hence two copies of Z^* glued along ∂Z^* in the obvious way (f), admits a non-zero vector field. Thus $\chi(Z^* \cup_f Z^*) = 0 = 2\chi(Z^*)$. Because $\chi(Z^*) = 2\chi(Z)$, $\chi(Z) = 0$ contradicting $\chi(Z) \equiv 1 \bmod (2)$.

The immediate corollary of Lemma 3 is the following

THEOREM 2. *In dimensions $\not\equiv 1 \bmod (4)$, any oriented compact boundary is the oriented boundary of a compact connected oriented manifold admitting a line bundle normal on the boundary. In dimensions $\equiv 1 \bmod (4)$, if an oriented boundary bounds a manifold of odd Euler number, it cannot be the oriented boundary of an oriented manifold admitting a line bundle normal on the boundary.*

(2) *L^0-Cobordism.* Although L^0-cobordism deals with oriented normal line bundles with no boundary conditions, Reinhart's results on V-cobordism are used to obtain the following easy theorems. Note that we do not impose any algebraic structure on the cobordism classes in this section.

THEOREM 3. *Two compact, closed, connected manifolds X_1 and X_2 are L^0-cobordant iff they have the same Stiefel–Whitney numbers and if $|\chi(X_1)| = |\chi(X_2)|$.*

https://doi.org/10.1017/S0305004100048556 Published online by Cambridge University Press

Proof. If X_1 and X_2 are L^0-cobordant, they are O-cobordant and hence have the same Stiefel–Whitney numbers. Moreover, either X_1 is V-cobordant to X_2 implying that $\chi(X_1) = \chi(X_2)$ by theorem 1 of (1), or X_1 is the V-cobordism inverse of X_2 implying $\chi(X_1) = -\chi(X_2)$. Conversely, if X_1 and X_2 have the same Stiefel–Whitney numbers and $|\chi(X_1)| = |\chi(X_2)|$, either X_1 is V-cobordant to X_2 or is its V-cobordism inverse. In either case, X_1 is L-cobordant to X_2.

There is a canonical surjection from n-dimensional L^0-cobordism classes of connected compact manifolds onto unoriented cobordism classes $j^n\colon M^n_{L^0} \to M^n_0$. ($j^n$ is onto because any manifold is cobordant to a connected manifold.) Let $K^n_{L^0}$ be the 'kernel' $j^{n-1}(0)$.

THEOREM 4. *$K^n_{L^0} = 0$ for $n \equiv 1 \bmod (2)$. For $n \equiv 0 \bmod (2)$, $K^n_{L^0}$ is the set of positive powers of $|S^n_{2H}|_{L^0}$ under connected sums.*

Proof. (i) Dimensions $\equiv 1 \bmod (2)$. If $|X|_{L^0} \in K^n_{L^0}$, $|X|_V \in K^n_V = 0$. Hence $|X|_{L^0} = 0$ and $K^n_{L^0} = 0$. (ii) Dimensions $\equiv 0 \bmod (2)$. If

$$|X|_{L^0} \in K^{2k}_{L^0}, \quad |X|_0 = 0 \quad \text{and} \quad \chi(X) \equiv 0 \bmod (2).$$

Now if $r.S^{2k}_{2H} = S^{2k}_{2rH} = S^{2k}_{2H} * \ldots * S^{2k}_{2H}$, for $r > 0$ and $0.S^{2k}_{2H} = S^{2k}_H$, then $|X|_0 = |r.S^{2k}_{2H}|_0$ and $\chi(X) = \chi(k_X.S^{2k}_{2H})$ for $k_X = |\chi(X)|/2$. Thus $|X|_{L^0} = |k_X.S^{2k}_{2H}|_L$. There are the following analogous results for SL^0-cobordism.

THEOREM 5. *Two compact, connected, closed, oriented manifolds X_1 and X_2 are SL^0-cobordant in dimensions $\not\equiv 1 \bmod (4)$ iff they have the same Stiefel–Whitney and Pontryagin numbers and if $|\chi(X_1)| = |\chi(X_2)|$. In dimensions $\equiv 1 \bmod (4)$ two such manifolds are SL^0-cobordant iff they are cobordant in the oriented sense and together bound a manifold of even Euler number.*

Proof. The non-exceptional dimensions $n \not\equiv 1 \bmod (4)$, are treated exactly analogously to the L^0-cobordism case. In the exceptional dimensions $\equiv 1 \bmod (4)$, we proceed as follows. If X_1 and X_2 are SL^0-cobordant, either X_1 and X_2 are SV-cobordant and hence $X_1^+ \cup X_2^- = \partial_0 Z$ for some compact, connected, oriented manifold Z with even Euler number, or $X_{1,}^+ \cup X_2^- = \partial_0 Z$ for Z admitting a non-singular vector field inner normal on its boundary. In the latter case, $\chi(Z) = 0$ follows from the Poincaré–Hopf theorem. Conversely, if $X_1^+ \cup X_2^- = \partial_0 Z$ for $\chi(Z)$ even, X_1 and X_2 are SV-, and hence SL^0-cobordant.

THEOREM 6. *If $K^n_{SL^0}$ is the 'kernel' of the canonical surjection $M^n_{SL^0} \to M^n_S$ in dimension 'n', then*

(*a*) *If $n \equiv 0 \bmod (2)$, $K^n_{SL^0}$ is 'generated' by powers of S^n_{2H}.*

(*b*) *If $n \equiv -1 \bmod (4)$, $K^n_{SL^0} = 0$.*

(*c*) *If $n \equiv 1 \bmod (4)$, $K^n_{SL^0}$ consists of two elements 0 and the class of S^n.*

Proof. Parts (*a*) and (*b*) are directly analogous to the L^0-cobordism case. (*c*) In dimensions $\equiv 1 \bmod (4)$, if $|X|_{SL^0} \in K^{4k+1}_{SL^0}$, X is the oriented boundary of a compact, connected manifold Z of odd Euler number. Excise an open disc $\overset{0}{D}$ from Z to obtain an oriented manifold Z^* with boundary $X \cup S^{n-}$. But $\chi(Z^*) = \chi(Z) - 1$ which is even

https://doi.org/10.1017/S0305004100048556 Published online by Cambridge University Press

and thus by theorem 5, $|X|_{SL^0} = |S^n|_{SL^0}$. Now $|S^n|_{SL^0} \neq 0$, because, as we have already mentioned, if $S^{4k+1} = \partial_0 Z$, $\chi(Z)$ is odd.

We both thank Prof. T. J. Willmore for discussions, and the S.R.C. for financial support.

REFERENCES

(1) Bruce L. Reinhart. Cobordism and the Euler number. *Topology* 2 (1962), 173–177.
(2) Bruce L. Reinhart. Cobordism and foliations. *Ann. Inst. Fourier (Grenoble)* **14** (1964), 49–52.

https://doi.org/10.1017/S0305004100048556 Published online by Cambridge University Press