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A dynamical system approach to the Kakutani–Fibonacci sequence

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Abstract. In this paper we consider the sequence of Kakutani's α-refinements corresponding to the inverse of the golden ratio (which we call the *Kakutani–Fibonacci sequence of partitions*) and associate to it an ergodic interval exchange (which we call the *Kakutani–Fibonacci transformation*) using the 'cutting–stacking' technique. We prove that the orbit of the origin under this map coincides with a low discrepancy sequence (which we call the *Kakutani–Fibonacci sequence of points*), which has also been considered by other authors.

1. *Introduction and preliminaries*

In this paper we study with a dynamical system approach a sequence of points which arises from a beautiful geometric construction due to Kakutani [**16**]: the α-splitting procedure. He proved the uniform distribution of successive α-refinements using ergodic theory methods. It has been proved in [**4**] that when α is the inverse of the golden ratio, the sequence of α-refinements (called the *Kakutani–Fibonacci sequence of partitions* for reasons which will be clear later) has *low discrepancy*.

In [**4, 3**] the first author proposed two algorithms which associate to this sequence of α-refinements a sequence of points (called the *Kakutani–Fibonacci sequence of points*) which has *low discrepancy*, too.

In this section we present some basic definitions on uniform distribution, discrepancy, ergodic theory and the well-known Birkhoff's theorem. We also recall the definition of the *Kakutani–Fibonacci sequence*, with the addition of some properties, the most important of which being Theorem 7.

In §2 we introduce an interval exchange T associated to the cutting–stacking procedure for this sequence and we prove that this transformation is ergodic (Theorem 15). By means of this result and Birkhoff's theorem, we conclude that the orbit of almost every $x \in [0, 1[$ is a uniformly distributed sequence of points (Theorem 16). With the last results (Theorem 17) we obtain the Kakutani–Fibonacci sequence as the orbit of the

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origin under T. In a way, our results parallel Lambert's observation concerning the von Neumann–Kakutani mapping and the van der Corput sequence [**20**]. We give a synthetic description of both concepts at the beginning of §2.

Both sections benefit from figures that should be of some help to the reader.

Definition 1. Let $\alpha \in]0, 1[$ be a real number and π be any partition of $[0, 1[$. Kakutani's α-refinement of π, denoted by $\alpha\pi$, is obtained by splitting all the intervals of π having maximal length into subintervals of lengths proportional to α and $1 - \alpha$, respectively.

The sequence of partitions $\{\alpha^n\omega\}_{n\geq 1}$ obtained by successive α-refinements of the trivial partition $\omega = \{[0, 1[\}$ is called the *Kakutani α-sequence*.

Definition 2. Let $\{\pi_n\}_{n\geq 1}$ be a sequence of partitions of $[0, 1[$, represented by $\pi_n = ([y_j^{(n)}, y_{j+1}^{(n)}[\ : \ 1 \leq j \leq t_n)$. For any fixed n the *discrepancy* of π_n is defined by

$$D(\pi_n) = \sup_{0\leq a<b\leq 1} \left| \frac{1}{t_n} \sum_{j=1}^{t_n} \chi_{[a,b[}(y_j^{(n)}) - (b - a) \right|.$$

We say that $\{\pi_n\}_{n\geq 1}$ is *uniformly distributed (u.d.)* if $D(\pi_n) \to 0$ when $n \to \infty$.

If there exists a constant $C > 0$ such that $t_n \ D(\pi_n) \leq C$ for any n, we say that $\{\pi_n\}_{n\geq 1}$ has *low discrepancy*.

Definition 3. The *discrepancy* of the first N points of a sequence $X = \{x_n\}_{n\geq 1}$ in $[0, 1[$ is defined by

$$D_N(X) = \sup_{0\leq a<b\leq 1} \left| \frac{1}{N} \sum_{j=1}^{N} \chi_{[a,b[}(x_j) - (b - a) \right|$$

and the *star-discrepancy* is defined as

$$D_N^\star(X) = \sup_{0<b\leq 1} \left| \frac{1}{N} \sum_{j=1}^{N} \chi_{[0,b[}(x_j) - b \right|.$$

We say that X is *uniformly distributed (u.d.)* if $D_N(X) \to 0$ when $n \to \infty$.

If there exists a constant $C > 0$ such that $N D_N(X) \leq C \ \log N$ for any N, we say that X has *low discrepancy*.

Since Schmidt [**22**] proved that, given any sequence X, there exists an absolute constant $c > 0$ such that $N D_N^*(X) > c \log N$ for infinitely many N, it is clear that the bound for low discrepancy sequences is the best possible.

It is well known that $D^\star(X_N) \leq D(X_N) \leq 2D^\star(X_N)$ (for further information on uniform distribution and discrepancy, we refer to [**18**]).

The following theorems show the link between u.d. sequences of points and Quasi-Monte Carlo methods (although the most interesting applications appear in higher dimension): the first theorem is due to Weyl [**24**], the second to Koksma [**17**].

THEOREM 4. *A sequence of points $\{x_n\}_{n\geq 1}$ in $[0, 1[$ is* uniformly distributed (u.d.) *if and only if for any continuous function f on $[0, 1]$ it is*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} f(x_j) = \int_0^1 f(t)\, dt.$$

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THEOREM 5. *If f is a function on* $[0, 1]$ *with bounded variation* $V(f)$ *and* $x_1, \ldots, x_N$ *are* N *points in* $[0, 1[$ *with discrepancy* D_N^*, *then*

$$\left| \frac{1}{N} \sum_{j=1}^{N} f(x_j) - \int_0^1 f(t)\, dt \right| \leq V(f) D_N^*.$$

Kakutani proved [**16**] that the sequence of partitions $\{\alpha^n \omega\}_{n \geq 1}$ is uniformly distributed for any $\alpha \in]0, 1[$.

This result attracted considerable attention in the subsequent decades and in recent years many extensions of Kakutani's idea have been considered.

The concept of u.d. sequences of partitions has been extended to compact metric probability spaces [**8**] and to fractals [**15**]. Recently, the first and third authors showed how to rearrange a dense sequence of partitions in order to obtain a u.d. one [**7**].

The third author [**23**] extended Kakutani's α-refinement by introducing the concept of *ρ-refinement* of a partition π of $[0, 1[$. It is obtained by splitting the longest intervals of π into a finite number of parts homothetically to a fixed finite partition ρ of $[0, 1[$. It has been proved that the sequence of subsequent ρ-refinements $\{\rho^n \omega\}_{n \geq 1}$ is u.d. for every ρ.

Kakutani's splitting procedure has also been extended to the multidimensional case in [**6**].

The general case in which the successive ρ-refinements are applied to an arbitrary non-trivial partition π has been studied in [**1**]. The authors give necessary and sufficient conditions on π and ρ which assure that $\{\rho^n \pi\}_{n \geq 1}$ is u.d.

Drmota and Infusino [**9**] derived bounds for the discrepancy of the sequences of ρ-refinements $\{\rho^n \omega\}_{n \geq 1}$ considering a new approach based on the Khodak algorithm for parsing trees.

A specific countable class of sequences of ρ-refinements, called LS-sequences of partitions, has been studied in [**4**]. In that paper, an explicit algorithm was also presented that orders the points determining the LS-sequences of partitions to obtain a u.d. sequence of points, called an LS-sequence of points, providing estimates for the discrepancy.

The paper [**3**] contains a new explicit algorithm 'à la van der Corput' to generate these sequences of points using the representation of natural numbers in base $L + S$. This algorithm is explicit and allows us to compute directly the points of each LS-sequence.

Some interesting extensions of LS-sequences of points to dimension two were first introduced in [**14**] and presented in [**5**].

The sequence of points we want to study in this paper is a special case of an LS-sequence for $L = S = 1$ which turns out to be associated to Kakutani's α-sequence $\{\alpha^n \omega\}_{n \geq 1}$, where $\alpha = (\sqrt{5} - 1)/2$ is the inverse of the golden ratio. It is easy to see that, since $\alpha^2 + \alpha = 1$, each partition $\alpha^n \omega$ is formed only by intervals of length α^n (long) and α^{n+1} (short).

If we denote by $\{l_n\}_{n \geq 1}$ and $\{s_n\}_{n \geq 1}$ the number of long and short intervals, respectively, and by $\{t_n\}_{n \geq 1}$ the total number of intervals of the nth partition, they satisfy the difference equation

$$a_{n+2} = a_{n+1} + a_n$$

with initial conditions $l_0 = 1$ and $l_1 = 1$, $s_0 = 0$ and $s_1 = 1$ and, respectively, $t_0 = 1$ and $t_1 = 2$. Therefore $\{t_n\}_{n \geq 1}$ is the Fibonacci sequence. For this reason, in [**4**] this sequence

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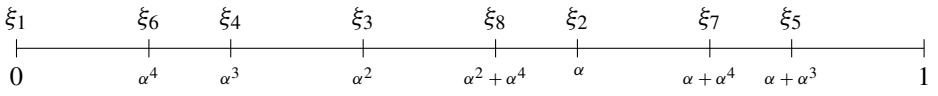


FIGURE 1. The first eight points of the Kakutani–Fibonacci sequence $\{\xi_n\}_{n\geq 1}$.

has been called the *Kakutani–Fibonacci sequence of partitions*, and the following theorem was also proved.

THEOREM 6. *The Kakutani–Fibonacci sequence of partitions* $\{\alpha^n\omega\}_{n\geq 1}$ *has low discrepancy.*

In [**4, 3**] the first author proposed two algorithms which order in a natural way the left endpoints of $\{\alpha^n\omega\}_{n\geq 1}$ and associate to them a sequence of points, called the *Kakutani–Fibonacci sequence of points,* which we shall denote by $\{\xi_n\}_{n\geq 1}$.

The sequence $\{\xi_n\}_{n\geq 1}$ can be constructed inductively as follows.

Begin with the first block

$$\Lambda_1 = (\xi_1, \xi_2) = (0, \alpha).$$

If we denote by $\Lambda_n = (\xi_1, \ldots, \xi_{t_n})$ the set of the t_n points of $\alpha^n\omega$ ordered in the appropriate way, the $t_{n+1} = t_n + l_n$ points of $\alpha^{n+1}\omega$ are defined by

$$\Lambda_{n+1} = (\xi_1, \ldots, \xi_{t_n}, \phi_{n+1}(\xi_1), \ldots, \phi_{n+1}(\xi_{l_n})), \tag{1}$$

where $\phi_{n+1}(x) = x + \alpha^{n+1}$.

In Figure 1 we see the first eight elements of the sequence $\{\xi_n\}_{n\geq 1}$.

It is interesting to notice that the sequence $\{\xi_n\}_{n\geq 1}$ has been constructed by Ninomiya in [**21**] using a totally different approach, without observing that it is one of the possible reorderings of the points of a Kakutani α-sequence.

The next result is due to Ninomiya in [**21**], who extended to a larger class of irrational numbers β some results of Barat and Grabner [**2**] about the so-called β-adic van der Corput sequences $\{\phi_\beta(n)\}$, where ϕ_β denotes the *Monna map*. An alternative proof, using completely different methods, has been given in [**4**].

THEOREM 7. *The Kakutani–Fibonacci sequence of points* $\{\xi_n\}_{n\geq 1}$ *has low discrepancy.*

In the next section we shall obtain this sequence using a dynamical system approach.

This approach to uniform distribution theory goes back to von Neumann and Kakutani (see [**11, 12, 20**]). For a recent survey on this matter see [**13**].

Definition 8. Let X be a set and $T : X \to X$ be a map. Given $x \in X$, the sequence $x, T(x), T^2(x), \ldots$ is called the *orbit* of x in X.

Definition 9. A map $T : [0, 1[\to [0, 1[$ is said to be an *interval exchange* on $[0, 1[$ if there exist a finite or countable family $\mathcal{F}$ of non-empty subintervals $I_k = [a_k, b_k[$ of $[0, 1[$ and a family of corresponding real numbers c_k with $1 \leq k \leq \text{card}(\mathcal{F})$, such that:

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(i) $I_h \cap I_k = \emptyset$ and $(I_h + c_h) \cap (I_h + c_k) = \emptyset$ if $h \neq k$;
(ii) $T(x) = x + c_k$ if $x \in I_k$;
(iii) $\lambda(\bigcup_{k\geq 0} I_k) = \lambda(\bigcup_{k\geq 0}(I_k + c_k)) = 1$.

It is easy to see that an interval exchange is a map of $[0, 1[$ into itself which is one-to-one, preserves the Lebesgue measure λ and is continuous λ-almost everywhere.

A special role is played by the *ergodic* transformations (see [**11**], for instance).

Definition 10. Let $(X, \mathcal{A}, \mu)$ be a probability space. A measurable transformation $T : X \to X$ is called *ergodic* if for every $A \in \mathcal{A}$ such that $T^{-1}(A) = A$, either $\mu(A) = 0$ or $\mu(A) = 1$.

We shall use later an equivalent property: T is ergodic if and only if for any measurable set B such that $\mu(B) > 0$, the set $B^T = \bigcup_{i=-\infty}^{+\infty} T^i(B)$ has measure 1.

We shall also assume that the ergodic transformation T is measure preserving, as most (but not all) authors do.

The system $(X, \mathcal{A}, \mu, T)$ is called a *measure theoretical dynamical system*, or *dynamical system*, for short. If T is ergodic, the system is called *ergodic*.

The link between dynamical systems and uniform distribution is given by the following theorem.

THEOREM 11. (Birkhoff's theorem) *Let $(X, \mathcal{A}, \mu, T)$ be a measure theoretical dynamical system. Then, for every $f \in \mathcal{L}^1(X)$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x)$$

exists for μ-almost every $x \in X$ (here $T^0 x = x$).

If $T : X \to X$ is ergodic, then for every $f \in \mathcal{L}^1(X)$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} f(T^j x) = \int_X f(x)\, d\mu(x),$$

for μ-almost every $x \in X$.

The conclusion of the previous theorem says that the orbit $\{T^j x\}_{j\geq 0}$ of x is u.d. for almost every $x \in X$ whenever T is ergodic.

2. *Main results*

We now turn to the main purpose of this paper showing that the Kakutani–Fibonacci sequence can be obtained as the orbit of the origin generated by an interval exchange $T : [0, 1[\to [0, 1[$, using the so-called *cutting–stacking* technique (see [**11, 20, 13**]). We briefly describe this procedure in the well-known case of the van der Corput sequence, which is a special case of LS-sequences corresponding to $L = 2$ and $S = 0$.

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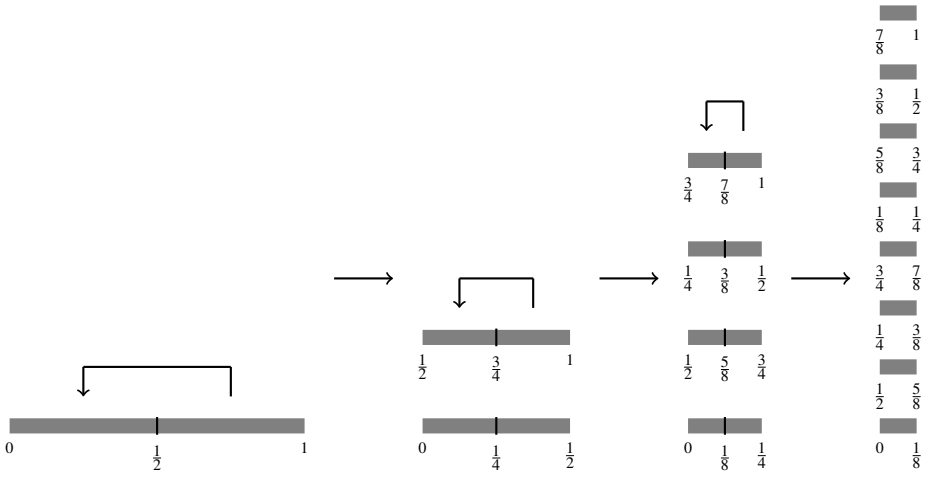


FIGURE 2. Partial graph of the dyadic odometer.

Von Neumann–Kakutani odometer. Let us consider the unit interval as a column and let us denote it by $S_0 = \{[0, 1[\}$. Let us split S_0 into two subintervals, $[0, 1/2[$ and $[1/2, 1[$ and stack them one onto the other in order to form the column $S_1 = \{[0, 1/2[, [1/2, 1[\}$. The *height* of S_1 is $h(S_1) = 2$ and its *width* is $w(S_1) = 1/2$. We define the map $T_{|[0,1/2[}(x) = x + (1/2)$ as the translation of the first subinterval of S_1 onto the second one. At the second step we cut each interval of S_1 into equal parts, take the right half and stack it onto the top of the left half. So we get the column $S_2 = \{[0, 1/4[, [1/2, 3/4[, [1/4, 1/2[, [3/4, 1[\}$, with $h(S_2) = 4$ and $w(S_2) = 1/4$. To this second step we associate the map $T_{|[1/2,3/4[}(x) = x + (1/4)$. We can visualize the above steps of the cutting–stacking procedure in Figure 2.

The procedure goes on this way, splitting each interval in half and stacking the right column of intervals onto the left column. So we obtain the sequence of columns

$$S_n = \{[\phi_2(0), \phi_2(0) + 2^{-n}[, [\phi_2(1), \phi_2(1) + 2^{-n}[, \ldots, [\phi_2(n-1), 1[\},$$

with $h(S_n) = 2^{n-1}$ and $w(S_n) = 1/2^{n-1}$. Here $\phi_2(n)$ denotes, as usual, the radical inverse function of n in base 2, defined by

$$\phi_2(n) = \sum_{j=0}^{\infty} a_j 2^{-j-1},$$

where $n = \sum_{j=0}^{\infty} a_j 2^j$ and $a_j \in \{0, 1\}$ (with the usual notation $\phi_2(0) = 0$).

The associated sequence $(\phi_2(n))_{n\geq 0}$ is the well-known van der Corput sequence.

Inductively, the transformation T associated to this construction and called the *Kakutani–von Neumann odometer* is defined on the countable sequence of intervals considered above by

$$T_k(x) = x - 1 + 2^{-k} + 2^{-k-1} \quad \text{for } x \in [1 - 2^{-k}, 1 - 2^{-k-1}[\text{ and } k \geq 0.$$

Figure 3 shows the transformation T on $[0, 1/2[\cup [1/2, 3/4[\cup [3/4, 7/8[$.

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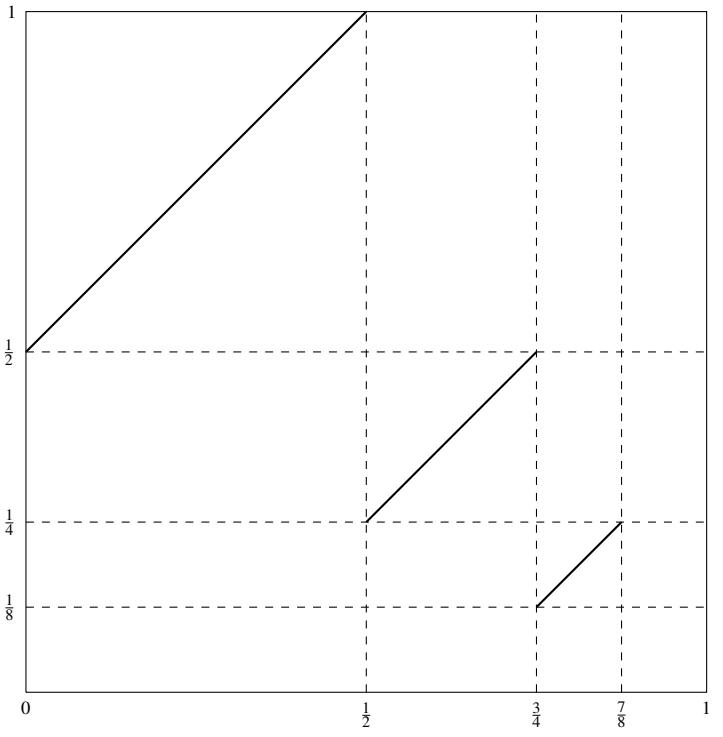


FIGURE 3. Partial graph of the von Neumann–Kakutani transformation.

Lambert [**19, 20**] proved that the sequence obtained as the orbit of 0 under the von Neumann–Kakutani transformation T is the van der Corput sequence, i.e.

$$(T^n 0)_{n\geq 0} = (\phi_2(n))_{n\geq 0}.$$

Let us recall, furthermore, that the von Neumann–Kakutani odometer is the simplest example of a *rank one* transformation and, consequently, it is ergodic.

The next construction represents an example of a *rank two* transformation which is, as we shall show in Theorem 15, also ergodic. For definition and properties of *systems of finite rank*, we refer the reader to the complete article of Ferenczi [**10**].

Definition 12. (Cutting–stacking technique for the Kakutani–Fibonacci sequence) For every fixed n, the intervals of the nth Kakutani–Fibonacci partition $\alpha^n\omega$ are represented by two columns $C_n = \{L_n, S_n\}$ constructed as follows.

Let us start with two columns $C_1 = \{L_1, S_1\}$, where $L_1 = [0, \alpha[$, $S_1 = [\alpha, 1[$. If we divide L_1 proportionally to α and α^2, we can write $L_1 = \{L_1^0, L_1^1\}$, where $L_1^0 = [0, \alpha^2[$ and $L_1^1 = [\alpha^2, \alpha[$. Now we stack the interval S_1 over L_1^0 (and use the common notation $L_1^0 * S_1$) as they have the same *width*, namely $w(L_1^0) = w(S_1) = \alpha^2$. So we get two new columns $C_2 = \{L_2, S_2\}$, where $L_2 = L_1^0 * S_1$ and $S_2 = L_1^1$. We denote by $b(L_2) = [0, \alpha^2[$ the *bottom* of L_2 and by $h(L_2) = 2 = l_2$ its *height*.

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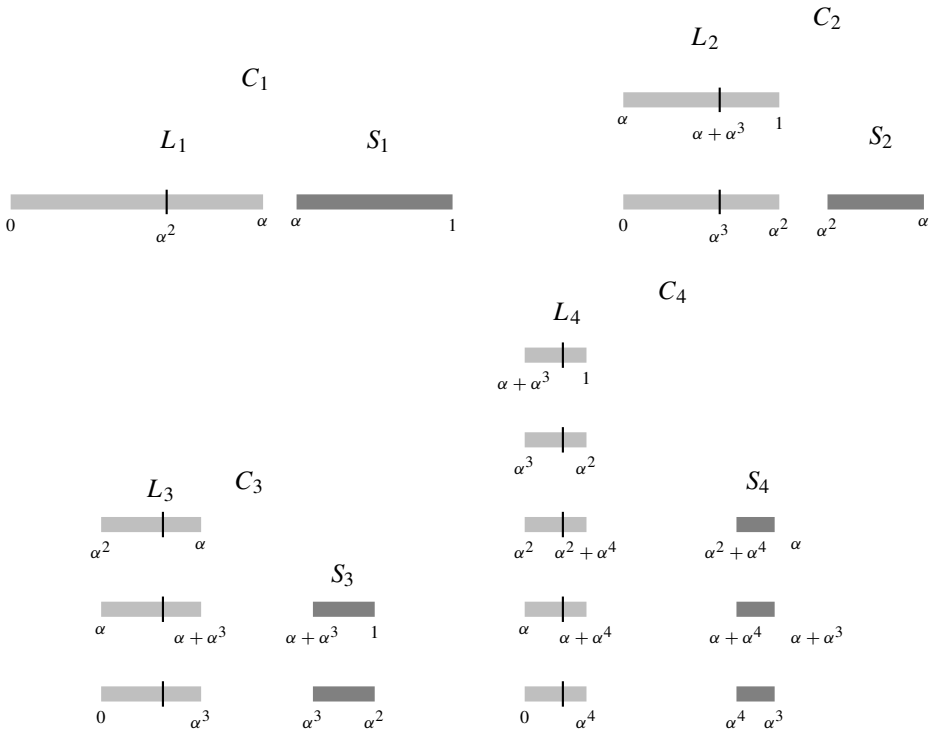


FIGURE 4. Partial graph of the cutting–stacking method for $\{\alpha^n\omega\}_{n\geq 1}$.

If we continue this way, at the nth step we get two columns denoted by $C_n = \{L_n, S_n\}$, with $h(L_n) = l_n$ and $h(S_n) = s_n$. We divide L_n into two columns, say L_n^0 and L_n^1, where $w(L_n^0) = \alpha^{n+1}$ and $w(L_n^1) = \alpha^{n+2}$, and stack S_n over L_n^0, obtaining

$$C_{n+1} = \{L_{n+1}, S_{n+1}\},$$

where

$$L_{n+1} = L_n^0 * S_n \quad \text{and} \quad S_{n+1} = L_n^1,$$

with

$$w(L_{n+1}) = \alpha^{n+1}, \quad h(L_{n+1}) = l_{n+1}, \quad b(L_{n+1}) = [0, \alpha^{n+1}[$$

and

$$w(S_{n+1}) = \alpha^{n+2}, \quad h(S_{n+1}) = s_{n+1}, \quad b(C_{n+1}) = [\alpha^{n+1}, \alpha^n[\,.$$

The first steps of the procedure are visualized in Figure 4.

To the above cutting–stacking construction is naturally associated an interval exchange, whose explicit expression is given by the following result.

PROPOSITION 13. *The interval exchange corresponding to the cutting–stacking procedure described in Definition 12 is the map* $T : [0, 1[\to [0, 1[$ *whose restriction to* I_k *is* T_k, *where*

$$T_1(x) = x + \alpha \quad \text{if } x \in I_1 = [0, \alpha^2[$$

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and, for every $k \geq 1$,

$$T_{2k}(x) = x + \alpha^{2k} - \sum_{j=0}^{k-1} \alpha^{2j+1} \quad \text{if } x \in I_{2k} = \left[\sum_{j=0}^{k-1} \alpha^{2j+1}, \sum_{j=0}^{k} \alpha^{2j+1}\right[$$

and

$$T_{2k+1}(x) = x + \alpha^{2k+1} - \sum_{j=0}^{k-1} \alpha^{2(j+1)} \quad \text{if } x \in I_{2k+1} = \left[\sum_{j=0}^{k-1} \alpha^{2(j+1)}, \sum_{j=0}^{k} \alpha^{2(j+1)}\right[.$$

Proof. If we write $T(x) = x + c_k$ whenever $x \in I_k$ for some $k \geq 1$, we simply observe that $I_k + c_k = [\alpha^k, \alpha^k + \alpha^{k+1}[$ for all $k \geq 1$. Therefore, $\lambda(\bigcup_{k\geq 0}(I_k + c_k)) = 1$ and $(I_h + c_h) \cap (I_k + c_k) = \emptyset$ whenever $h \neq k$, which proves that T is an interval exchange.

Now we show how the map T acts in the cutting–stacking procedure, proving that in each column $C_n = \{L_n, S_n\}$ the transformation T maps each interval of L_n (respectively, S_n) onto the interval above it and the *top* interval of L_n^0, denoted as usual by $\mathrm{top}(L_n^0)$, onto the bottom interval of S_n.

Let us start with $C_1 = \{L_1, S_1\}$. When we divide the column L_1 into two sub-columns L_1^0 and L_1^1, made up of one interval each, we notice that $\mathrm{top}(L_1^0) = I_1$ and $b(S_1) = T(I_1)$. Therefore, we stack S_1 onto L_1^0 using T_1, which maps I_1 onto $b(S_1)$ and gives birth to the column $C_2 = \{L_2, S_2\}$, where $L_2 = L_1^0 * S_1$ and $S_2 = L_1^1$.

Now we consider $n = 2k$ and $C_{2k} = \{L_{2k}, S_{2k}\}$. When we divide L_{2k} proportionally to α and α^2, obtaining therefore L_{2k}^0 and L_{2k}^1, we notice that $b(S_{2k}) = T_{2k}(I_{2k}) = [\alpha^{2k}, \alpha^{2k-1}[$ and, consequently, $\mathrm{top}(L_{2k}^0) = I_{2k}$ because T_{2k} is a bijection. In other words, T_{2k} stacks the bottom of S_{2k} onto the top of L_{2k}^0 because $T_{2k}(\mathrm{top}(L_{2k}^0)) = b(S_{2k}) = [\alpha^{2k}, \alpha^{2k-1}[$.

A simple calculation shows that if we consider the case $n = 2k + 1$ we have $T_{2k+1}(I_{2k+1}) = b(S_{2k+1}) = [\alpha^{2k+1}, \alpha^{2k}[$ and, therefore, $\mathrm{top}(L_{2k+1}^0) = I_{2k+1}$.

The proposition is now completely proved. □

Definition 14. The map $T : [0, 1[\to [0, 1[$, whose expression is given in Proposition 13, is called the *Kakutani–Fibonacci transformation*.

Figure 5 shows the graph of the maps T_k, with $1 \leq k \leq 4$.

Note that in [**13**] a transformation T_f is introduced that is called the Fibonacci transformation, related to the translation $x \to x + \alpha \pmod 1$ by means of the cutting–stacking technique. At each step of the cutting–stacking procedure, the two columns obtained by means of T_f have the same height and width as the two columns obtained by means of the Kakutani–Fibonacci transformation T, but the bottoms are different as well as the order of the intervals in each column. Moreover, T_f is not defined in 0.

THEOREM 15. *The Kakutani–Fibonacci transformation T is ergodic.*

Proof. The arguments we use are inspired by [**11**].

Let us denote by C_m^* the *stack* made by $L_m * S_m$ and let us define the mapping $\tau_m : [0, 1[\to [0, 1[$ as follows. For $1 \leq i \leq l_m - 1$ it is the translation of the interval J_i of L_m onto J_{i+1}, the interval just above J_i, and hence it coincides with T. If $i = l_m$, it is a

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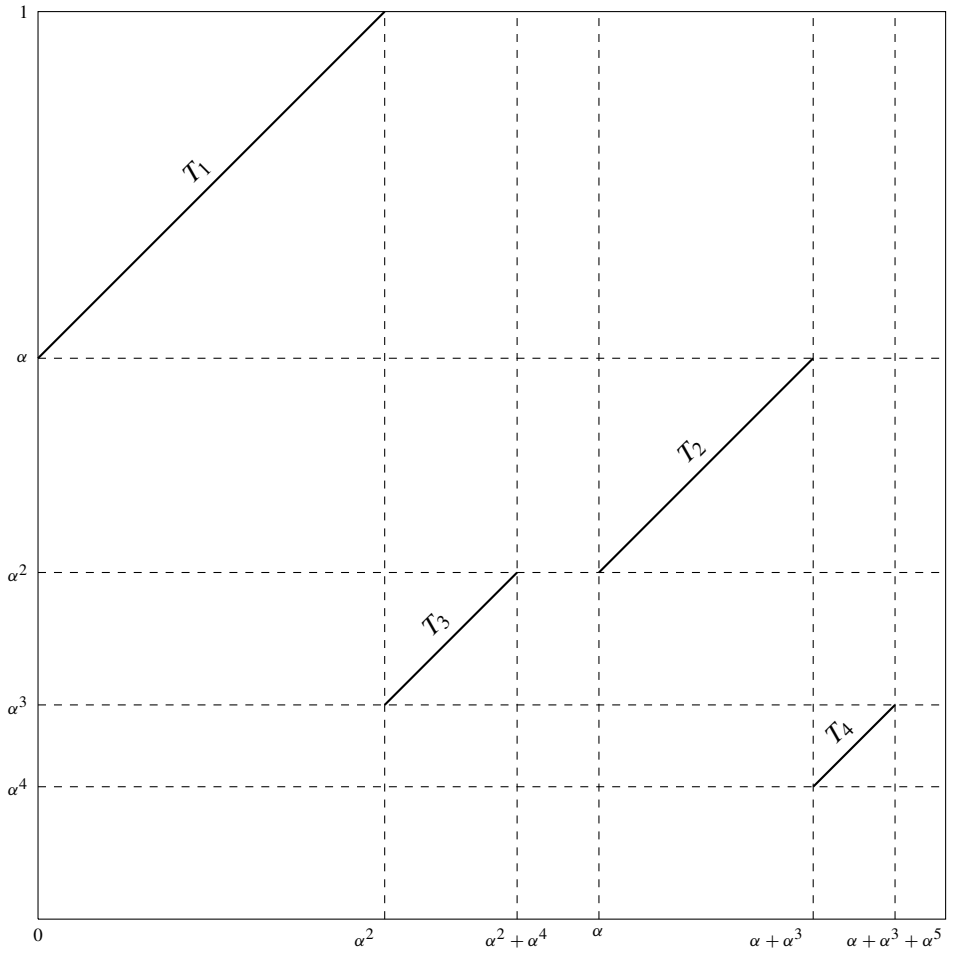


FIGURE 5. Partial graph of the Kakutani–Fibonacci transformation T.

contraction by parameter α of top(L_m) onto $b(S_m)$. For $1 \le i \le s_m - 1$ it is the translation of the interval $\tilde{J}_i$ of S_m onto $\tilde{J}_{i+1}$, the interval just above $\tilde{J}_i$, and hence it coincides with T again.

Now we fix a measurable set B in $[0, 1[$ with $\lambda(B) > 0$, such that $T^{-1}(B) = B$, and we prove that $\lambda(B^T) = 1$, where $B^T = \bigcup_{i=-\infty}^{+\infty} T^i(B)$.

From the Lebesgue density theorem, there exists a point x_0 having density 1 for B. This implies that for every fixed $\epsilon > 0$ there exists $\delta > 0$ such that if I is any interval containing x_0 with $\lambda(I) < \delta$, we have

$$\lambda(B \cap I) > (1 - \epsilon)\lambda(I). \tag{2}$$

Due to the fact that the diameter of the Kakutani–Fibonacci sequence of partitions $(\alpha^m \omega)_{m \ge 1}$ tends to 0 when $m \to \infty$, there exists $m > 0$ such that $\alpha^m < \delta$ and, therefore,

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there exists an interval of C_m^* for which (2) holds. For the sake of brevity, we shall denote it by I.

As $C_m^* = L_m * S_m$, the interval I could belong to the column L_m or to the column S_m. If $I \in S_m$, this is equivalent to saying that $I \in L_{m+1}$. For this reason, without any loss of generality we may suppose that $I \in L_m$.

Suppose now that I is the rth interval from below in the column L_m and observe that

$$\lambda\Bigg(\bigcup_{i=-r+1}^{t_m-r} \tau_m^i(I)\Bigg) = 1.$$

Taking the above identity and (2) into account, we have the following inequalities:

$$\begin{aligned}
\lambda(B^T) &\geq \lambda((B \cap I)^T) \\
&\geq \lambda\Bigg(\bigcup_{i=1-r}^{l_m-r} T^i(B \cap I)\Bigg) + \lambda\Bigg(\bigcup_{j=0}^{s_m-1} (T^j(B \cap b(S_m)))\Bigg) \\
&= \lambda\Bigg(\bigcup_{i=1-r}^{l_m-r} \tau_m^i(B \cap I)\Bigg) + \lambda\Bigg(\bigcup_{j=l_m+1-r}^{t_m-r} \tau_m^j(B \cap I)\Bigg) \\
&= \sum_{i=1-r}^{l_m-r} \lambda(\tau_m^i(B \cap I)) + \sum_{j=l_m+1-r}^{t_m-r} \lambda(\tau_m^j(B \cap I)) \\
&> (1-\epsilon) l_m \alpha^m + (1-\epsilon) s_m \alpha^{m+1} = 1 - \epsilon.
\end{aligned}$$

As ϵ is arbitrary, we conclude that $\lambda(B) = 1$. Therefore, T is ergodic and the theorem is proved. □

A direct consequence of the above theorem and Theorem 11 is the following theorem.

THEOREM 16. *The sequence $\{T^n(x)\}_{n\geq 0}$ is u.d. for almost every $x \in [0, 1[$.*

We conclude this paper by showing how to get the points of the Kakutani–Fibonacci sequence by means of the Kakutani–Fibonacci transformation.

THEOREM 17. *The Kakutani–Fibonacci sequence of points $\{\xi_n\}_{n\geq 1}$ coincides with $\{T^n(0)\}_{n\geq 0}$.*

Proof. We want to show by induction on $n \geq 1$ that

$$(\xi_1, \xi_2, \ldots, \xi_{t_n}) = (0, T(0), T^2(0), \ldots, T^{t_n-1}(0)). \tag{3}$$

If $n = 1$, (3) is obviously verified.

We suppose that (3) is true for n and prove that

$$\begin{aligned}
&(\xi_1, \xi_2, \ldots, \xi_{t_n}, \phi_{n+1}(\xi_1), \ldots, \phi_{n+1}(\xi_{l_n})) \\
&\quad = (0, T(0), T^2(0), \ldots, T^{t_n-1}(0), T^{t_n}(0), T^{t_n+1}(0), \ldots, T^{t_n+l_n-1}(0)).
\end{aligned}$$

Due to (1) and the inductive assumption, it is sufficient to prove that

$$(\phi_{n+1}(0), \phi_{n+1}(T(0)), \ldots, \phi_{n+1}(T^{t_n-1}(0))) = (T^{t_n}(0), T^{t_n+1}(0), \ldots, T^{t_n+l_n-1}(0)).$$

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As $\phi_{n+1}(x) = x + \alpha^{n+1}$, the above identity is equivalent to

$$(\alpha^{(n+1)}, T(0) + \alpha^{n+1}, \ldots, T^{t_n-1}(0) + \alpha^{n+1}) = (T^{t_n}(0), T^{t_n+1}(0), \ldots, T^{t_n+l_n-1}(0)). \tag{4}$$

In order to prove (4), we focus our attention on the intervals of the column S_{n+1} and on their left endpoints.

We note that, due to the cutting–stacking procedure described in Definition 12 and to the nature of T described in the proof of Proposition 13, and specifically to the fact that $b(S_{n+1}) = T(\mathrm{top}(L_{n+1})) = T(T^{l_n+s_n-1}([0, \alpha^{n+1}[))$, the columns $C_{n+1} = \{L_{n+1}, S_{n+1}\}$ can be written as follows:

$$\begin{aligned} L_{n+1} &= L_n^0 * S_n = (b(L_n^0) * \cdots * \mathrm{top}(L_n^0)) * (b(S_n) * \cdots * \mathrm{top}(S_n)) \\ &= ([0, \alpha^{n+1}[* \cdots * T^{l_n-1}([0, \alpha^{n+1}[)) \\ &\quad * (T^{l_n}([0, \alpha^{n+1}[) * \cdots * T^{l_n+s_n-1}([0, \alpha^{n+1}[)) \end{aligned}$$

and

$$\begin{aligned} S_{n+1} = L_n^1 &= b(L_n^1) * \cdots * \mathrm{top}(L_n^1) \\ &= ([\alpha^{n+1}, \alpha^n[* \cdots * T^{l_n-1}([\alpha^{n+1}, \alpha^n[)) \\ &= (T^{t_n}([0, \alpha^{n+1}[) * \cdots * T^{t_n+l_n-1}([0, \alpha^{n+1}[)). \end{aligned}$$

Therefore, as T is an interval exchange, the t_n left endpoints of L_{n+1} are $0, T(0), \ldots, T^{t_n-1}(0)$ and the left endpoints of S_{n+1}, which are a right shift by the constant α^{n+1} of the left endpoints of L_{n+1}, are $T^{t_n}(0), T^{t_n+1}(0), \ldots, T^{t_n+s_n-1}(0)$, which proves (4).

The theorem is now completely proved. □

Addendum. After the submission of the paper, the second author, in a joint work with M. Hofer and R. Tichy, proved (in particular) that the Kakutani–Fibonacci transformation is in fact uniquely ergodic.

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