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A class of simple C^*-algebras arising from certain non-sofic subshifts

KENGO MATSUMOTO†

Department of Mathematical Sciences, Yokohama City University, 22-2 Seto, Kanazawa-ku, Yokohama, 236-0027, Japan
(e-mail: kengo@yokohama-cu.ac.jp)

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Abstract. We present a class of subshifts Z_N, $N = 1, 2, \ldots$, whose associated C^*-algebras $\mathcal{O}_{Z_N}$ are simple, purely infinite and not stably isomorphic to any Cuntz–Krieger algebra nor to the Cuntz algebra $\mathcal{O}_\infty$. The class of the subshifts is the first example whose associated C^*-algebras are not stably isomorphic to any Cuntz–Krieger algebra nor to the Cuntz algebra $\mathcal{O}_\infty$. The subshifts Z_N are coded systems whose languages are context free. We compute the topological entropy for the subshifts and show that a KMS-state (a state satisfying the Kubo–Martin–Schwinger condition) for gauge action on the associated C^*-algebra $\mathcal{O}_{Z_N}$ exists if and only if the logarithm of the inverse temperature is the topological entropy for the subshift Z_N, and the corresponding KMS-state is unique.

1. *Introduction*

Let Σ be a finite set with its discrete topology, which is called an alphabet. Each element of Σ is called a symbol. Let $\Sigma^{\mathbb{Z}}$ be the infinite product space $\prod_{i\in\mathbb{Z}}\Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}$ given by $\sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$ is called the full shift over Σ. Let Λ be a shift-invariant closed subset of $\Sigma^{\mathbb{Z}}$, i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift or a symbolic dynamical system. It is written as Λ for brevity. The theory of symbolic dynamical systems gives a basic method to study general dynamical systems (cf. [**16, 24**]). It also has significant uses in coding for information theory. There is a class of subshifts called sofic shifts that contains the topological Markov shifts. Sofic shifts are presented by finite square matrices with entries in formal sums of symbols [**11, 17, 18, 43**]. Such a matrix is called a symbolic matrix. It is an equivalent object to a finite labeled graph called a λ-graph. In [**29**], the author has introduced the notions of the symbolic matrix system and the λ-graph system. They are presentations of subshifts and generalizations of symbolic matrices and λ-graphs respectively (cf. [**20**]). A symbolic matrix system $(\mathcal{M}, I)$ consists of a sequence of pairs $(\mathcal{M}_{l,l+1}, I_{l,l+1})$, $l \in \mathbb{Z}_+$, of rectangular symbolic matrices $\mathcal{M}_{l,l+1}$ and rectangular $\{0, 1\}$-matrices $I_{l,l+1}$, where $\mathbb{Z}_+$ denotes the set of all non-negative integers. Both the matrices $\mathcal{M}_{l,l+1}$ and $I_{l,l+1}$ have the same size for each $l \in \mathbb{Z}_+$. The

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column size of $\mathcal{M}_{l,l+1}$ is the same as the row size of $\mathcal{M}_{l+1,l+2}$. They satisfy the following commutation relations as symbolic matrices:

$$I_{l,l+1}\mathcal{M}_{l+1,l+2} = \mathcal{M}_{l,l+1}I_{l+1,l+2}, \quad l \in \mathbb{Z}_+. \tag{1.1}$$

We further assume that for i there exists j such that the (i, j)-component $I_{l,l+1}(i, j) = 1$, and for j there uniquely exists i such that $I_{l,l+1}(i, j) = 1$. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ consists of a vertex set $V = V_0 \cup V_1 \cup V_2 \cup \cdots$, an edge set $E = E_{0,1} \cup E_{1,2} \cup E_{2,3} \cup \cdots$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $\iota_{l,l+1} : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$. An edge $e \in E_{l,l+1}$ has its source vertex $s(e)$ in V_l, its terminal vertex $t(e)$ in V_{l+1} and its label $\lambda(e)$ in Σ. For a symbolic matrix system $(\mathcal{M}, I)$, the labeled edges from a vertex $v_i^l \in V_l$ to a vertex $v_j^{l+1} \in V_{l+1}$ are given by the symbols appearing in the (i, j)-component $\mathcal{M}_{l,l+1}(i, j)$ of $\mathcal{M}_{l,l+1}$. The matrix $I_{l,l+1}$ defines a surjection $\iota_{l,l+1}$ from V_{l+1} to V_l for each $l \in \mathbb{Z}_+$. By this observation, the symbolic matrix systems and the λ-graph systems are the same objects. They give rise to subshifts by taking the set of all label sequences appearing in the labeled Bratteli diagram. For a symbolic matrix system $(\mathcal{M}, I)$, let $M_{l,l+1}$ be the non-negative rectangular matrix obtained from $\mathcal{M}_{l,l+1}$ by setting all the symbols equal to 1 for each $l \in \mathbb{Z}_+$. Then the resulting pair (M, I) satisfies the following relations by (1.1):

$$I_{l,l+1}M_{l+1,l+2} = M_{l,l+1}I_{l+1,l+2}, \quad l \in \mathbb{Z}_+. \tag{1.2}$$

We call (M, I) the non-negative matrix system for $(\mathcal{M}, I)$. Let $m(l)$ be the row size of the matrix $I_{l,l+1}$ for each $l \in \mathbb{Z}_+$. Let $\mathbb{Z}_{I^t}$ be the abelian group defined by the inductive limit $\mathbb{Z}_{I^t} = \varinjlim\{I_{l,l+1}^t \mid \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}\}$. The sequence $M_{l,l+1}^t$, $l \in \mathbb{Z}_+$ of the transposes of $M_{l,l+1}$ naturally acts on $\mathbb{Z}_{I^t}$ by the relation (1.2), which is denoted by $\lambda_{(M,I)}$. The K-groups for (M, I) have been defined in [**29**] as

$$K_0(M, I) = \mathbb{Z}_{I^t}/(\mathrm{id} - \lambda_{(M,I)})\mathbb{Z}_{I^t}, \quad K_1(M, I) = \mathrm{Ker}(\mathrm{id} - \lambda_{(M,I)}) \text{ in } \mathbb{Z}_{I^t}.$$

In [**33**], the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a λ-graph system $\mathfrak{L}$ has been introduced. The C^*-algebras are generalizations of the Cuntz–Krieger algebras and the C^*-algebras associated with subshifts (cf. [**4, 6, 7, 9, 25**]). They are universal unique concrete C^*-algebras generated by finite families of partial isometries and sequences of projections subject to certain operator relations encoded by the structure of the λ-graph systems. Let (M, I) be the non-negative matrix system for the symbolic matrix system $(\mathcal{M}, I)$ of $\mathfrak{L}$. The K-theory formulae for $\mathcal{O}_{\mathfrak{L}}$ have been obtained in the following way:

$$K_0(\mathcal{O}_{\mathfrak{L}}) = K_0(M, I), \quad K_1(\mathcal{O}_{\mathfrak{L}}) = K_1(M, I). \tag{1.3}$$

There is a canonical method to construct a λ-graph system $\mathfrak{L}$ from a given subshift Λ. The λ-graph system is called the canonical λ-graph system for Λ and is written as $\mathfrak{L}^\Lambda$. The C^*-algebra $\mathcal{O}_{\mathfrak{L}^\Lambda}$ associated with the canonical λ-graph system $\mathfrak{L}^\Lambda$ coincides with the C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift Λ (cf. [**4, 25**]). If in particular Λ is a topological Markov shift Λ_A for a finite square matrix A with entries in $\{0, 1\}$, the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\Lambda_A}}$ is canonically isomorphic to the Cuntz–Krieger algebra $\mathcal{O}_A$.

In this paper, we present a class of non-sofic subshifts Z_N, $N \in \mathbb{N}$, whose associated C^*-algebras $\mathcal{O}_{Z_N}$ are simple, purely infinite and not stably isomorphic to any Cuntz–Krieger algebra nor to any Cuntz algebra. The subshifts Z_N, $N \in \mathbb{N}$, are coded systems whose languages are context free (cf. [**1, 2, 5, 13**]). In studying a topological dynamical system, the topological entropy is a very important quantity to measure complexity for

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the topological dynamical system. In [**10**], it was shown that the topological entropy for irreducible Markov shifts appears as the logarithm of the inverse temperature for admitting a KMS-state (a state satisfying the Kubo–Martin–Schwinger condition) for gauge action on the corresponding Cuntz–Krieger algebras. This result has been generalized to more general subshifts in [**37**] (cf. [**23, 39**]). Corresponding to these results, we will compute the topological entropy for the subshifts Z_N, $N \in \mathbb{N}$, and show that the KMS-state for gauge action on the associated C^*-algebra $\mathcal{O}_{Z_N}$ exists if and only if the logarithm of the inverse temperature is the topological entropy for the subshift Z_N, and the corresponding KMS-state is unique. We will prove the following theorem.

THEOREM 1.1. (Theorems 3.7, 4.9 and 5.15)

(i) *For $N \in \mathbb{N}$, the C^*-algebra $\mathcal{O}_{Z_N}$ associated with the subshift Z_N is simple and purely infinite. It is the universal concrete C^*-algebra generated by two isometries T_1, T_2 and N partial isometries S_i, $i = 1, \ldots, N$, satisfying the relations*

$$T_1T_1^* + T_2T_2^* + \sum_{j=1}^{N} S_jS_j^* = 1, \tag{1.4}$$

$$S_i^*S_i = 1 - \sum_{k=1}^{\infty} \sum_{m=0,k\neq 2m}^{k} \sum_{j=1}^{N} T_{2^m1^{k-m}} S_j S_j^* T_{2^m1^{k-m}}^*, \quad i = 1, \ldots, N \tag{1.5}$$

where $T_{2^m1^{k-m}}$ denotes $\underbrace{T_2 \cdots T_2}_{m \text{ times}} \underbrace{T_1 \cdots T_1}_{k-m \text{ times}}$. The infinite sum of the right-hand side of the relation (1.5) is taken under strong operator topology on a Hilbert space.

(ii) *The K-groups for the C^*-algebra $\mathcal{O}_{Z_N}$ are as follows:*

$$K_0(\mathcal{O}_{Z_N}) = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad K_1(\mathcal{O}_{Z_N}) = 0.$$

The position of the unit [1] *in $K_0(\mathcal{O}_{Z_N}) = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}$ is* 0. *Hence $\mathcal{O}_{Z_N}$ is not stably isomorphic to any Cuntz–Krieger algebra nor to the Cuntz algebra $\mathcal{O}_\infty$ for $N \geq 2$.*

(iii) *There is a KMS-state for gauge action on $\mathcal{O}_{Z_N}$ if and only if the logarithm of the inverse temperature is* $\log \beta_N$*: the topological entropy of the subshift Z_N, where β_N is the unique solution β satisfying $\beta > N$ of the equation*

$$\beta^6 - (N+3)\beta^5 + (3N+1)\beta^4 - 2(N-1)\beta^3 - (N+2)\beta^2 + N - 1 = 0. \tag{1.6}$$

The admitted KMS-state is unique.

The value β_N is increasing on N and satisfies $N < \beta_N < N + 1$ such that

$$\lim_{N\to\infty} (\beta_N - N) = 0.$$

For $N = 1$, the subshift Z_1 is nothing but the subshift Z named as the context-free shift in [**24**, Example 1.2.9] and the associated C^*-algebra $\mathcal{O}_{Z_1}$ is the C^*-algebra $\mathcal{O}_Z$ studied in [**27**]. The algebra $\mathcal{O}_Z$ has been proved to be not stably isomorphic to any Cuntz–Krieger algebra $\mathcal{O}_A$ for a finite square matrix with entries in {0, 1} and to be stably isomorphic to the Cuntz algebra $\mathcal{O}_\infty$ [**27**]. Similarly, the C^*-algebras $\mathcal{O}_\beta$, $1 < \beta \in \mathbb{R}$, studied in [**14**] are stably isomorphic to $\mathcal{O}_\infty$ if the associated β-shifts are not sofic. These C^*-algebras $\mathcal{O}_Z$ and $\mathcal{O}_\beta$ are constructed from the canonical λ-graph systems for the presented subshifts. We have many examples of λ-graph systems $\mathfrak{L}$ whose associated C^*-algebras are not stably isomorphic to any Cuntz–Krieger algebra $\mathcal{O}_A$ nor to the Cuntz algebra $\mathcal{O}_\infty$ as in,

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for example, [**33**, Theorem 7.7] and [**21, 34, 36**]. These examples of λ-graph systems that are not canonical λ-graph systems for the subshifts are called Cantor horizon λ-graph systems. The class of the subshifts Z_N, $N \in \mathbb{N}$, presented in this paper is the first example of the subshifts whose associated C^*-algebras, that come from the canonical λ-graph systems for Z_N, are not stably isomorphic to any Cuntz–Krieger algebra nor to the Cuntz algebra $\mathcal{O}_\infty$. For other types of coded systems, see for example [**19, 22**].

We will finally mention an application of our discussions to a classification problem in the theory of symbolic dynamical systems. By [**31, 32**], the K-groups $K_i(\mathcal{O}_\Lambda)$ are invariant under not only topological conjugacy but also flow equivalence of subshifts Λ. Hence we know that the subshifts Z_N, $N \in \mathbb{N}$, are not flow equivalent to one another (Proposition 6.1).

2. *Subshifts and the C^*-algebras associated with λ-graph systems*

For a subshift Λ over alphabet Σ, we denote by $X_\Lambda \subset \Sigma^{\mathbb{N}}$ the set of all right-infinite sequences that appear in Λ. The topological dynamical system (X_Λ, σ) is called the one-sided subshift for Λ. A finite sequence $\mu = (\mu_1, \dots, \mu_k)$ of elements $\mu_j \in \Sigma$ is called a word. We denote by $|\mu|$ the length k of μ. A block $\mu = (\mu_1, \dots, \mu_k)$ is said to appear in $x = (x_i) \in \Sigma^{\mathbb{Z}}$ if $x_m = \mu_1, \dots, x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$. For a number $k \in \mathbb{N}$, let $B_k(\Lambda)$ be the set of all words of length k in $\Sigma^{\mathbb{Z}}$ appearing in some $x \in \Lambda$. Put $B_*(\Lambda) = \bigcup_{k=0}^{\infty} B_k(\Lambda)$ where $B_0(\Lambda)$ denotes the empty word $\emptyset$. Set

$$\Lambda_l(x) = \{\mu \in B_l(\Lambda) | \mu x \in X_\Lambda\} \quad \text{for } x \in X_\Lambda, l \in \mathbb{Z}_+.$$

We define a nested sequence of equivalence relations in the space X_Λ as follows [**26, 28, 29**]. Two points $x, y \in X_\Lambda$ are said to be *l-past equivalent*, written as $x \sim_l y$, if $\Lambda_l(x) = \Lambda_l(y)$. Denote by $\Omega_l = X_\Lambda / \sim_l$ the quotient space of X_Λ under $\sim_l$. For $x, y \in X_\Lambda$ and $\mu \in B_k(\Lambda)$, one sees the following.

(i) If $x \sim_l y$, we have $x \sim_m y$ for $m < l$.

(ii) If $x \sim_l y$ and $\mu x \in X_\Lambda$, we have $\mu y \in X_\Lambda$ and $\mu x \sim_{l-k} \mu y$ for $l > k$.

We have the following sequence of surjections in a natural way:

$$\Omega_0 \leftarrow \Omega_1 \leftarrow \Omega_2 \leftarrow \cdots \leftarrow \Omega_l \leftarrow \Omega_{l+1} \leftarrow \cdots,$$

where Ω_0 is a singleton. The subshift Λ is a sofic shift if and only if $\Omega_l = \Omega_{l+1}$ for some $l \in \mathbb{N}$. For a fixed $l \in \mathbb{Z}_+$, let F_i^l, $i = 1, 2, \dots, m(l)$, be the set of all l-past equivalence classes of X_Λ so that X_Λ is a disjoint union of F_i^l, $i = 1, 2, \dots, m(l)$. Then the canonical λ-graph system $\mathfrak{L}^\Lambda = (V^\Lambda, E^\Lambda, \lambda^\Lambda, \iota^\Lambda)$ for Λ is defined as follows [**29**]. The vertex set V_l at level l consists of the sets F_i^l, $i = 1, \dots, m(l)$. We write an edge with label a from the vertex $F_i^l \in V_l$ to $F_j^{l+1} \in V_{l+1}$ if $ax \in F_i^l$ for some $x \in F_j^{l+1}$. We denote by $E_{l,l+1}$ the set of all edges from V_l to V_{l+1}. There exists a natural map $\iota_{l,l+1}^\Lambda$ from V_{l+1} to V_l by mapping F_j^{l+1} to F_i^l when F_i^l contains F_j^{l+1}. Set

$$V^\Lambda = \bigcup_{l=0}^{\infty} V_l \quad \text{and} \quad E^\Lambda = \bigcup_{l=0}^{\infty} E_{l,l+1}.$$

The labeling of edges is denoted by $\lambda^\Lambda : E \to \Sigma$.

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over Σ. The C^*-algebra $\mathcal{O}_\mathfrak{L}$ associated with $\mathfrak{L}$ is the universal C^*-algebra generated by partial isometries S_α, $\alpha \in \Sigma$, and projections

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E_i^l, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$, satisfying the following operator relations:

$$\sum_{\alpha\in\Sigma} S_\alpha S_\alpha^* = 1, \tag{2.1}$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \quad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j)E_j^{l+1}, \tag{2.2}$$

$$S_\alpha S_\alpha^* E_i^l = E_i^l S_\alpha S_\alpha^*, \tag{2.3}$$

$$S_\alpha^* E_i^l S_\alpha = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j)E_j^{l+1}, \tag{2.4}$$

for $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$, $\alpha \in \Sigma$, where $V_l = \{v_1^l, \ldots, v_{m(l)}^l\}$,

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v_i^l,\ \lambda(e) = \alpha,\ t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise,} \end{cases}$$

for $i = 1, 2, \ldots, m(l)$, $j = 1, 2, \ldots, m(l+1)$, $\alpha \in \Sigma$. If $\mathfrak{L}$ satisfies λ-condition (I) and is λ-irreducible, the C^*-algebra $\mathcal{O}_\mathfrak{L}$ is simple and purely infinite [**33, 35**]. By the universality, the correspondence $\rho_z^\mathfrak{L}$, $z \in \mathbb{C}$, with $|z| = 1$ defined by $\rho_z^\mathfrak{L}(S_\alpha) = zS_\alpha$, $\rho_z^\mathfrak{L}(E_i^l) = E_i^l$ for $\alpha \in \Sigma$, $i = 1, \ldots, m(l)$, $l \in \mathbb{Z}_+$, yields an action $\rho^\mathfrak{L} : z \in \mathbb{T} \longrightarrow \rho_z^\mathfrak{L} \in \mathrm{Aut}(\mathcal{O}_\mathfrak{L})$ called gauge action.

Let $\mathcal{A}_{\mathfrak{L},l}$ be the C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by the projections E_i^l, $i = 1, \ldots, m(l)$. We denote by $\mathcal{A}_\mathfrak{L}$ the C^*-subalgebra of $\mathcal{O}_\mathfrak{L}$ generated by the all projections E_i^l, $i = 1, \ldots, m(l)$, $l \in \mathbb{Z}_+$. Let $\lambda_\mathfrak{L}$ be the positive operator on the algebra $\mathcal{A}_\mathfrak{L}$ defined by

$$\lambda_\mathfrak{L}(X) = \sum_{\alpha\in\Sigma} S_\alpha^* X S_\alpha \quad \text{for } X \in \mathcal{A}_\mathfrak{L}\,.$$

Let $\lambda_l : \mathcal{A}_{\mathfrak{L},l} \longrightarrow \mathcal{A}_{\mathfrak{L},l+1}$ be the restriction of $\lambda_\mathfrak{L}$ to $\mathcal{A}_{\mathfrak{L},l}$ and $\iota_l : \mathcal{A}_{\mathfrak{L},l} \hookrightarrow \mathcal{A}_{\mathfrak{L},l+1}$ the natural inclusion.

In this paper we will study the C^*-algebras $\mathcal{O}_{\mathfrak{L}^{Z_N}}$ associated with the canonical λ-graph systems $\mathfrak{L}^{Z_N}$ for non-sofic subshifts Z_N, $N \in \mathbb{N}$, defined in the next section. The C^*-algebras $\mathcal{O}_{\mathfrak{L}^{Z_N}}$ will be denoted by $\mathcal{O}_{Z_N}$.

3. *The C^*-algebra $\mathcal{O}_{Z_N}$*

For a finite set Σ, denote by Σ^* the set of all words of Σ. A (finite or infinite) collection $\mathcal{C}(\subset \Sigma^*)$ of words over Σ is said to be uniquely decipherable if whenever $\alpha_1\alpha_2\cdots\alpha_n = \gamma_1\gamma_2\cdots\gamma_m$ with $\alpha_i, \gamma_j \in \mathcal{C}$, then $n = m$ and $\alpha_i = \gamma_i$ for $i = 1, \ldots, n$. A uniquely decipherable set $\mathcal{C}$ is called a code. Blanchard and Hansel [**2**] have introduced the notion of a coded system. A subshift Λ is called a *coded system* if Λ is the closure of the set of biinfinite sequences obtained by freely concatenating the words in a code $\mathcal{C}$. It is denoted by $\Lambda_\mathcal{C}$. In this section, we will study certain coded systems written as Z_N, $N \in \mathbb{N}$. We fix a natural number $N \in \mathbb{N}$. Let Σ_N be a set $\{c, b, \alpha_1, \ldots, \alpha_N\}$ of symbols. The subshift Z_N is defined to be the subshift over Σ_N whose forbidden words are

$$\mathcal{F}_N = \{\alpha_i b^m c^k \alpha_j \mid i, j = 1, \ldots, N,\ m, k = 0, 1, \ldots \text{ with } m \neq k\}$$

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where the word $\alpha_i b^m c^k \alpha_j$ means $\alpha_i \underbrace{b \cdots b}_{m \text{ times}} \underbrace{c \cdots c}_{k \text{ times}} \alpha_j$. It is not a sofic subshift and hence not a Markov shift. Put

$$W_{b,c} = \{w \in \{b, c\}^* \mid w \neq b^m c^k \text{ for } m, k = 0, 1, \dots \text{ with } m \neq k\},$$

the set of all words of b, c that are not of the form $b^m c^k$ for $m, k = 0, 1, \dots$ with $m \neq k$.

PROPOSITION 3.1. *The set* $\mathcal{C}_N := \{\alpha_i w \mid i = 1, \dots, N,\ w \in W_{b,c}\}$ *is a code such that* $\Lambda_{\mathcal{C}_N} = Z_N$.

Proof. It is clear that $\mathcal{C}_N$ is a code. The inclusion relation $\Lambda_{\mathcal{C}_N} \subset Z_N$ is obvious. Conversely, the forbidden words $\mathcal{F}_N$ are not admissible in $\Lambda_{\mathcal{C}_N}$. Hence $\Lambda_{\mathcal{C}_N} = Z_N$. □

Define sequences of subsets of X_{Z_N} in the following way:

$$P_0 = \{c^k b^\infty \mid k \geq 0\} \cup \{b^k c^m b y \in X_{Z_N} \mid k \geq 0, m \geq 1, y \in X_{Z_N}\}$$

and

$$E_n = \{c^n \alpha_j y \in X_{Z_N} \mid y \in X_{Z_N},\ j = 1, \dots, N\},$$
$$Q_l = \bigcup_{n>l} E_n,$$
$$F_n = \{b^m c^{m+n} \alpha_j y \in X_{Z_N} \mid m \geq 1, y \in X_{Z_N},\ j = 1, \dots, N\},$$
$$R_l = \{b^m c^k \alpha_j y \in X_{Z_N} \mid m \geq 1, k \geq 0, m + n \neq k \text{ for } n = 0, 1, \dots, l, j = 1, \dots, N\}$$

for $l, n = 0, 1, \dots$.

LEMMA 3.2. *For each* $l \in \mathbb{N}$, *the space* X_{Z_N} *is decomposed as a disjoint union:*

$$X_{Z_N} = P_0 \bigsqcup_{n=0}^{l-1} E_n \sqcup Q_{l-1} \bigsqcup_{n=0}^{l-1} F_n \sqcup R_{l-1}.$$

This decomposition of X_{Z_N} *into* $2l + 3$*-components corresponds to the* l*-past equivalence classes of* X_{Z_N}.

Irreducibility and condition (I) for square matrices with entries in $\{0, 1\}$ have been generalized to λ-graph systems as λ-irreducibility and λ-condition (I) respectively [**35**].

LEMMA 3.3. *The canonical* λ*-graph system* $\mathfrak{L}^{Z_N}$ *satisfies* λ*-condition (I) and is* λ*-irreducible.*

Proof. For any natural number l, set $L = l + 4$. For a word $\nu \in B_{l+2}(Z_N)$, put $\mu = \nu cb \in B_L(Z_N)$. Then we have for $x \in X_{Z_N}$,

$$\begin{aligned}
\mu x \in E_k \quad & \text{if } \nu = c^k \alpha_j^{l-k+2} \text{ for some } j = 1, \dots, N \ (k = 0, 1, \dots, l-1), \\
\mu x \in Q_{l-1} \quad & \text{if } \nu = c^l \alpha_j^2 \text{ for some } j = 1, \dots, N, \\
\mu x \in F_k \quad & \text{if } \nu = b c^{k+1} \alpha_j^{l-k} \text{ for some } j = 1, \dots, N \ (k = 0, 1, \dots, l-1), \\
\mu x \in R_{l-1} \quad & \text{if } \nu = b^2 c \alpha_j^{l-1} \text{ for some } j = 1, \dots, N, \\
\mu x \in P_0 \quad & \text{if } \nu = cb^2.
\end{aligned}$$

Since the family $E_0, F_0, E_1, F_1, \dots E_{l-1}, F_{l-1}, Q_{l-1}, R_{l-1}, P_0$ represents the set of all l-past equivalence classes, the canonical λ-graph system $\mathfrak{L}^{Z_N}$ for the subshift Z_N is λ-irreducible. It is easy to see that $\mathfrak{L}^{Z_N}$ satisfies λ-condition (I). □

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Therefore we conclude the following by Lemma 3.3.

COROLLARY 3.4. *The C^*-algebra $\mathcal{O}_{Z_N}$ is simple and purely infinite.*

The C^*-algebra $\mathcal{O}_{Z_N}$ is generated by $N+2$ partial isometries S_c, S_b, $S_{\alpha_1}, \ldots, S_{\alpha_N}$. We set $T_1 := S_c$, $T_2 := S_b$, $S_j := S_{\alpha_j}$, $j = 1, \ldots, N$. Since, for any $x \in X_{Z_N}$, both cx and bx are admissible and hence belong to X_{Z_N}, both the operators T_1 and T_2 are isometries. Since, for any $x \in X_{Z_N}$, $\alpha_i x \in X_{Z_N}$ if and only if $\alpha_j x \in X_{Z_N}$ for $i, j = 1, \ldots, N$, one has $S_i^* S_i = S_j^* S_j$ for $i, j = 1, \ldots, N$. It has been proved in [**30**] that, for a subshift Λ in general, the associated C^*-algebra $\mathcal{O}_\Lambda$ can be realized as a universal C^*-algebra in the following way.

LEMMA 3.5. (cf. [**4, 30**]) *For a subshift Λ over $\Sigma = \{1, 2, \ldots, n\}$, the C^*-algebra $\mathcal{O}_\Lambda$ associated with Λ is the universal concrete C^*-algebra generated by n partial isometries S_i, $i = 1, 2, \ldots, n$, subject to the following relations.*

(i) $\sum_{j=1}^{n} S_j S_j^* = 1$.

(ii) $S_i^* S_i = 1 - \sum_{k=1}^{\infty} \sum_{\nu \in L_i^k} S_\nu S_\nu^*$, $i = 1, 2, \ldots, n$, *where*

$$L_i^k = \{\nu_1 \cdots \nu_k \in B_k(\Lambda) \mid i\nu_1 \cdots \nu_{k-1} \in B_k(\Lambda),\ i\nu_1 \cdots \nu_{k-1}\nu_k \notin B_{k+1}(\Lambda)\}.$$

The infinite sum of the right-hand side of the relation (ii) is taken under strong operator topology on a Hilbert space.

The above lemma means that there exists a representation of $\mathcal{O}_\Lambda$ in operators on a Hilbert space such that the canonical generators satisfy both the relations (i) and (ii). Conversely, if there exist n partial isometries on a Hilbert space satisfying the above relations, then there exists a canonical surjective homomorphism from $\mathcal{O}_\Lambda$ to the C^*-algebra generated by them.

We apply the preceding lemma to our C^*-algebra $\mathcal{O}_{Z_N}$. The following lemma is clear.

LEMMA 3.6. $L_{\alpha_i}^k = \{b^m c^l \alpha_j \mid m + l = k - 1,\ m \neq l,\ \text{and } j = 1, \ldots, N\}$ *for* $i = 1, \ldots, N$ *and* $k = 2, 3, \ldots$.

Thus we obtain the following theorem.

THEOREM 3.7. *For $N \in \mathbb{N}$, the C^*-algebra $\mathcal{O}_{Z_N}$ associated with the subshift Z_N is simple and purely infinite. It is the universal concrete C^*-algebra generated by two isometries T_1, T_2 and N partial isometries S_j, $j = 1, \ldots, N$, satisfying the following relations.*

(i) $T_1 T_1^* + T_2 T_2^* + \sum_{j=1}^{N} S_j S_j^* = 1$.

(ii) $S_i^* S_i = 1 - \sum_{k=1}^{\infty} \sum_{m=0, k \neq 2m}^{k} \sum_{j=1}^{N} T_{2^m 1^{k-m}} S_j S_j^* T_{2^m 1^{k-m}}^*$, $i = 1, \ldots, N$, *where* $T_{2^m 1^{k-m}}$ *denotes* $\underbrace{T_2 \cdots T_2}_{m \text{ times}} \underbrace{T_1 \cdots T_1}_{k-m \text{ times}}$.

The infinite sum of the right-hand side of the relation (ii) is taken under strong operator topology on a Hilbert space.

4. *The K-theory for $\mathcal{O}_{Z_N}$*

In this section, we compute the K-groups for $\mathcal{O}_{Z_N}$. Let $(M_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+}$ be the non-negative matrix system for the symbolic matrix system $(\mathcal{M}_{l,l+1}, I_{l,l+1})_{l \in \mathbb{Z}_+}$ of the

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canonical λ-graph system $\mathfrak{L}^{Z_N}$. Let $m(l)$ be the row size of the matrix $M_{l,l+1}$. The main tool is the following K-theory formulae proved in [**26, 33**]. The formulae hold for the C^*-algebras associated with λ-graph systems in general.

LEMMA 4.1. [**26, 33**]

(i) $K_0(\mathcal{O}_{Z_N}) = \varinjlim\{\mathbb{Z}^{m(l+1)}/(M_{l,l+1}^t - I_{l,l+1}^t)\mathbb{Z}^{m(l)}, \overline{I_{l,l+1}^t}\}$.

(ii) $K_1(\mathcal{O}_{Z_N}) = \varinjlim\{\mathrm{Ker}(M_{l,l+1}^t - I_{l,l+1}^t)$ *in* $\mathbb{Z}^{m(l)}, I_{l,l+1}^t\}$.

In (i) and (ii), $M_{l,l+1}^t$, $I_{l,l+1}^t$ *are the transposes of the matrices* $M_{l,l+1}$, $I_{l,l+1}$ *respectively, and*

$$\overline{I_{l,l+1}^t} : \mathbb{Z}^{m(l)}/(M_{l-1,l}^t - I_{l-1,l}^t)\mathbb{Z}^{m(l-1)} \longrightarrow \mathbb{Z}^{m(l+1)}/(M_{l,l+1}^t - I_{l,l+1}^t)\mathbb{Z}^{m(l)}$$

is the natural quotient map induced by $I_{l,l+1}^t$.

We will first find the symbolic matrix system $(\mathcal{M}_{l,l+1}, I_{l,l+1})_{l\in\mathbb{Z}_+}$ of the canonical λ-graph system $\mathfrak{L}^{Z_N}$. We have the following.

(1) $\alpha_i P_0 \subset P_0$ and $\alpha_i E_0, \alpha_i F_0 \subset E_0$ for $i = 1, \ldots, N$.

(2) $bP_0 \subset P_0$, $bE_0, bF_0, bQ_l, bR_l \subset R_{l-1}$ and $bE_n, bF_n \subset F_{n-1}$ for $n = 1, 2, \ldots, l$.

(3) $cP_0, cF_n, cR_l \subset P_0$ for $n = 0, 1, \ldots, l$, $cE_n \subset E_{n+1}$ for $n = 0, 1, \ldots, l-2$ and $cE_{l-1}, cE_l, cQ_l \subset Q_{l-1}$.

We also have

$$Q_{l-1} = E_l \sqcup Q_l \quad \text{and} \quad R_{l-1} = F_l \sqcup R_l.$$

Then we can represent the transposes $\mathcal{M}_{l,l+1}^t$ and $I_{l,l+1}^t$ of the matrices $\mathcal{M}_{l,l+1}$ and $I_{l,l+1}$ respectively as

$$\mathcal{M}_{l,l+1}^t = \begin{bmatrix}
b+c & \alpha_1+\cdots+\alpha_N & 0 & \cdots & & & & & & & & 0 & 0 \\
0 & \alpha_1+\cdots+\alpha_N & 0 & c & 0 & \cdots & & & & & & 0 & b \\
c & \alpha_1+\cdots+\alpha_N & 0 & \cdots & & & & & & & & 0 & b \\
0 & 0 & b & 0 & 0 & c & 0 & \cdots & & & & & 0 \\
c & 0 & b & 0 & \cdots & & & & & & & & 0 \\
0 & 0 & 0 & 0 & b & 0 & 0 & c & 0 & \cdots & & & 0 \\
c & 0 & 0 & 0 & b & 0 & \cdots & & & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & c & 0 & \cdots & 0 \\
c & 0 & 0 & 0 & 0 & 0 & b & 0 & \cdots & & & & 0 \\
\vdots & & & & & & \ddots & \ddots & \ddots & & \ddots & & \vdots \\
\vdots & & & & & & & \ddots & \ddots & \ddots & & \ddots & \vdots \\
0 & \cdots & & & & & & & 0 & b & 0 & 0 & c & 0 \\
c & 0 & \cdots & & & & & & 0 & b & 0 & 0 & 0 & 0 \\
0 & \cdots & & & & & & & & 0 & b & c & 0 \\
c & 0 & \cdots & & & & & & & 0 & b & 0 & 0 \\
0 & \cdots & & & & & & & & & 0 & c & b \\
c & 0 & \cdots & & & & & & & & 0 & 0 & b
\end{bmatrix},$$

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and

$$I^t_{l,l+1} = \begin{bmatrix}
1 & 0 & \cdots & & & & & & & & & & 0 \\
0 & 1 & 0 & \cdots & & & & & & & & \cdots & 0 \\
0 & 0 & 1 & 0 & \cdots & & & & & & & & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & & & & & & & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & & & & & & & & & \ddots & \ddots & \ddots & \vdots \\
\vdots & & & & & & & & & & \ddots & \ddots & \vdots \\
\vdots & & & & & & & & & & & \ddots & \vdots \\
0 & \cdots & & & & & & & & \cdots & 0 & 1 & 0 \\
0 & \cdots & & & & & & & & & & 0 & 1 \\
0 & \cdots & & & & & & & & \cdots & 0 & 1 & 0 \\
0 & \cdots & & & & & & & & & & 0 & 1
\end{bmatrix}$$

along the ordered basis for the row

$$(P_0, E_0, F_0, E_1, F_1, \ldots, E_l, F_l, Q_l, R_l),$$

and the column

$$(P_0, E_0, F_0, E_1, F_1, \ldots, E_{l-1}, F_{l-1}, Q_{l-1}, R_{l-1}).$$

The commutative C^*-algebras $\mathcal{A}_{\mathfrak{L}^{Z_N},l}$ and $\mathcal{A}_{\mathfrak{L}^{Z_N}}$ are denoted by $\mathcal{A}_{Z_N,l}$ and $\mathcal{A}_{Z_N}$ respectively. The set of the minimal projections $\mathcal{A}_{Z_N,l}$ corresponds to the set of all l-past equivalence classes $P_0, E_0, F_0, E_1, F_1, \ldots, E_{l-1}, F_{l-1}, Q_{l-1}, R_{l-1}$, which are denoted by $p_0, e_0, f_0, e_1, f_1, \ldots, e_{l-1}, f_{l-1}, q_{l-1}, r_{l-1}$ respectively. Hence $\dim(\mathcal{A}_l) = m(l) = 2l + 3$ for $l \geq 2$. We can then represent the induced matrix on the K_0-groups

$$M^t_{l,l+1}(= \lambda_{l*}) : K_0(\mathcal{A}_{Z_N,l}) = \mathbb{Z}^{2l+3} \to K_0(\mathcal{A}_{Z_N,l+1}) = \mathbb{Z}^{2l+5}$$

of the operator λ_l to be the matrix $M^t_{l,l+1}$ as follows:

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$$M^t_{l,l+1} = \begin{bmatrix}
2 & N & 0 & \cdots & & & & & & & & & 0 & 0 \\
0 & N & 0 & 1 & 0 & \cdots & & & & & & & 0 & 1 \\
1 & N & 0 & \cdots & & & & & & & & & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & & & & & & 0 \\
1 & 0 & 1 & 0 & \cdots & & & & & & & & & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & & & & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & \cdots & & & & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots & & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & & & 0 \\
\vdots & & & & & & \ddots & \ddots & \ddots & & & \ddots & & \vdots \\
\vdots & & & & & & & \ddots & \ddots & \ddots & & \ddots & & \vdots \\
0 & \cdots & & & & & & & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & \cdots & & & & & & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & \cdots & & & & & & & & 0 & 1 & 1 & 0 \\
1 & 0 & \cdots & & & & & & & 0 & 1 & 0 & 0 \\
0 & \cdots & & & & & & & & & 0 & 1 & 1 \\
1 & 0 & \cdots & & & & & & & & 0 & 0 & 1
\end{bmatrix}.$$

The natural inclusion ι_l from $\mathcal{A}_{Z_N,l}$ to $\mathcal{A}_{Z_N,l+1}$ induces the matrix

$$I^t_{l,l+1}(=\iota_{l*}) : K_0(\mathcal{A}_{Z_N,l}) = \mathbb{Z}^{2l+3} \to K_0(\mathcal{A}_{Z_N,l+1}) = \mathbb{Z}^{2l+5}.$$

We then see

$$M^t_{l,l+1} - I^t_{l,l+1} = \begin{bmatrix}
1 & N & 0 & \cdots & & & & & & & & & 0 & 0 \\
0 & N-1 & 0 & 1 & 0 & \cdots & & & & & & & 0 & 1 \\
1 & N & -1 & 0 & \cdots & & & & & & & & 0 & 1 \\
0 & 0 & 1 & -1 & 0 & 1 & 0 & \cdots & & & & & & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & \cdots & & & & & & & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & \cdots & & & & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & \cdots & & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & \cdots & & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & \cdots & & & 0 \\
\vdots & & & & & & & \ddots & \ddots & \ddots & & \ddots & & \vdots \\
\vdots & & & & & & & & \ddots & \ddots & \ddots & & \ddots & \vdots \\
0 & \cdots & & & & & & & & 0 & 1 & -1 & 0 & 1 & 0 \\
1 & 0 & \cdots & & & & & & & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & \cdots & & & & & & & & & 0 & 1 & 0 & 0 \\
1 & 0 & \cdots & & & & & & & & 0 & 1 & 0 & -1 \\
0 & \cdots & & & & & & & & & & 0 & 0 & 1 \\
1 & 0 & \cdots & & & & & & & & & & & 0
\end{bmatrix},$$

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that is, for $i = 1, 2, \ldots, 2l+5$, $j = 1, 2, \ldots, 2l+3$,

$$(M^t_{l,l+1} - I^t_{l,l+1})(i, j) = \begin{cases} N & \text{if } j = 2,\ i = 1, 3, \\ N-1 & \text{if } i = j = 2, \\ 1 & \text{if } j = 1,\ i = 1, 3, 5, \ldots, 2l+5, \\ 1 & \text{if } j = 2l+3,\ i = 2, 3, 2l+4, \\ 1 & \text{if } j = i+2,\ i = 2, 4, 6, \ldots, 2l, \\ 1 & \text{if } i = j+1,\ j+2,\ j = 3, 5, 7, \ldots, 2l+1, \\ -1 & \text{if } i = j,\ i = 3, 4, 5, \ldots, 2l+1, 2l+3. \end{cases}$$

We easily see the following lemma.

LEMMA 4.2. *We have*

$$\mathrm{Ker}(M^t_{l,l+1} - I^t_{l,l+1}) = 0 \quad \textit{for } 2 \le l \in \mathbb{N}.$$

Thus we have the following proposition, by Lemma 4.1.

PROPOSITION 4.3. *We have*

$$K_1(\mathcal{O}_{Z_N}) \cong 0.$$

We will next compute $K_0(\mathcal{O}_{Z_N})$.

Let $B_{l,l+1}$ be the $(2l+5) \times (2l+3)$ matrix defined by setting, for $i = 1, 2, \ldots, 2l+5$, $j = 1, 2, \ldots, 2l+3$,

$$B_{l,l+1}(i, j) = \begin{cases} N & \text{if } j = 2,\ i = 1, \\ 1 & \text{if } j = 1,\ i = 1, 5, 7, 9, \ldots, 2l+5, \\ 1 & \text{if } j = 2l+3,\ i = 3, 2l+4, \\ 1 & \text{if } j = i+2,\ i = 2, 4, 6, \ldots, 2l, \\ 1 & \text{if } i = j+2,\ j = 3, 5, 7, \ldots, 2l+1, \\ -1 & \text{if } i = j,\ i = 3, 5, 7, \ldots, 2l+1, 2l+3, \end{cases}$$

which is written as

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$$B_{l,l+1} = \begin{bmatrix}
1 & N & 0 & \cdots & & & & & & & & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & & & & & & 0 & 0 \\
0 & 0 & -1 & 0 & \cdots & & & & & & & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & & & 0 \\
1 & 0 & 1 & 0 & -1 & 0 & \cdots & & & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & & & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & \cdots & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & \cdots & & 0 \\
\vdots & & & & & & \ddots & \ddots & \ddots & & & \ddots & \vdots \\
\vdots & & & & & & & \ddots & \ddots & \ddots & \ddots & & \vdots \\
0 & \cdots & & & & & & & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & \cdots & & & & & & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & \cdots & & & & & & & & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & \cdots & & & & & & & 0 & 1 & 0 & -1 \\
0 & \cdots & & & & & & & & & 0 & 0 & 1 \\
1 & 0 & \cdots & & & & & & & & & & 0
\end{bmatrix}.$$

Take a $(2l+3) \times (2l+3)$ matrix P_l by setting, for $i, j = 1, \ldots, 2l+3$,

$$P_l(i, j) = \begin{cases} 1 & \text{if } i = j = 1, 2, 3, \ldots, 2l+3, \\ -1 & \text{if } j = 1, i = 2, 3. \end{cases}$$

We then have

$$P_{l+1}(M_{l,l+1}^t - I_{l,l+1}^t)\mathbb{Z}^{2l+3} = B_{l,l+1}\mathbb{Z}^{2l+3}.$$

We denote by $\bar{P}_{l+1}$ the induced isomorphism from $\mathbb{Z}^{2l+5}/(M_{l,l+1}^t - I_{l,l+1}^t)\mathbb{Z}^{2l+3}$ onto $\mathbb{Z}^{2l+5}/B_{l,l+1}\mathbb{Z}^{2l+3}$. The following lemma is clear.

LEMMA 4.4. *The diagram*

$$\begin{array}{ccc}
\mathbb{Z}^{2l+3}/(M_{l-1,l}^t - I_{l-1,l}^t)\mathbb{Z}^{2l+1} & \xrightarrow{\overline{I_{l,l+1}^t}} & \mathbb{Z}^{2l+5}/(M_{l,l+1}^t - I_{l,l+1}^t)\mathbb{Z}^{2l+3} \\
\downarrow{\scriptstyle \bar{P}_l} & & \downarrow{\scriptstyle \bar{P}_{l+1}} \\
\mathbb{Z}^{2l+3}/B_{l-1,l}\mathbb{Z}^{2l+1} & \xrightarrow{\overline{I_{l,l+1}^t}} & \mathbb{Z}^{2l+5}/B_{l,l+1}\mathbb{Z}^{2l+3}
\end{array}$$

is commutative.

For an integer n, we denote by $q(n) \in \mathbb{Z}$ the quotient of n by N and by $r(n) \in \{0, 1, \ldots, N-1\}$ its residue such as $n = q(n)N + r(n)$.

LEMMA 4.5. *Fix $l = 2, 3, \ldots$. For*

$$z = \begin{bmatrix} z_1 \\ \vdots \\ z_{2l+5} \end{bmatrix} \in \mathbb{Z}^{2l+5},$$

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put inductively

$$x_1 = z_{2l+5},$$
$$x_2 = q(z_1 - z_{2l+5}),$$
$$x_{2k+2} = z_{2k} \quad \textit{for } k = 1, 2, \ldots, l,$$
$$x_{2l+3} = z_{2l+4},$$
$$x_{2k-1} = z_{2k+1} - z_{2l+5} + x_{2k+1} \quad \textit{for } k = 2, 3, \ldots, l+1,$$

and set

$$r_{l,l+1}(z) = r(z_1 - z_{2l+5}) \in \{0, 1, \ldots, N-1\},$$
$$\eta_{l,l+1}(z) = z_3 + z_5 + \cdots + z_{2l+1} + z_{2l+2} - z_{2l+4},$$
$$\psi_{l,l+1}(z) = -z_{2l+2} + z_{2l+3} + z_{2l+4} - z_{2l+5}.$$

Then we have

$$\begin{bmatrix} z_1 \\ \vdots \\ z_{2l+5} \end{bmatrix} = B_{l,l+1} \begin{bmatrix} x_1 \\ \vdots \\ x_{2l+3} \end{bmatrix} + \begin{bmatrix} r_{l,l+1}(z) \\ 0 \\ \eta_{l,l+1}(z) \\ 0 \\ \vdots \\ 0 \\ \psi_{l,l+1}(z) \\ 0 \\ 0 \end{bmatrix}.$$

Proof. For $[y_i]_{i=1}^{2l+3} \in \mathbb{Z}^{2l+3}$, one sees

$$B_{l,l+1} \begin{bmatrix} y_1 \\ \vdots \\ y_{2l+3} \end{bmatrix} = \begin{bmatrix} y_1 + Ny_2 \\ y_4 \\ -y_3 + y_{2l+3} \\ y_6 \\ y_1 + y_3 - y_5 \\ \vdots \\ y_{2k+2} \\ y_1 + y_{2k-1} - y_{2k+1} \\ \vdots \\ y_{2l+2} \\ y_1 + y_{2l-1} - y_{2l+1} \\ y_{2l+1} \\ y_1 + y_{2l+1} - y_{2l+3} \\ y_{2l+3} \\ y_1 \end{bmatrix}.$$

Hence the assertion is straightforward. □

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The following lemma is also obvious.

LEMMA 4.6. *For* $z = [z_i]_{i=1}^{2l+5} \in \mathbb{Z}^{2l+5}$, *one has*

$$r_{l+1}(z) = 0 \text{ in } \{0, 1, \ldots, N-1\} \quad \text{and} \quad \eta_{l+1}(z) = \psi_{l+1}(z) = 0 \text{ in } \mathbb{Z}$$

if and only if there exists $y = [y_i]_{i=1}^{2l+3} \in \mathbb{Z}^{2l+3}$ *such that* $z = B_{l,l+1}(y)$.

LEMMA 4.7. *The map*

$$\xi_{l+1} : [z_i]_{i=1}^{2l+5} \in \mathbb{Z}^{2l+5} \longrightarrow (r_{l,l+1}(z), \eta_{l,l+1}(z), \psi_{l,l+1}(z)) \in \{0, 1, \ldots, N-1\} \oplus \mathbb{Z} \oplus \mathbb{Z}$$

induces an isomorphism from $\mathbb{Z}^{2l+5}/B_{l,l+1}\mathbb{Z}^{2l+3}$ *onto* $\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Proof. It suffices to show the surjectivity of ξ_{l+1}. For $(g, m, k) \in \{0, 1, \ldots, N-1\} \oplus \mathbb{Z} \oplus \mathbb{Z}$, put $z = [g, 0, m, 0, \ldots, 0, k, 0, 0]^t \in \mathbb{Z}^{2l+5}$. One then sees that

$$r_{l,l+1}(z) = g, \quad \eta_{l,l+1}(z) = m, \quad \psi_{l,l+1}(z) = k. \qquad \square$$

We denote by $\bar{\xi}_{l+1}$ the above isomorphism from $\mathbb{Z}^{2l+5}/B_{l,l+1}\mathbb{Z}^{2l+3}$ onto $\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ induced by ξ_{l+1}.

LEMMA 4.8. *The diagram*

$$\begin{array}{ccc} \mathbb{Z}^{2l+3}/B_{l-1,l}\mathbb{Z}^{2l+1} & \xrightarrow{\overline{I^t_{l,l+1}}} & \mathbb{Z}^{2l+5}/B_{l,l+1}\mathbb{Z}^{2l+3} \\ \downarrow{\scriptstyle \bar{\xi}_l} & & \downarrow{\scriptstyle \bar{\xi}_{l+1}} \\ \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{L} & \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \end{array}$$

is commutative, where

$$L = \mathrm{id} \oplus \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Proof. For $z = [z_i]_{i=1}^{2l+3} \in \mathbb{Z}^{2l+3}$, one sees that

$$\begin{aligned} &r_{l,l+1}(I^t_{l,l+1}(z)) = r_{l-1,l}, \\ &\eta_{l,l+1}(I^t_{l,l+1}(z)) = \eta_{l-1,l}(z) + \psi_{l-1,l}(z), \\ &\psi_{l,l+1}(I^t_{l,l+1}(z)) = 0. \end{aligned}$$

Hence the above diagram is commutative. $\square$

Therefore we conclude the following theorem.

THEOREM 4.9. $K_0(\mathcal{O}_{Z_N}) \cong \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}$, *and the class* [1] *of the unit* 1 *of the algebra* $\mathcal{O}_{Z_N}$ *in* $K_0(\mathcal{O}_{Z_N})$ *is* $0 \oplus 0$ *in* $\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}$.

Proof. By Lemmas 4.1, 4.4 and 4.8, one sees that

$$\begin{aligned} K_0(\mathcal{O}_{Z_N}) &= \varinjlim\{\mathbb{Z}^{2l+5}/(M^t_{l,l+1} - I^t_{l,l+1})\mathbb{Z}^{2l+3}, \overline{I^t_{l,l+1}}\} \\ &= \varinjlim\{\mathbb{Z}^{2l+5}/B_{l,l+1}\mathbb{Z}^{2l+3}, \overline{I^t_{l,l+1}}\} \\ &= \varinjlim\{\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, L\} \\ &\cong \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

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The class of the unit [1] in $K_0(\mathcal{O}_{Z_N})$ corresponds to [1] in $K_0(\mathcal{A}_{\mathfrak{L},l})$ which is represented by the vector $[1, \dots, 1] \in \mathbb{Z}^{2l+3}$. Since

$$r_{l,l+1}(P_l[1, \dots, 1]) = 0$$

and

$$\eta_{l,l+1}(P_l[1, \dots, 1]) = \psi_{l,l+1}(P_l[1, \dots, 1]) = 0,$$

we see that $\xi_{l+1}([1]) = (0, 0, 0)$. Thus the projection [1] in $K_0(\mathcal{O}_{Z_N})$ represents $0 \oplus 0$ in $\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}$. □

By [**8**], we know that $K_0(\mathcal{O}_\infty) = \mathbb{Z}$, $K_1(\mathcal{O}_\infty) = 0$ and the position of [1] in $K_0(\mathcal{O}_\infty)$ is 1 in $\mathbb{Z}$. By the classification theorem of purely infinite simple C^*-algebras proved by Kirchberg [**15**] and Philips [**40**] (cf. [**41, 42**]), we have the following corollary.

COROLLARY 4.10.

(i) *The C^*-algebra $\mathcal{O}_{Z_N}$ for $N \geq 2$ is not stably isomorphic to any Cuntz–Krieger algebra nor to the Cuntz algebra $\mathcal{O}_\infty$ of order infinity.*

(ii) *The C^*-algebra $\mathcal{O}_{Z_N}$ for $N = 1$ is not isomorphic to any Cuntz–Krieger algebra nor to the Cuntz algebra $\mathcal{O}_\infty$ of order infinity but is stably isomorphic to $\mathcal{O}_\infty$. In fact, $\mathcal{O}_{Z_1}$ is isomorphic to $(\mathcal{O}_\infty)_{1-s_1 s_1^*}$ where $\mathcal{O}_\infty$ is generated by a sequence s_i, $i = 1, 2, \dots$, of isometries with mutually orthogonal ranges.*

5. *KMS-states for gauge action and topological entropy*

In this section, we study a KMS-state for gauge action on $\mathcal{O}_{Z_N}$ and compute the topological entropy for the subshift Z_N.

In studying the KMS-state for gauge action on the C^*-algebras associated with subshifts, the lemma below proved in [**37**] plays a crucial role (cf. [**23, 39**]).

LEMMA 5.1. [**37**] *For a λ-graph system $\mathfrak{L}$, if a state φ on $\mathcal{O}_\mathfrak{L}$ is a KMS-state for gauge action on $\mathcal{O}_\mathfrak{L}$ at the inverse temperature* $\log \beta$ *for some $1 < \beta \in \mathbb{R}$, it satisfies the condition $\varphi \circ \lambda_\mathfrak{L} = \beta\varphi$ on $\mathcal{A}_\mathfrak{L}$. Conversely, a state φ on $\mathcal{A}_\mathfrak{L}$ satisfying the condition $\varphi \circ \lambda_\mathfrak{L} = \beta\varphi$ can be uniquely extended to a KMS-state for gauge action on $\mathcal{O}_\mathfrak{L}$.*

We will apply this lemma to our C^*-algebra $\mathcal{O}_{Z_N}$ for the subshift Z_N. Let us find a state on $\mathcal{A}_{Z_N}$ satisfying the condition $\varphi \circ \lambda_{Z_N} = \beta\varphi$ for some real number β with $1 < \beta \in \mathbb{R}$, where λ_{Z_N} denotes $\lambda_{\mathfrak{L}^{Z_N}}$.

Let $\hat{p}_0, \hat{e}_0, \hat{f}_0, \hat{e}_1, \hat{f}_1, \dots, \hat{e}_l, \hat{f}_l, \hat{q}_l, \hat{r}_l$ be real numbers for $l \in \mathbb{N}$ satisfying the condition

$$\hat{p}_0 + \sum_{j=0}^{l} (\hat{e}_j + \hat{f}_j) + \hat{q}_l + \hat{r}_l = 1. \tag{5.1}$$

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We consider the following equations:

$$M^t_{l,l+1}\begin{bmatrix}\hat{p}_0\\ \hat{e}_0\\ \hat{f}_0\\ \hat{e}_1\\ \hat{f}_1\\ \hat{e}_2\\ \hat{f}_2\\ \vdots\\ \hat{e}_{l-1}\\ \hat{f}_{l-1}\\ \hat{e}_l\\ \hat{f}_l\\ \hat{q}_l\\ \hat{r}_l\end{bmatrix} = \beta\begin{bmatrix}\hat{p}_0\\ \hat{e}_0\\ \hat{f}_0\\ \hat{e}_1\\ \hat{f}_1\\ \hat{e}_2\\ \hat{f}_2\\ \vdots\\ \hat{e}_{l-1}\\ \hat{f}_{l-1}\\ \hat{q}_{l-1}\\ \hat{r}_{l-1}\end{bmatrix} \quad \text{and} \quad I^t_{l,l+1}\begin{bmatrix}\hat{p}_0\\ \hat{e}_0\\ \hat{f}_0\\ \hat{e}_1\\ \hat{f}_1\\ \hat{e}_2\\ \hat{f}_2\\ \vdots\\ \hat{e}_{l-1}\\ \hat{f}_{l-1}\\ \hat{e}_l\\ \hat{f}_l\\ \hat{q}_l\\ \hat{r}_l\end{bmatrix} = \begin{bmatrix}\hat{p}_0\\ \hat{e}_0\\ \hat{f}_0\\ \hat{e}_1\\ \hat{f}_1\\ \hat{e}_2\\ \hat{f}_2\\ \vdots\\ \hat{e}_{l-1}\\ \hat{f}_{l-1}\\ \hat{q}_{l-1}\\ \hat{r}_{l-1}\end{bmatrix}$$

for a real number $\beta > 1$. This means that

$$2\hat{p}_0 + \sum_{i=0}^{l} \hat{f}_i + \hat{r}_l = \beta\hat{p}_0, \tag{5.2}$$

$$N(\hat{p}_0 + \hat{e}_0 + \hat{f}_0) = \beta\hat{e}_0, \tag{5.3}$$

$$\hat{e}_n + \hat{f}_n = \beta\hat{f}_{n-1}, \quad n = 1, 2, \ldots, l, \tag{5.4}$$

$$\hat{e}_{n-1} = \beta\hat{e}_n, \quad n = 1, 2, \ldots, l-1, \tag{5.5}$$

$$\hat{e}_{l-1} + \hat{e}_l + \hat{q}_l = \beta\hat{q}_{l-1}, \tag{5.6}$$

$$\hat{e}_0 + \hat{f}_0 + \hat{q}_l + \hat{r}_l = \beta\hat{r}_{l-1} \tag{5.7}$$

and the equations

$$\hat{e}_l + \hat{q}_l = \hat{q}_{l-1}, \tag{5.8}$$

$$\hat{f}_l + \hat{r}_l = \hat{r}_{l-1}. \tag{5.9}$$

These equations can be solved in the following way by straightforward calculation. Put

$$g_N(\beta) = (\beta-1)\beta\{(2N-1)\beta - (N+1)\},$$
$$h_N(\beta) = (\beta+1)\{(\beta-N)(\beta-2)(\beta-1)^2\beta + (N-2)(\beta-1)\beta - N(\beta-1) - 1\}$$

and

$$F_N(\beta) = h_N(\beta) - g_N(\beta).$$

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LEMMA 5.2.

$$\hat{p}_0 = \frac{1}{N}\{(\beta - N)\hat{e}_0 - N\hat{f}_0\}, \tag{5.10}$$

$$(\beta - 1)(\beta + 1)g_N(\beta)\hat{f}_0 = h_N(\beta)\hat{e}_0, \tag{5.11}$$

$$\hat{e}_n = \frac{\beta - 2}{\beta^{n+1}}, \tag{5.12}$$

$$\hat{f}_n = \beta^n \hat{f}_0 - (\beta^{n-1}\hat{e}_1 + \beta^{n-2}\hat{e}_2 + \cdots + \beta\hat{e}_{n-1} + \hat{e}_n), \tag{5.13}$$

$$\hat{q}_n = \frac{\beta - 2}{\beta^{n+1}(\beta - 1)}, \tag{5.14}$$

$$\hat{r}_n = (\beta - 2)\hat{p}_0 - (\hat{f}_0 + \hat{f}_1 + \cdots + \hat{f}_n), \quad n = 0, 1, 2 \ldots l. \tag{5.15}$$

We henceforth assume that the real numbers $\hat{p}_0, \hat{e}_0, \hat{f}_0, \hat{e}_1, \hat{f}_1, \ldots, \hat{e}_l, \hat{f}_l, \hat{q}_l, \hat{r}_l$ are all non-negative for all $l \in \mathbb{N}$.

LEMMA 5.3.

(i) $\beta > 2$ *and* $\hat{e}_n, \hat{q}_n > 0$ *for* $n = 0, 1, \ldots, l$.

(ii) $\beta > N$.

(iii) $g_N(\beta) > 0$.

Proof. (i) Since $\hat{e}_1 \geq 0$, one sees that $\beta \geq 2$ by (5.12). Suppose that $\beta = 2$. One has $\hat{e}_n = \hat{q}_n = 0$ for $n = 0, 1, \ldots, l$. Equation (5.15) implies that $\hat{r}_n = \hat{f}_n = 0$ for $n = 0, 1, \ldots, l$. It follows that $\hat{p}_0 = 0$ by (5.10), a contradiction to (5.1). Hence $\beta > 2$ and $\hat{e}_n, \hat{q}_n > 0$ for $n = 0, 1, \ldots, l$.

(ii) Suppose next that $\beta \leq N$. Equation (5.10) implies that $\hat{p}_0 = 0$, $\hat{f}_0 = 0$. Then (5.13) implies that $\hat{f}_n < 0$, which contradicts the hypothesis.

(iii) As

$$\frac{g_N(\beta)}{(\beta - 1)\beta} = (2N - 1)\beta - (N + 1) > (2N - 1)N - (N + 1) = (N - 1)^2 + N^2 - 2, \tag{5.16}$$

one sees that $g_N(\beta) > 0$ for $N \geq 2$ and $g_1(\beta) = (\beta - 2)(\beta - 1)\beta > 0$. □

LEMMA 5.4. *The conditions* $\hat{f}_n \geq 0$ *for all* $n = 0, 1, \ldots$ *imply* $F_N(\beta) \geq 0$.

Proof. Equations (5.12) and (5.13) imply

$$\hat{f}_n = \beta^n \left\{ \hat{f}_0 - \frac{1}{\beta^2 - 1}\left(1 - \frac{1}{\beta^{2n}}\right)\hat{e}_0 \right\} \tag{5.17}$$

so that the conditions $\hat{f}_n \geq 0$ for all $n = 0, 1, \ldots$ imply

$$(\beta^2 - 1)\hat{f}_0 - \hat{e}_0 \geq -\frac{1}{\beta^{2n}}\hat{e}_0 \quad \text{for all } n = 0, 1, \ldots. \tag{5.18}$$

As $g_N(\beta) > 0$, by (5.11), the above equalities imply the inequality $h_N(\beta) - g_N(\beta) \geq 0$. □

LEMMA 5.5. *The conditions* $\hat{r}_n \geq 0$ *for all* $n = 0, 1, \ldots$ *imply* $F_N(\beta) \leq 0$.

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Proof. By (5.12) and (5.13), one sees that

$$\sum_{k=0}^{n} \hat{f}_k = \frac{\beta^{n+1}-1}{\beta-1}\hat{f}_0 + \left\{\frac{1}{(\beta-1)^2} - \frac{1}{(\beta-1)^2(\beta+1)}\left(\beta^{n+1}+\frac{1}{\beta^n}\right)\right\}\hat{e}_0.$$

Equation (5.15) implies that

$$\begin{aligned}
&(\beta-1)^2(\beta+1)g_N(\beta)\hat{r}_n \\
&= \left[\frac{\beta-N}{N}(\beta-2)(\beta-1)^2(\beta+1) - \left\{(\beta+1) - \left(\beta^{n+1}+\frac{1}{\beta^n}\right)\right\}\right]g_N(\beta)\hat{e}_0 \\
&\quad - [(\beta-2)(\beta-1)^2(\beta+1) + (\beta-1)(\beta+1)(\beta^{n+1}-1)]g_N(\beta)\hat{f}_0 \\
&= \left[\frac{\beta-N}{N}(\beta-2)(\beta-1)^2(\beta+1) - \left\{(\beta+1) - \left(\beta^{n+1}+\frac{1}{\beta^n}\right)\right\}\right]g_N(\beta)\hat{e}_0 \\
&\quad - [(\beta-2)(\beta-1) + (\beta^{n+1}-1)]h_N(\beta)\hat{e}_0 \\
&= -F_N(\beta)\beta^{n+1}\hat{e}_0 \\
&\quad + \left[\frac{\beta-N}{N}(\beta-2)(\beta-1)^2(\beta+1) - \left\{(\beta+1) - \frac{1}{\beta^n}\right\}\right]g_N(\beta)\hat{e}_0 \\
&\quad - [(\beta-2)(\beta-1)-1]h_N(\beta)\hat{e}_0.
\end{aligned}$$

Since

$$\liminf_{n\to\infty} \frac{(\beta-1)^2(\beta+1)g_N(\beta)\hat{r}_n}{\beta^{n+1}} = \liminf_{n\to\infty} \frac{(\beta-1)^2(\beta+1)g_N(\beta)\hat{r}_n}{\beta^{n+1}} = -F_N(\beta)\hat{e}_0 \geq 0$$

and $\hat{e}_0 > 0$, we see that $F_N(\beta) \leq 0$. □

Therefore we have the following corollary.

COROLLARY 5.6. *Suppose that* $\hat{p}_0, \hat{e}_n, \hat{f}_n, \hat{q}_n, \hat{r}_n$ *are all non-negative for all* $n = 0, 1, \ldots$. *Then* $\beta > N$ *and* $F_N(\beta) = 0$.

Conversely, for $N \geq 2$, we have the following lemma.

LEMMA 5.7. *For* $\beta > N$, $F_N(\beta) = 0$ *implies* $\hat{f}_n, \hat{r}_n > 0$ *for all* $n = 0, 1, \ldots$.

Proof. Assume that $F_N(\beta) = 0$. As $\beta > N$, (5.16) implies $g_N(\beta) > 0$. Since $h_N(\beta) = g_N(\beta)$, the inequality (5.18) holds, which means $\hat{f}_n > 0$ for all $n = 0, 1, \ldots$ by (5.17).

We will next show that $\hat{r}_n > 0$ for all $n = 0, 1, \ldots$. By the equality $g_N(\beta) = h_N(\beta)$, one has

$$\begin{aligned}
&(\beta-1)^2(\beta+1)g_N(\beta)\hat{r}_n \\
&= \left[\frac{\beta-N}{N}(\beta-2)(\beta-1)^2(\beta+1) - \left\{(\beta+1) - \frac{1}{\beta^n}\right\}\right]g_N(\beta)\hat{e}_0 \\
&\quad - [(\beta-2)(\beta-1)-1]g_N(\beta)\hat{e}_0.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\left[\frac{\beta-N}{N}(\beta-2)(\beta-1)^2(\beta+1) - (\beta+1)\right] - [(\beta-2)(\beta-1)-1] \\
&= \frac{1}{N\beta}(\beta^3 - N + 1) > 0,
\end{aligned}$$

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so that $(\beta-1)^2(\beta+1)g_N(\beta)\hat{r}_n$ is positive. As $g_N(\beta)>0$, one sees that $\hat{r}_n>0$ for all $n=0, 1, \ldots$. □

Therefore we have the following corollary.

COROLLARY 5.8. *For $\beta>N$, $F_N(\beta)=0$ if and only if $\hat{p}_0, \hat{e}_n, \hat{f}_n, \hat{q}_n, \hat{r}_n$ are all non-negative for all $n=0, 1, \ldots$. In this case, they are all positive.*

Proof. For $N=1$, the statement has been shown in [**27**]. □

We note that the identity

$$F_N(\beta)=\beta^6-(N+3)\beta^5+(3N+1)\beta^4-2(N-1)\beta^3-(N+2)\beta^2+N-1$$

holds. For $N=1$, the above identity goes to

$$F_1(\beta)=\beta^6-4\beta^5+4\beta^4-3\beta^2.$$

The unique positive solution of $F_1(\beta)=0$ is $\beta=1+\sqrt{1+\sqrt{3}}=2.652\,891\cdots$ [**27**].

We will next study positive solutions of the equations

$$F_N(\beta)=0 \quad \text{for } N\geq 2.$$

LEMMA 5.9.

(i) $F_2(3)<0<F_2(4)$.

(ii) $F_N(N)<0<F_N(N+1)$ *for* $N\geq 3$.

Proof. By the identities

$$F_N(N)=-(N^2+1)(N^2-N+1),$$
$$F_N(N+1)=\{N(N-2)(N+1)-1\}(N+1)^2+N-1$$

one sees that (iii) holds. □

The following lemma is straightforward.

LEMMA 5.10.

(i) *For $N=2$, $F_2'(\beta)>0$ for all $\beta\geq 3$. Hence $F_2(\beta)=0$ has a unique solution in $[3, \infty)$.*

(ii) *For $N=3$, $F_3'(3)<0$ and $F_3''(\beta)>0$ for all $\beta\geq 3$. Hence $F_3(\beta)=0$ has a unique solution in $[3, \infty)$.*

(iii) *For $N\geq 4$, $F_N'(\beta)>0$ for all $\beta\geq N$. Hence $F_N(\beta)=0$ has a unique solution in $[N, \infty)$.*

Therefore we have the following proposition.

PROPOSITION 5.11. *For $N\geq 1$, the equation $F_N(\beta)=0$ has a unique solution in $[N, \infty)$. It satisfies $N<\beta<N+1$ for $N\geq 3$.*

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LEMMA 5.12. *For the unique $\beta > N$ satisfying $F_N(\beta) = 0$, one has*

$$\hat{p}_0 = \frac{(\beta-2)(\beta^2 - N\beta - 1)}{N(\beta^2-1)},$$
$$\hat{e}_n = \frac{\beta-2}{\beta^{n+1}},$$
$$\hat{f}_n = \frac{\beta-2}{(\beta^2-1)\beta^{n+1}},$$
$$\hat{q}_n = \frac{\beta-2}{\beta^{n+1}(\beta-1)},$$
$$\hat{r}_n = \frac{\beta-2}{N(\beta-1)^2(\beta+1)}\left\{(\beta-2)(\beta-1)(\beta^2-N\beta-1) - N + \frac{1}{\beta^{n+1}}\right\}.$$

We remark that for $N = 1$, under the condition $F_1(\beta) = 0$, one sees that $\hat{p}_0 = 1/(\beta - 1)^2$.

Let $p_0, e_0, f_0, e_1, f_1, \ldots, e_{l-1}, f_{l-1}, q_{l-1}, r_{l-1}$ be the set of minimal projections of $\mathcal{A}_{Z_N,l}$ considered in the previous section. For a state φ on $\mathcal{A}_{Z_N}$, put

$$\hat{t} := \varphi(t) \quad \text{for } t = p_0, e_0, f_0, e_1, f_1, \ldots, e_{l-1}, f_{l-1}, q_{l-1}, r_{l-1}.$$

Then φ satisfies the condition $\varphi \circ \lambda_{Z_N} = \beta\varphi$ on $\mathcal{A}_{Z_N}$ if and only if the real numbers $\hat{p}_0, \hat{e}_0, \hat{f}_0, \hat{e}_1, \hat{f}_1, \ldots, \hat{e}_{l-1}, \hat{f}_{l-1}, \hat{q}_{l-1}, \hat{r}_{l-1}$ are all non-negative and satisfy all the equations from (5.1) to (5.9) for all $l = 2, 3, \ldots$. Therefore, we have the following proposition.

PROPOSITION 5.13. *A state φ on $\mathcal{A}_{Z_N}$ satisfies the condition $\varphi \circ \lambda_{Z_N} = \beta\varphi$ for some real number β if and only if $\beta > N$ and β is the unique solution of the equation*

$$\beta^6 - (N+3)\beta^5 + (3N+1)\beta^4 - 2(N-1)\beta^3 - (N+2)\beta^2 + N - 1 = 0.$$

Moreover, a state φ that satisfies the above condition is unique.

The topological entropies of some classes of subshifts, including irreducible topological Markov shifts, β-shifts for the real number $\beta > 1$ and the subshift Z_N for $N = 1$, have appeared as the logarithm of the inverse temperature of the admitted KMS-states for gauge actions on the associated C^*-algebras [**10, 14, 27**]. For a subshift (Λ, σ) and a natural number k, let $\theta_k(\Lambda)$ be the cardinal number of the words $B_k(\Lambda)$ of length k appearing in Λ. The topological entropy $h_{\mathrm{top}}(\Lambda)$ for (Λ, σ) is given by

$$h_{\mathrm{top}}(\Lambda) = \lim_{k\to\infty} \frac{1}{k} \log \theta_k(\Lambda)$$

(cf. [**24**]).

For the subshifts Z_N, we have the following lemma.

LEMMA 5.14. *If there exists a $\log\beta$-KMS state on $\mathcal{O}_{Z_N}$ for gauge action for some real number $\beta > N$, we have*

$$\log\beta = \log r(\lambda_{Z_N}) = h_{\mathrm{top}}(Z_N)$$

where $r(\lambda_{Z_N})$ denotes the spectral radius of the operator λ_{Z_N} on $\mathcal{A}_{Z_N}$.

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Proof. The proof is similar to [**27**, Proof of Lemma 6.7]. For the sake of completeness, we will give a proof. A word $\mu = \mu_1 \cdots \mu_n$ in Σ_N appears in the subshift Z_N if and only if $S_\mu (= S_{\mu_1} \cdots S_{\mu_n}) \neq 0$. Let φ be a log β-KMS state on $\mathcal{O}_{Z_N}$ for gauge action for some positive real number β. For $k \in \mathbb{N}$, it follows that

$$\begin{aligned}\beta^k &= \varphi\Big(\sum_{\mu \in B_k(Z_N)} S_\mu^* S_\mu\Big) \leq \Big\| \sum_{\mu \in B_k(Z_N)} S_\mu^* S_\mu \Big\| \\ &= \|\lambda_{Z_N}^k(1)\| \leq \sum_{\mu \in B_k(Z_N)} \|S_\mu^* S_\mu\| = \theta_k(Z_N).\end{aligned}$$

As $\lambda_{Z_N}^k$ is a completely positive map on the unital C^*-algebra $\mathcal{A}_{Z_N}$, we have $\|\lambda_{Z_N}^k(1)\| = \|\lambda_{Z_N}^k\|$ so that we see

$$\beta^k \leq \|\lambda_{Z_N}^k\| \leq \theta_k(Z_N). \tag{5.19}$$

On the other hand, by the inequality $\beta^k \geq \theta_k(Z_N) \min_{\mu \in Z^k} \varphi(S_\mu^* S_\mu)$, we obtain

$$\min_{\mu \in B_k(Z_N)} \varphi(S_\mu^* S_\mu)^{1/k} \cdot \theta_k(Z_N)^{1/k} \leq \beta \leq \theta_k(Z_N)^{1/k}.$$

Now we have $S_\mu^* S_\mu \geq P_0$ for any word $\mu \in B_*(Z_N)$. It follows that

$$\varphi(S_\mu^* S_\mu) \geq \varphi(P_0) = \frac{(\beta - 2)(\beta^2 - N\beta - 1)}{N(\beta^2 - 1)} > 0 \quad \text{for } \mu \in B_*(Z_N).$$

Hence we obtain

$$\lim_{k \to \infty} \min_{\mu \in B_k(Z_N)} \varphi(S_\mu^* S_\mu)^{1/k} = 1$$

and $\lim_{k \to \infty} \theta_k(Z_N)^{1/k} = \beta$. Thus we get the desired equalities from (5.19). □

Therefore we conclude the following theorem.

THEOREM 5.15.

(i) *For a positive real number β, there exists a* log β*-KMS state for gauge action on $\mathcal{O}_{Z_N}$ if and only if $\beta > N$ and β is the unique solution of the equation*

$$\beta^6 - (N+3)\beta^5 + (3N+1)\beta^4 - 2(N-1)\beta^3 - (N+2)\beta^2 + N - 1 = 0.$$

(ii) *The above KMS-state is unique.*

(iii) $\log \beta = h_{\mathrm{top}}(Z_N)$ *: the topological entropy for the subshift Z_N.*

We finally mention an asymptotic behavior of the solution $\beta > N$ for $F_N(\beta) = 0$. Let $\beta_N > N$ be the unique solution for the equation $F_N(\beta) = 0$. We know them by numerical

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calculation as in the following way:

$$\begin{aligned}
\beta_1 &= 2.652\,891\,650\cdots,\\
\beta_2 &= 3.063\,607\,825\cdots,\\
\beta_3 &= 3.670\,666\,991\cdots,\\
\beta_4 &= 4.446\,202\,651\cdots,\\
\beta_5 &= 5.321\,226\,229\cdots,\\
\beta_6 &= 6.247\,124\,025\cdots,\\
\beta_7 &= 7.199\,582\,119\cdots,\\
\beta_8 &= 8.166\,942\,400\cdots,\\
&\cdots.
\end{aligned}$$

PROPOSITION 5.16. *Let $\beta_N > N$ be the unique solution of the equation $F_N(\beta) = 0$. Then we have*

$$\lim_{N\to\infty}(\beta_N - N) = 0.$$

Proof. Put $t_N = (\beta_N/N)$. It follows that

$$\begin{aligned}
0 = \frac{F_N(\beta_N)}{N^6} = {t_N}^6 - \left(1+\frac{3}{N}\right){t_N}^5 + \left(\frac{3}{N}+\frac{1}{N^2}\right){t_N}^4 - 2\left(\frac{1}{N^2}-\frac{1}{N^3}\right){t_N}^3\\
- \left(\frac{1}{N^3}+\frac{2}{N^4}\right){t_N}^2 + \frac{1}{N^5} - \frac{1}{N^6}
\end{aligned}$$

and hence

$$\begin{aligned}
N(1-t_N) = -3 + \left(3+\frac{1}{N}\right)\frac{1}{t_N} - 2\left(\frac{1}{N}-\frac{1}{N^2}\right)\frac{1}{{t_N}^2}\\
- \left(\frac{1}{N^2}+\frac{2}{N^3}\right)\frac{1}{{t_N}^3} + \left(\frac{1}{N^4}-\frac{1}{N^5}\right)\frac{1}{{t_N}^5}.
\end{aligned}$$

As $N < \beta_N < N+1$ and hence $\lim_{N\to\infty}|t_N - 1| = 0$, we get

$$\lim_{N\to\infty}|\beta_N - N| = \lim_{N\to\infty}|N(1-t_N)| = 0. \qquad \square$$

6. *Flow equivalence classes of the subshifts Z_N, $N \in \mathbb{N}$*

We will finally apply our discussions to a classification problem in symbolic dynamical systems under flow equivalence (cf. [**3, 12, 38**]). In [**31, 32**], we have defined the K-groups $K_i(\Lambda)$, $i = 0, 1$ and the Bowen–Franks groups $BF^i(\Lambda)$, $i = 0, 1$ for the subshift Λ by the K-groups $K_i(\mathcal{O}_\Lambda)$, $i = 0, 1$, and the Ext-groups $\mathrm{Ext}^i(\mathcal{O}_\Lambda)$, $i = 0, 1$, for the associated C^*-algebra $\mathcal{O}_\Lambda$ respectively. We have then proved that the groups $K_i(\Lambda)$, $BF^i(\Lambda)$, $i = 0, 1$, are invariant under not only the topological conjugacy class but also the flow equivalence class of subshifts. In particular, there is no known computable invariant under flow equivalence of subshifts other than the groups. Since the Ext-groups $\mathrm{Ext}^i(\mathcal{O}_\Lambda)$, $i = 0, 1$, can be computed by the universal coefficient theorem for the K-theory of C^*-algebra $\mathcal{O}_\Lambda$, we have, by Theorem 4.9,

$$\begin{aligned}
K_0(Z_N) = \mathbb{Z}/N\mathbb{Z}\oplus\mathbb{Z}, \quad K_1(Z_N) = \{0\},\\
BF^0(Z_N) = \mathbb{Z}/N\mathbb{Z}, \quad BF^1(Z_N) = \mathbb{Z}.
\end{aligned}$$

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Therefore we have the following proposition.

PROPOSITION 6.1. *The subshifts Z_N, $N = 1, 2, \ldots$, are not flow equivalent to each other.*

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