Math. Struct. in Comp. Science (2017), *vol.* 27, *pp.* 516–529. © Cambridge University Press 2015
doi:10.1017/S0960129515000201 First published online 18 May 2015

Lattice-valued Scott topology on dcpos

WEI YAO

Department of Mathematics, Hebei University of Science and Technology, Shijizhuang 050018, P.R. China
Email: yaowei0516@163.com

Received 18 November 2013; revised 11 August 2014

This paper studies the fuzzy Scott topology on dcpos with a $*$-continuous semigroup $(L, *)$ as the truth value table. It is shown that the fuzzy Scott topological space on a continuous dcpo is an ι_L-sober space. The fuzzy Scott topology is completely distributive iff L is completely distributive and the underlying dcpo is continuous. For $(L, *)$ being an integral quantale, semantics of L-possibility of computations is studied by means of a duality.

1. Introduction

As is documented in Tix *et al.* (2005), the semantics of programming languages has been intensively studied by mathematicians and computer scientists. In the late sixties, Dana Scott invented appropriate semantic domains for that purpose (Scott 1970, 1972). Domains are a kind of directed complete partially ordered sets, dcpos for short, which are often interpreted as data types. Therein, the order can be thought of as an information ordering, which means the greater an element is, the more information it carried about the object it approximates. In this approach, computable functions are the so-called (Scott) continuous functions on dcpos. Moreover, within domains, recursion can be interpreted via least fixed points of continuous functions and Scott open sets as predicates. Thus, basic elements of computation can be discovered from their semantic aspects in the Cartesian closed category of dcpos and continuous functions.

It is well known that on a dcpo X, a subset $A \subseteq X$ is Scott open iff $A : X \longrightarrow \{0, 1\}$ is Scott continuous. Now we replace $\{0,1\}$ by a complete lattice L. Then the family of all Scott continuous maps from X to L forms an L-topology on X (Lu 2012), which is indeed a lattice-valued topological space and will be called the fuzzy Scott topology on X. The fuzzy Scott topology also could be arisen from the following two issues.

1. In Warner (1990), it is shown that for a continuous frame L (i.e., a frame which is a continuous lattice as well), the family of all continuous maps from a topological space $(X, \mathcal{T})$ to the Scott topological space $(L, \sigma(L))$ is an L-topology on X. If X itself is also a dcpo and $\mathcal{T}$ is the Scott topology on X, then a continuous map from $(X, \mathcal{T})$ to $(L, \sigma(L))$ is exactly a Scott continuous map from X to L and of course the results in Warner (1990) still hold. In the case, the condition L is continuous frame may could be weaken. In fact, Theorem II-4.19 in Giertz *et al.* (2003) points out that L is a frame is already a sufficient condition.
2. For a crisp topology $\mathcal{T}$ on a set X, by using the Lowen functor ω_L (see in Kubiak (1992) for a detail discussion), we can construct an L-topology $\omega_L(\mathcal{T})$ on X, called the

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

L-topology generated by $\mathcal{T}$. In fact, $\omega_L(\mathcal{T})$ is a set of all continuous maps from $(X, \mathcal{T})$ to $(L, v(L))$, where $v(L)$ is the upper topology on L. If L is completely distributive, then $v(L) = \sigma(L)$. Thus in this case, $\omega_L(\mathcal{T})$ is the set of all continuous maps from $(X, \mathcal{T})$ to $(L, \sigma(L))$. Also, if X is a dcpo and $\mathcal{T}$ is the Scott topology on X, then $\omega_L(\mathcal{T})$ is the family of all Scott continuous maps from X to L.

The main of this paper is to continue to study the fuzzy Scott topology on dcpos for $(L, *)$ a $*$-continuous semigroup. This paper is arranged as follows. In Section 2, we recall some basic preliminaries and results which will be used throughout this paper. In Section 3, we show that: if a dcpo is continuous, then the fuzzy Scott topological space is an ι_L-sober space; the fuzzy Scott topology is completely distributive iff L is completely distributive and the underlying dcpo is continuous. In the last section, for $(L, *)$ being an integral quantale, we apply fuzzy Scott open sets in studying the semantics of L-possibility of computations and establish a duality between denotational semantics and logical semantics.

2. Preliminaries

2.1. *Continuous dcpos and the Scott topology*

We refer to Giertz *et al.* (2003) for domain theory related to continuous dcpos and the classical Scott topology.

A poset X is called *directed complete* (or a *dcpo*, in short) if each directed subset of X has a join. Let x, y be two elements in a dcpo X, x is said to be *way below* y, in symbols $x \ll y$, if for any directed subset $D \subseteq X$, $y \leqslant \bigvee D$ implies $x \leqslant d$ for some $d \in D$. For every $x \in X$, denote $\Downarrow x = \{a \in X |\ a \ll x\}$ ($\Uparrow x$ has the dual meaning). A dcpo X is called *continuous* if for each $x \in X$, $\Downarrow x$ is directed and $x = \bigvee \Downarrow x$.

A subset $U \subseteq X$ is called *Scott open* if U is an upper set and $\bigvee D \in U$ implies $D \cap U \neq \varnothing$ for all directed subset $D \subseteq X$. The family of all Scott open sets forms a topology on X, called the *Scott topology*, in symbols $\sigma(X)$ and by $\Sigma(X)$ the corresponding topological space. A map $f : X \longrightarrow Y$ between two dcpos is called *Scott continuous* if f preserves joins of directed subsets.

Proposition 2.1 (Giertz *et al.* 2003).

1. In a dcpo X, for every $x \in X$, $L \backslash \downarrow x$ is a Scott open set.
2. A map $f : X \longrightarrow Y$ is Scott continuous iff $f : \Sigma(X) \longrightarrow \Sigma(Y)$ is continuous.

A Scott open set F of a dcpo X is called an *open filter* if it is filtered in X, that is for any $a, b \in F$, there exists $c \in F$ such that $c \leqslant a, b$. The set of all open filters of X is denoted by $OFil(X)$.

Let C be a complete lattice and $pt(C)$ denote the set of all maps from C to $\{0, 1\}$ which preserve arbitrary joins and finite meets. Let $\mathcal{T}$ be a topology on a set X. Define $pt_X : X \longrightarrow pt(\mathcal{T})$ by for any $x \in X$ and any $A \in \mathcal{T}$, $pt_X(x)(A) = 1$ if $x \in A$ and 0 otherwise. A topological space $(X, \mathcal{T})$ is called a *sober space* if pt_X is a bijection.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

Proposition 2.2 (Giertz *et al.* 2003). Let X be a continuous dcpo. Then

1. The way below relation has the property of interpolation, i.e., $x \ll y$ always implies that there exists z such that $x \ll z \ll y$.
2. Both $\{\Uparrow x |\ x \in X\}$ and $OFil(X)$ are bases of $\Sigma(X)$.
3. $\Sigma(X)$ is a sober space.

A complete lattice is called *completely distributive* if for every family of subsets $X_i = \{a_{ij} |\ j \in J_i\}$ $(i \in I)$ of L, it holds that

$$\bigwedge_{i\in I}\bigvee_{j\in J_i} a_{ij} = \bigvee_{f\in\prod_{i\in I}J_i}\bigwedge_{i\in I} a_{i,f(i)} \quad or \quad \bigvee_{i\in I}\bigwedge_{j\in J_i} a_{ij} = \bigwedge_{f\in\prod_{i\in I}J_i}\bigvee_{i\in I} a_{i,f(i)}.$$

Let x, y be two elements in a complete lattice L, x is said to be *wedge below* y, in symbols $x \lhd y$, if for any subset $D \subseteq L$, $y \leqslant \bigvee D$ implies $x \leqslant d$ for some $d \in D$. We denote $\overset{\triangle}{\uparrow} x = \{a \in L |\ x \lhd a\}$ and $\overset{\triangle}{\downarrow} x = \{a \in L |\ a \lhd x\}$.

Proposition 2.3 (Giertz *et al.* 2003).

1. A complete lattice L is completely distributive iff for each $x \in L$, $x = \bigvee \overset{\triangle}{\downarrow} x$.
2. In every completely distributive lattice, the wedge below relation has the property of interpolation.
3. A dcpo X is continuous iff $\sigma(X)$ is completely distributive (under the order of set inclusion).
4. Every completely distributive lattice is a continuous lattice.
5. If L is a completely distributive lattice, then every $\overset{\triangle}{\uparrow} a$ is a Scott open set.

Proof. (1–4) can be found in Giertz *et al.* (2003). For 5, clearly $\overset{\triangle}{\uparrow} a$ is an upper set. If $D \subseteq L$ is directed and $\bigvee D \in \overset{\triangle}{\uparrow} a$, then $a \lhd \bigvee D$. By 2, there exists an element $b \in L$ such that $a \lhd b \lhd \bigvee D$. It follows that $b \leqslant d$ and then $a \lhd d$ for some $d \in D$. Thus $D \cap \overset{\triangle}{\uparrow} a \neq \varnothing$. Hence, $\overset{\triangle}{\uparrow} a$ is a Scott open set. □

2.2. *Lattices, fuzzy sets and L-topologies*

We refer to Höhle and Šostak (1999) and Liu and Luo (1997) for fuzzy sets and fuzzy topology.

Let L be a complete lattice (with 1,0 as the top and bottom elements respectively) and let $* : L \times L \longrightarrow L$ be a semigroup operation. If for every $a \in L$ and every directed subset $D \subseteq L$, it holds that

$$a * (\bigvee D) = \bigvee_{d\in D}(a * d) \quad and \quad (\bigvee D) * a = \bigvee_{d\in D}(d * a),$$

then we call the pair $(L, *)$ is $*$-*continuous* or $(L, *)$ is a $*$-*continuous semigroup*. Clearly, $*$-continuity guarantees the monotonicity of $*$. Notice that a $*$-continuous semigroup is called a *join-continuous semigroup* with a further condition of $a * 0 = 0 * a = 0$ for every $a \in L$ in Höhle and Kubiak (2007). Here, we give this name since $(L, *)$ is $*$-continuous iff the operation $*$ is Scott continuous at both arguments.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

A $*$-continuous semigroup $(L, *)$ is called a *quantale* (Rosenthal 1990) if $*$ is distributive over arbitrary joins. A quantale $(L, *)$ is called *integral* or *strictly two-sided* (Rosenthal 1990) if the top element 1 is the unit of $*$, i.e., $1 * a = a = a * 1$ holds for every $a \in L$. It is a routine to check that if the operation $*$ is commutative, then $(L, *)$ is an integral quantale iff it is a complete residuated lattice (Hájek 1998), that is, an integral quantale can be considered as a noncommutative complete residuated lattice.

Let L be a complete lattice and X be set. We shall make no difference between a crisp subset and its characterization function. For $a \in L$, a_X denotes the constant map from X to L with the value a. For $A, B \in L^X$, $A * B \in L^X$ denotes the L-subset of X sending every $x \in X$ to $A(x) * B(x)$. For $a \in L$, we write $a_X * B$ as $a * B$ for short. For $a \in L$, $S \subseteq X$, we use the symbol aS is the L-subset of X with the value a for $x \in S$ and 0 otherwise. Notice that aS and $a * S$ need not be the same in a $*$-continuous semigroup L unless it is integral.

Let L be a complete lattice. Let $\prec$ be a binary relation on X. For any $A \in L^X$, denote $[A \succ a] = \{x \in X |\ a \prec A(x)\}$. For a map $f : X \longrightarrow Y$, define $f_L^{\rightarrow} : L^X \longrightarrow L^Y$ by $f_L^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ $(\forall A \in L^X)$ and define $f_L^{\leftarrow} : L^Y \longrightarrow L^X$ by $f_L^{\leftarrow}(B) = B \circ f$ $(\forall B \in L^Y)$.

A *stratified L-topology* on a set X is a subfamily $\delta \subseteq L^X$ that satisfies (TO) $a_X \in \delta$ for every $a \in L$; (T1) $\forall A, B \in \delta,\ A * B \in \delta$; (T2) $\forall \{A_i |\ i \in I\} \subseteq \delta,\ \bigvee_i A_i \in \delta$. A subfamily $\mathcal{S}$ of a stratified L-topological space (X, δ) is called a *basis* if for every $U \in \delta$, it holds that $U = \bigvee \{S \in \mathcal{S} |\ S \leqslant U\}$. A map $f : (X, \delta_X) \longrightarrow (Y, \delta_Y)$ between two stratified L-topological spaces is called *continuous* if $f_L^{\leftarrow}(B) \in \delta_X$ for all $B \in \delta_Y$.

For L a complete lattice, the pair of Lowen functors (ω_L, ι_L) is studied in Kubiak (1992). Let X, Y be two topological spaces. Denote by $\Omega[X, Y]$ the set of all continuous maps from X to Y. For a topology $\mathcal{T}$ on a set X, $\omega_L(\mathcal{T})$ be the L-topology generated by the family $\Omega[X, v(L)]$, where $v(L)$ is the upper topology generated by the subbasis $\{L \backslash \downarrow a |\ a \in L\}$. For an L-topological space (X, δ), $\iota_L(\delta)$ is the topology generated by the subbase $\bigcup \{A^{-1}(v(L)) |\ A \in \delta\}$.

In Pultr and Rodabaugh (2003), different types of fuzzy sobriety for L-topological spaces are discussed. Let (X, δ) be an L-topology and $Lpt(\delta)$ be the set of all maps from δ to L which preserve arbitrary joins and finite meets. Define $\Psi_L : X \longrightarrow Lpt(\delta)$ by $\Psi_L(x)(A) = A(x)$ $(\forall A \in \delta)$. The space (X, δ) is called *L-sober* if Ψ_L is a bijection. If we modify $Lpt(\delta)$ to $Lpt_{\mathrm{mod}}(\delta) = \{p \in Lpt(\delta) |\ \forall a_X \in \delta,\ p(a_X) = a\}$, then the space (X, δ) is called *modified L-sober* if Ψ_L is a bijection onto $Lpt_{\mathrm{mod}}(\delta)$. An L-topological space (X, δ) is called *ι_L-sober* if $(X, \iota_L(\delta))$ is a sober space.

3. A lattice-valued Scott topology on dcpos

In this section, we assume that $(L, *)$ is a $*$-continuous semigroup.

For the ordinary Scott topology on a dcpo X, it is well known that a subset U is Scott open iff $U : X \longrightarrow \{0, 1\}$ is Scott continuous. It motivates us to define a fuzzy Scott topology on X.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

Proposition 3.1. For every dcpo X, the family $\sigma_L(X)$ of all Scott continuous maps from X to L is a stratified L-topology on X.

Proof. (T0) For any $a \in L$, $a_X : X \longrightarrow L$ is obviously a Scott continuous map. Thus, $a_X \in \sigma_L(X)$.

(T1) Suppose that $A, B \in \sigma_L(X)$. Clearly $A * B$ is monotone. For each directed subset $D \subseteq X$, we have

$$\begin{aligned}(A * B)(\bigvee D) &= \bigvee_{d_1 \in D} A(d_1) * \bigvee_{d_2 \in D} B(d_2)\\ &= \bigvee_{d_1, d_2 \in D} A(d_1) * B(d_2)\\ &\leqslant \bigvee_{d \in D} A(d) * B(d)\\ &= \bigvee_{d \in D} (A * B)(d).\end{aligned}$$

Hence, $A * B \in \sigma_L(X)$.

(T2) Suppose that $\{A_i |\ i \in I\} \subseteq \sigma_L(X)$. For each directed subset $D \subseteq X$,

$$\bigvee_i A_i(\bigvee D) = \bigvee_i \bigvee_{d \in D} A_i(d) = \bigvee_{d \in D} \bigvee_i A_i(d).$$

Thus, $\bigvee_i A_i \in \sigma_L(X)$. □

For $L = [0, 1]$ with $* = \wedge$, $\sigma_L(X)$ turns out to be the fuzzy Scott topology in Chen and Wu (2008); and for a complete residuated lattice, $\sigma_L(X)$ turns out to be the fuzzy Scott topology in Lu (2012).

It is easy to verify that for every $U \in \sigma(X)$ and every $a \in L$, the L-subset $aU \in \sigma_L(X)$ and so $U \in \sigma_L(X)$ (considering U as its characterization function). Thus, classical Scott topology $\sigma(X)$ can be considered as a subfamily of $\sigma_L(X)$. The following proposition will show the more closer relations between the fuzzy Scott topology and the ordinary Scott topology.

Proposition 3.2. Suppose that X is a dcpo. Then

1. $\sigma_L(X) = \Omega[\Sigma(X), \nu(L)]$, thus $\Omega[\Sigma(X), \nu(L)]$ precisely is the L-topology generated by $\sigma(X)$;
2. $\sigma(X) = \bigcup\{A^{-1}(\nu(L)) |\ A \in \sigma_L(X)\}$, thus $\sigma(X)$ precisely equals to $\iota_L(\sigma_L(X))$.

In a summary, $\sigma_L(X) = \omega_L(\sigma(X))$ and $\sigma(X) = \iota_L(\sigma_L(X))$.

Proof.

1. Let $A \in \Omega[\Sigma(X), \nu(L)]$. Suppose that $x \leqslant y$, if $A(x) \not\leqslant A(y)$, then $A(x) \in L \backslash \downarrow A(y)$ and $x \in A^{-1}(L \backslash \downarrow A(y)) \in \sigma(X)$. It follows that $y \in A^{-1}(L \backslash \downarrow A(y))$ and $A(y) \not\leqslant A(y)$. This is a contradiction. Thus, $A : X \longrightarrow L$ is monotone. Let D be a directed subset of X and $\bigvee_{d \in D} A(d) = a$. If $A(\bigvee D) \not\leqslant a$, then $\bigvee D \in A^{-1}(L \backslash \downarrow a) \in \sigma(X)$. It follows that there exists $d \in D$ such that $d \in A^{-1}(L \backslash \downarrow a)$ and $A(d) \not\leqslant a$. It is also a contradiction. Thus, $A(\bigvee D) \leqslant \bigvee_{d \in D} A(d)$ and then $A(\bigvee D) = \bigvee_{d \in D} A(d)$. Hence, $A \in \sigma_L(X)$.
   Conversely, suppose that $A \in \sigma_L(X)$, that is $A : X \longrightarrow L$ is Scott continuous. Let $a \in L$ and $U = A^{-1}(L \backslash \downarrow a) = [A \not\leqslant a]$. If $x \leqslant y$ and $x \in U$, then $A(x) \not\leqslant a$ and $A(y) \not\leqslant a$ since A is monotone. Thus, $y \in A^{-1}(L \backslash \downarrow a) = U$ and U is an upper set. Let D be a direct subset of X and $\bigvee D \in U$. Then, $\bigvee_{d \in D} A(d) = A(\bigvee D) \not\leqslant a$. Then, there

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

exists $d \in D$ such that $A(d) \not\leqslant a$ and $d \in A^{-1}(L\backslash\downarrow a) = U$. Thus, $A^{-1}(L\backslash\downarrow a) \in \sigma(X)$. Hence, $A \in \Omega[\Sigma(X), v(L)]$.

Therefore, $\Omega[\Sigma(X), v(L)] = \sigma_L(X)$. The family $\Omega[\Sigma(X), v(L)]$ is already an L-topology and then it is precisely $\omega_L(\sigma(X))$.

2. For every $a \in L$ and every $A \in \sigma_L(X)$, since $L\backslash\downarrow a$ is Scott open in L and $A : \Sigma(X) \longrightarrow \Sigma(L)$ is continuous, we have that $A^{-1}(L\backslash\downarrow a)$ is a Scott open set.

Conversely, for any $U \in \sigma(X)$, considered its characterization function, belongs to $\sigma_L(X)$. It is easy to show that $U = U^{-1}(L\backslash\downarrow 0) \in \bigcup\{A^{-1}(v(L))|\ A \in \sigma_L(X)\}$. Hence, $\sigma(X) = \bigcup\{A^{-1}(v(L))|\ A \in \sigma_L(X)\}$.

Therefore, $\sigma(X) = \bigcup\{A^{-1}(v(L))|\ A \in \sigma_L(X)\}$. The family $\bigcup\{A^{-1}(v(L))|\ A \in \sigma_L(X)\}$ is already a topology and then it is precisely $\iota_L(\sigma_L(X))$. □

Remark 3.3.

1. In Proposition 3.2, there does not need any further condition for the lattice L. While for a general topological space $(X, \mathcal{T})$ and a general L-topological space (Y, δ), the two equalities $\omega_L(\mathcal{T}) = \Omega[X, v(L)]$ and $\iota_L(\delta) = \bigcup\{A^{-1}(v(L))|\ A \in \delta\}$ do not hold generally. One sufficient condition is that L is completely distributive and $* = \wedge$ (Proposition 3.4 in Kubiak (1992)).
2. In Höhle and Kubiak (2007), based on a join-continuous (that is, $*$-continuous) semigroup $(L, *)$ with an arbitrary topology $\gamma(L)$, Höhle and Kubiak defined a new kind of lower semicontinuous maps with respect to a crisp topological space X and showed that $LSC(X, L)$, the family of all lower semicontinuous maps, is a stratified L-topology on X (Lemma 4.5 in Höhle and Kubiak (2007)). This situation gives rise to a functor ω_L from the category of crisp topological spaces to the category of stratified L-topological spaces. This functor ω_L coincides with the one in Kubiak (1992) only in the case of L being a continuous lattice (Remark 3.11(2) in Kubiak (1992)).
3. In Theorem 4.4 in Höhle and Kubiak (2007), for every topological space X, it is shown that if L is a continuous lattice with the Scott topology $\sigma(L)$, then $LSC(X, L) = \Omega[X, L]$, that is, lower semicontinuity coincides with continuity with respect to $\sigma(L)$, for every topological space X. Clearly, if L is a continuous lattice, then by Proposition 3.2, our fuzzy Scott topology $\sigma_L(X)$ is a special case of $LSC(X, L)$ in Höhle and Kubiak (2007) for X being a dcpo. But we here do not need the condition of L being a continuous lattice, since we only consider the restricted case of X being a dcpo with the Scott topology.

By Propositions 2.2(3) and 3.2(2), we have

Corollary 3.4. If X is continuous, then $\Sigma_L(X)$ is an ι_L-sober space.

In Yao (2012), we have shown that L-sobriety always implies modified L-sobriety and if L is completely distributive, then modified L-sobriety implies ι_L-sobriety. We here have an example to show that the fuzzy Scott topology on a continuous dcpo need not be a modified L-sober space (and of course is nor an L-sober space) even if L is completely distributive and $* = \wedge$.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

Example 3.5. Let $X = \{0,1\}$, $L = \{0,a,1\}$ $(0 < a < 1)$. Since X is linear, a Scott continuous map from X to L exactly is a monotone map and then

$$\sigma_L(X) = \{0_X, a_X, 1_X, (0,a), (0,1), (a,1)\}.$$

For $p \in Lpt_{\mathrm{mod}}(\sigma_L(X))$, we have

$$p(0_X) = 0,\ p(a_X) = a,\ p(1_X) = 1.$$

Since $(a,1) = (0,1) \vee a_X$ and $(0,a) = (0,1) \wedge a_X$, the map p is completely determined by the value of $p(0,1)$. Thus, $|Lpt_{\mathrm{mod}}(\sigma_L(X))| = 3 > |X|$. Therefore, $\Sigma_L(X)$ is not a modified L-sober space.

The following proposition shows Scott continuous maps are the same as continuous maps with respect to the fuzzy Scott topology.

Proposition 3.6. Let $f : X \longrightarrow Y$ be a map between two dcpos. Then, f is Scott continuous iff $f : \Sigma_L(X) \longrightarrow \Sigma_L(Y)$ is continuous.

Proof. The necessity is clear since compositions of Scott continuous maps are still Scott continuous. The sufficiency: suppose that $V \in \sigma(Y)$. Then, $V \in \sigma_L(Y)$ and $f^{-1}(V) = f_L^{\leftarrow}(V) \in \sigma_L(X)$. Since $f^{-1}(V)$ is crisp, we have that $f^{-1}(V) \in \sigma(X)$. That is to say, f is topologically continuous with respect to the ordinary Scott topology. Hence, $f : X \longrightarrow Y$ is Scott continuous. □

Proposition 3.7. Let X be a dcpo and let L be continuous lattice. Then

1. $A \in \sigma_L(X)$ iff $[A \gg a] \in \sigma(X)$ for all $a \in L$;
2. if $\mathcal{S}$ is a basis of $\Sigma(X)$, then $\{aS |\ S \in \mathcal{S}, a \in L\}$ is a basis of $\Sigma_L(X)$.

Proof.

1. First of all, for any $A \in L^X$ and $a \in L$, we have $[A \gg a] = A^{-1}(\Uparrow a)$. If $A \in \sigma_L(X)$, then for every $a \in L$, we have $[A \gg a] = A^{-1}(\Uparrow a) \in \sigma(X)$ since $\Uparrow a$ is Scott open and $A : \Sigma(X) \longrightarrow \Sigma(L)$ is continuous. Conversely, for the continuous lattice L, we have that $\{\Uparrow a |\ a \in L\}$ is a basis of $\Sigma(L)$. If $A^{-1}(\Uparrow a) = [A \gg a] \in \sigma(X)$ for all $a \in L$, then $A : \Sigma(X) \longrightarrow \Sigma(L)$ is continuous and $A \in \sigma_L(X)$.
2. First of all, it is easy to see that $\{aS |\ S \in \mathcal{S}, a \in L\} \subseteq \sigma_L(X)$. Suppose that $A \in \sigma_L(X)$. For every $x \in X$ and for every $a \ll A(x)$, we have $x \in [A \gg a] \in \sigma(X)$ and then $x \in S \subseteq [A \gg a]$ for some $S \in \mathcal{S}$. Obviously, $aS \leqslant A$ and $(aS)(x) = a$. By the arbitrariness of x and a, we have $A = \bigvee\{aS |\ a \in L,\ S \in \mathcal{S},\ aS \leqslant A\}$. Hence, $\{aS |\ S \in \mathcal{S}, a \in L\}$ is a basis of $\Sigma_L(X)$.

□

Corollary 3.8. If both X and L are continuous, then both $\{a(\Uparrow x) |\ x \in X,\ a \in L\}$ and $\{aF |\ F \in OFil(X),\ a \in L\}$ are bases of $\Sigma_L(X)$.

In domain theory, X is a continuous dcpo iff the Scott topology $\sigma(X)$ is a completely distributive lattice; and for a completely distributive lattice, the set of all coprime elements forms a continuous dcpo under the induced order. Based on these two facts, categorical

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

duality was established between the category of continuous dcpos and that of completely distributive lattices both with suitable morphisms (Hoffmann 1981; Lawson 1979).

In a lattice L, an element r is called coprime if $r \leqslant b \vee c$ always implies $r \leqslant b$ or $r \leqslant c$. We denote $M(L)$ as the set of all coprime elements of L.

Lemma 3.9.

1. For every dcpo X, $OFil(\sigma(X)) = M(\sigma(X))$.
2. In a completely distributive lattice, the set $M(L)$ is join-dense in L, that is, $a = \bigvee\{r \in M(L) |\ r \leqslant a\}$ for every element $a \in L$.

Lemma 3.10. (Theorem I-3.16 in Giertz *et al.* (2003)) Let L be a complete lattice. Then, the following statements are equivalent:

1. L is completely distributive;
2. L is distributive and both L and L^{op} are continuous lattices;
3. L is continuous and every element is the join of coprime elements.

Lemma 3.11. Suppose that F is an open filter of a dcpo X and a is a coprime element of L. Then, aF is a coprime element of $\Sigma_L(X)$.

Proof. It is clear that $aF \in \Sigma_L(X)$. For any $A, B \in \Sigma_L(X)$ with $aF \leqslant A \vee B$, if $aF \not\leqslant A$ and $aF \not\leqslant B$, then there exist $x, y \in F$ such that $a \not\leqslant A(x)$ and $a \not\leqslant B(y)$. Since F is a filter, we can find a $z \in F$ such that $z \leqslant x, y$. It follows that $a \not\leqslant A(z)$ and $a \not\leqslant B(z)$ since A, B are monotone. Then, $a \not\leqslant (A \vee B)(z)$ since a is coprime. But this is a contradiction to $aF \leqslant A \vee B$. □

Corollary 3.12. Let L be a completely distributive lattice and let X be a continuous dcpo. Then, $\{aF |\ F \in OFil(X),\ a \in M(L)\}$ is a collection of coprime elements of $\sigma_L(X)$, which forms a basis of $\Sigma_L(X)$.

Proof. By Proposition 2.3(4) and Corollary 3.8, we know that $\{rF |\ F \in OFil(X),\ r \in L\}$ is a basis of $\Sigma_L(X)$. For every $F \in OFil(X)$ and every $r \in L$, it holds that $r = \bigvee\{a \in M(L) |\ a \leqslant r\}$ and then $rF = \bigvee\{aF |\ a \in M(L), a \leqslant r\}$. Hence, $\{aF |\ F \in OFil(X),\ a \in M(L)\}$ is a basis of $\Sigma_L(X)$. □

Lemma 3.13. 1. (Theorem II-4.7 in Giertz *et al.* (2003)) $\Omega[X, \Sigma(L)]$ is continuous iff both $\mathcal{O}(X)$ (the lattice of open sets of X) and L are continuous. Then, $\sigma_L(X)$ is continuous iff both $\sigma(X)$ and L are continuous.

A subset S of a complete lattice L is called subcomplete if for all $A \subseteq S$, $\bigvee A, \bigwedge A \in S$, that is, S itself is a complete lattice (as a subset of L) satisfying that $\bigvee_S A = \bigvee A$ and $\bigwedge_S A = \bigwedge A$, where $\bigvee_S A$ and $\bigwedge_S A$ are the joins and meets in S respectively.

Lemma 3.14. Let L be a completely distributive lattice and S a subcomplete lattice of L. Then, S is also completely distributive.

Proof. It is trivial since (i) for any subcomplete lattice S of L, we have for any $A \subseteq S$, $\bigvee_S A = \bigvee A$ and $\bigwedge_S A = \bigwedge A$; (ii) completely distributivity only depends on the computation of joins and meets. □

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

It is a routine to show that, both $\sigma(X)$ and L are subcomplete lattices of $\sigma_L(X)$ by considering Scott open sets as their characterization functions and $a \in L$ as the corresponding constant map a_X, respectively.

Theorem 3.15. $\sigma_L(X)$ is completely distributive if and only if L is completely distributive and X is continuous.

Proof. Suppose that L is completely distributive and X is continuous. We have both $\sigma(X)$ and L are continuous and then by Lemma 3.13, $\sigma_L(X)$ is continuous. Also by Corollary 3.12, $\sigma_L(X)$ has enough coprime elements. By Lemma 3.10, $\sigma_L(X)$ is completely distributive. Conversely, if $\sigma_L(X)$ is completely distributive, then both $\sigma(X)$ and L are subcomplete lattices of $\sigma_L(X)$, which are also completely distributive by Lemma 3.14. By Proposition 2.3(3), X is continuous. □

Remark 3.16. In Yao (2010), fuzzy dcpos, as a generalization of ordinary dcpos, were introduced. In Yao and Shi (2011), we have already defined and studied fuzzy Scott topology on fuzzy dcpos. It is a natural question that, whether or not the fuzzy Scott topology is a special case of that in Yao and Shi (2011)? The answer is negative. In fact in Yao *et al.* (2013), we have shown that if a set X contains more than two points and $L = \{0, a, b, 1\}$ is the simplest nontrivial Boolean algebra, then X equipped with any ordinary order could not be a fuzzy dcpo. And also in Yao and Zhao (2013), we showed that for a complete residuated lattice L, the family of fuzzy Scott open sets on a given fuzzy dcpo is a lattice-valued topology iff L is a complete Heyting algebra (i.e., a quantale with $* = \wedge$). While the results in this section are established on a $*$-continuous semigroup. There is no further restriction needed.

4. Domain semantics of possibility of computations

In domain theory, computation in general involves two classes: determinism and non-determinism. Deterministic computation means that the computed result is deterministic for a given input, however, non-deterministic one means that the results are more than one possibility. In order to describe this non-determinism, one introduces the concept of powerdomain (Hoare 1985; Plotkin 1976; Smyth 1978). Another method for description of non-determinism consists in quantifying this non-determinism by means of probability or possibility. For probabilistic model, probability distribution over a domain is introduced. Related works including He *et al.* (1997), Plotkin and Jones (Jones 1990; Plotkin 1982), and Tix *et al.* (2005).

Chen and Wu (2008) considered the model of possibility computation, in which the possibility measures/valuations are defined as maps from the family of Scott open sets to the unit interval [0,1], satisfying the axiomatic rule of possibility measures. The possibilistic powerdomain $P(D)$ of a dcpo D is introduced as the family of all possibility measures over the output space and the semantics of possibility computation is a Scott-continuous map from the input space to the possibilistic powerdomain of the output space. This possibility computation is also deterministic in the sense that, for a given input the output is a single possibility valuation.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

In this section, we suppose that $(L, *)$ is completely distributive, integral quantale. Clearly, the unit interval is such a lattice. We shall generalize the contents in Chen and Wu (2008) by replacing the unit interval by L as well as $\wedge$ by $*$.

Definition 4.1. Let D be a dcpo. A map $\Pi : \sigma(D) \longrightarrow L$ is called an *L-possibility valuation* of D if it preserves arbitrary unions. Denote by $\pi_L(D)$ the set of all L-possibility valuations of D.

We can consider L-possibility valuations as a computational model of non-deterministic computations.

Definition 4.2. Let D, E be two dcpos. The denotational semantics assigns to an L-possibility computation F from D to E a Scott continuous function $[\![F]\!]:\mathrm{D}\longrightarrow \pi_L(E)$.

From this definition, we can see that the result of a possibility computation at one input state, for example $d \in D$, is an L-possibility valuation on Scott topology $\sigma(E)$, which gives the possibility that one element, for example $e \in E$, belongs to Scott open set $U \in \sigma(E)$.

Definition 4.3. An *L-fuzzy predicate transformer* from D to E is a map $t : \sigma_L(E) \longrightarrow \sigma_L(D)$. An L-fuzzy predicate transformer t from D to E is called *healthy*, if it satisfies the following healthy conditions:

(H1) Sups-preserving: if $\{f_i |\ i \in I\} \subseteq \sigma_L(E)$, then $t(\bigvee_i f_i) = \bigvee_i t(f_i)$;

(H2) Level-preserving: for $r \in L$ and $U \in \sigma(E)$, $t(r * U) = r * t(U)$.

We denote $[\sigma_L(E) \longrightarrow_H \sigma_L(D)]$ the set of all healthy L-fuzzy predicate transformers from D to E.

Definition 4.4. Let D and E be dcpos. The logical semantics assigns to a possibility computation F from D to E a healthy fuzzy predicate transformer $\|F\|$ from D to E.

Proposition 4.5. The transformation $\alpha : [\sigma_L(E) \longrightarrow_H \sigma_L(D)] \longrightarrow [D \longrightarrow \pi_L(E)]$ by $\alpha(t)(x)(U) = t(U)(x)$ defines a map.

Proof. Let $t \in [\sigma_L(E) \longrightarrow \sigma_L(D)]$ and $x \in D$.

Step 1. $\alpha(t)(x) \in \pi_L(E)$, that is $\alpha(t)(x) : \sigma(E) \longrightarrow L$ preserves arbitrary unions. In fact, $\alpha(t)(x)(\varnothing) = t(\varnothing)(x) = t(0 * \varnothing)(x) = 0 * t(\varnothing)(x) = 0$ and let $\{U_i |\ i \in I\} \subseteq \sigma(E)$, we have

$$\alpha(x)(t)(\bigcup_{i\in I} U_i) = t(\bigcup_{i\in I} U_i)(x) = (\bigvee_{i\in I} t(U_i))(x) = \bigvee_{i\in I}(t(U_i)(x)).$$

Step 2. $\alpha(t) \in [D \longrightarrow \pi_L(E)]$, that is $\alpha(t) : D \longrightarrow \pi_L(E)$ is Scott continuous. In fact, first, for $x \leqslant y$ in D and for every $U \in \sigma(E)$, we have

$$\alpha(t)(x)(U) = t(U)(x) \leqslant t(U)(y) = \alpha(t)(y)(U).$$

Second, suppose that $\{x_i |\ i \in I\}$ is a directed subset of D. Then for every $U \in \sigma(E)$, we have

$$\alpha(t)(\bigvee_i x_i)(U) = t(U)(\bigvee_i x_i) = \bigvee_i t(U)(x_i) = \bigvee_i t(x_i)(U).$$

That is, $\alpha(t)(\bigvee_i x_i) = \bigvee_i t(x_i)$. □

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

The previous theorem states that for a possibility computation C, if we have got its fuzzy logical semantics $\|C\|$, then we can get the corresponding denotational semantics $[\![C]\!] = \alpha(\|C\|)$. In the rest of this section, we will consider the another respect—how to produce the corresponding logical semantics $\|C\| = \beta([\![C]\!])$ from a denotational semantics $[\![C]\!]$.

Lemma 4.6. Let X be a dcpo. Then

1. for every family $\{f_i : X \longrightarrow L |\ i \in I\}$ of monotone maps and any $r \in L$, it holds that $(\bigvee_i f_i)^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r) = \bigcup_i f_i^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r)$;
2. for every $f \in L^X$, it holds that $f = \bigvee_{r \in L} r * f^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r)$.

Proof.

1. On one hand, it is easy to see that $(\bigvee_i f_i)^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r) \supseteq \bigcup_i f_i^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r)$ since every f_i is monotone and every $\mathop{\uparrow}\!\!\!\!\triangle r$ is an upper set. On the other hand, suppose that $x \in (\bigvee_i f_i)^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r)$, that is, $r \lhd \bigvee_i f_i(x)$. By the property of interpolation, there is a $s \in L$ such that $r \lhd s \lhd \bigvee_i f_i(x)$. It follows that $r \lhd s \leqslant f_i(x)$ and $x \in f_i^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r)$ for some $i \in I$. Thus $x \in \bigcup_i f_i^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r)$. Hence, $(\bigvee_i f_i)^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r) \subseteq \bigcup_i f_i^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r)$.
2. For every $x \in X$,
$$(\bigvee_{r \in L} r * f^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r))(x) = \bigvee\{r \in L |\ f(x) \in \mathop{\uparrow}\!\!\!\!\triangle r\} = \bigvee(\mathop{\downarrow}\!\!\!\!\triangledown f(x)) = f(x),$$
here the condition $r * 1 = r$ is used. Hence, $f = \bigvee_{r \in L} r * f^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r)$.

□

For every $h \in [D \longrightarrow \pi_L(E)]$, define $\beta(h) : \sigma_L(E) \longrightarrow L^D$ by
$$\beta(h)(f)(x) = \bigvee_{r \in L} [r * h(x)(f^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r))] \quad (\forall f \in \sigma_L(E),\ \forall x \in D).$$

Lemma 4.7. Suppose that $\Pi \in \pi_L(E)$, $U \in \sigma(E)$ and $a \in L$. Then
$$\bigvee_{r \in L} [r * \Pi((a * U)^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r))] = a * \Pi(U).$$

Proof.
$$\bigvee_{r \in L} [r * \Pi((a * U)^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r))] = \bigvee_{r \lhd a} [r * \Pi(U)] = (\bigvee_{r \lhd a} r) * \Pi(U) = a * \Pi(U).$$

□

Proposition 4.8. The transformation $\beta : [D \longrightarrow \pi_L(E)] \longrightarrow [\sigma_L(E) \longrightarrow_H \sigma_L(D)]$ by
$$\beta(h)(f)(x) = \bigvee_{r \in L} [r * h(x)(f^{-1}(\mathop{\uparrow}\!\!\!\!\triangle r))],$$
defines a map.

Proof. Let $h \in [D \longrightarrow \pi_L(E)]$, $f \in \sigma_L(E)$ and $x \in D$.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

Step 1. $\beta(h)(f) \in \sigma_L(D)$. In fact, if $\{x_i |\ i \in I\}$ is a directed subset of D, then

$$\begin{aligned}\beta(h)(f)(\bigvee_i x_i) &= \bigvee_{r\in L}[r * h(\bigvee_i x_i)(f^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= \bigvee_{r\in L}[r * \bigvee_i h(x_i)(f^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= \bigvee_i \bigvee_{r\in L}[r * h(x_i)(f^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= \bigvee_i \beta(h)(f)(x_i).\end{aligned}$$

Step 2. $\beta(h) \in [\sigma_L(E) \longrightarrow_H \sigma_L(D)]$. Let $\{f_i |\ i \in I\} \subseteq \sigma_L(E)$. In fact, by Lemma 4.6(1), we have

$$\begin{aligned}\beta(h)(\bigvee_i f_i)(x) &= \bigvee_{r\in L}[r * h(x)((\bigvee_i f_i)^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= \bigvee_{r\in L}[r * h(x)(\bigcup_i f_i^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= \bigvee_{r\in L}[r * \bigvee_i h(x)(f_i^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= \bigvee_i \bigvee_{r\in L}[r * h(x)(f_i^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= \bigvee_i \beta(h)(f_i)(x).\end{aligned}$$

And for $a \in L,\ U \in \sigma(E)$, by Lemma 4.7, we have

$$\beta(h)(a * U)(x) = \bigvee_{r\in L}[r * h(x)((a * U)^{-1}(\overset{\triangle}{\uparrow} r))] = a * h(x)(U),$$

and

$$\beta(h)(U)(x) = \bigvee_{r\in L}[r * h(x)(U^{-1}(\overset{\triangle}{\uparrow} r))] = 1 * h(x)(U) = h(x)(U),$$

here the condition $1 * r = r$ is used. Hence, $\beta(h)(a * U) = a * \beta(h)(U)$. □

The definition of $\beta(h)(f)(x)$ is similar to the Sugeno's integral in a fuzzy measure space (Sugeno 1974).

Proposition 4.9. For all $h \in [D \longrightarrow \pi_L(E)]$ and $t \in [\sigma_L(E) \longrightarrow_H \sigma_L(D)]$, we have $\alpha(\beta(h)) = h$ and $\beta(\alpha(t)) = t$. That is $[\sigma_L(E) \longrightarrow_H \sigma_L(D)] \cong [D \longrightarrow \pi_L(E)]$ by the pair of functions (α, β).

Proof. First, let $h \in [D \longrightarrow \pi_L(E)],\ x \in D$ and $U \in \sigma(E)$, by Lemma 4.7, we have

$$\begin{aligned}(\alpha \circ \beta)(h)(x)(U) &= \alpha(\beta(h))(x)(U)\\ &= \beta(h)(U)(x)\\ &= \bigvee_{r\in L}[r * h(x)(U^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= 1 * h(x)(U)\\ &= h(x)(U).\end{aligned}$$

So, $\alpha(\beta(h)) = h$.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

Second, let $t \in [\sigma_L(E) \longrightarrow_H \sigma_L(D)]$, $f \in \sigma_L(E)$ and $x \in D$. By Lemma 4.6(2), we have

$$\begin{aligned}(\beta \circ \alpha)(t)(f)(x) &= \beta(\alpha(t))(f)(x)\\ &= \bigvee_{r\in L}[r * \alpha(t)(x)(f^{-1}(\overset{\triangle}{\uparrow} r))]\\ &= \bigvee_{r\in L}[r * t(f^{-1}(\overset{\triangle}{\uparrow} r))(x)]\\ &= t[\bigvee_{r\in L} r * f^{-1}(\overset{\triangle}{\uparrow} r)](x)\\ &= t(f)(x).\end{aligned}$$

So, $\beta(\alpha(t)) = t$. □

We now get a result which shows the equivalence between denotational semantics and logical semantics in such a way that both $\alpha(\beta([\![C]\!])) = [\![C]\!]$ and $\beta(\alpha(\|C\|)) = \|C\|$ will hold, for a given non-deterministic computation C.

Remark 4.10.

1. For the unit interval [0,1] with $* = \wedge$, our duality in Theorem 4.9 turns out to the one in Chen and Wu (2008).
2. In Wu and Chen (2012), with the unit interval [0,1] as the truth value table, Wu and Chen have considered the case of non-determinism of possibility computation and established a duality between the denotational semantics and the logic semantics. Therein, the denotational semantics is defined to be a Scott-continuous function from the input spaces to Hoare possibilistic powerdomain of output spaces; and the logical semantics is given by strongly healthy fuzzy predicate transformers from fuzzy predicates on the domain of output spaces to the ones on the domain of input spaces. Clearly, strongly healthy fuzzy predicate transformers are a special kind of healthy fuzzy predicate transformers. In the future, we may generalize Wu-Chen-duality for strongly healthy fuzzy predicate transformers in a more general setting by following a similar approach in this paper.

This paper is supported by the National Natural Science Foundation of China (11201112), the Natural Science Foundation of Hebei Province (A2013208175, A2014403008) and the Foundation of Hebei Education Department (Y2010020, BR2–210).

References

Chen, Y.X. and Wu, H.Y. (2008). Domain semantics of possibility computations. *Information Sciences* **178** 2661–2679.

Gierz, G., Hofmann, H.H., Keimel, K., Lawson, J.D., Mislove, M. and Scott, D.S. (2003). *Continuous Lattices and Domains*, Cambridge University Press, Cambridge.

Hájek, P. (1998). *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht.

He, J.F., Seidel, K. and McIver, A.K. (1997). Probabilistic models for the guarded command language. *Science of Computer Programming* **28** 171–192.

Hoare, C.A.R. (1985). *Communicating Sequential Process*, Prentice Hall International.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press

Hoffmann, R.-E. (1981). Continuous posets, prime spectra of completely distributive complete lattices, and Hausdorff compactifications. In: Continuous Lattices. *Lecture Notes in Mathematics* **871** 159–208 Springer-Verlag (Berlin–Heidelberg–New York).

Höhle, U. and Šostak, A.P. (1999). Axiomatic foundations of fixed-basis fuzzy topology. Chapter 3 In: Höhle, U. and Rodabaugh, S.E. (eds.) *Mathematics of Fuzzy Sets—Logic, Topology, and Measure Theory*, Kluwer Academic Publishers (Boston/Dordrecht/London) 123–272.

Höhle, U. and Kubiak, T. (2007). Many valued topologies and lower semicontinuity. *Semigroup Forum* **75** 1–17.

Jones, C. (1990). *Probabilistic Non-Determinism*, PhD Thesis, University of Edinburgh, Edinburgh.

Kubiak, T. (1992). The topological modification of the *L*-fuzzy unit interval. In: Rodabaugh, S.E., Klement, E.P. and Höhle, U. (eds.) *Applications of Category Theory to Fuzzy Subsets*, Kluwer, Dordrecht 276–305.

Lawson, J.D. (1979). The duality of continuous posets. *Houston Journal of Mathematics* **5** 357–386.

Liu, Y.M. and Luo, M.K. (1997). *Fuzzy Topology*, World Scientific, Singapore.

Lu, L.-X. (2012). Fuzzy Scott topology on directed complete posets. *Computer Engineering and Applications* **48** (25) 57–60. (In Chinese)

Plotkin, G.D. (1976). A powerdomains construction. *SIAM Journal on Computing* **5** 452–487.

Plotkin, G.D. (1982). Probabilistic powerdomains. In: *Proceedings CAAP* 271–287.

Pultr, A. and Rodabaugh, S.E. (2003). Examples of diffirent sobrieties in fixed-basis topology. In: Rodabaugh, S.E. and Klement, E.P. (eds.) *Topological and Algebraic Structures in Fuzzy Sets: A Handbook of Recent Developments in the Mathematics of Fuzzy Sets*, Trends in Logic volume 20, Kluwer Academic Publishers, Boston/Dordrecht/London 427–440.

Rosenthal, K. (1990). *Quantales and Their Applications*, Pitman Research Notes in Mathematics Series volume 234, Longman Scientific & Technical, Harlow.

Scott, D.S. (1970). Outline of a mathematical theory of computation. In: *The 4th Annual Princeton Conference on Information Science and Systems* 169–176.

Scott, D.S. (1972). Continuous lattices. In: Lawvere, E. (ed.) *Toposes, Algebraic Geometry and Logic*, Lecture Notes in Mathematics volume 274, Springer-Verlag pp. 97–136.

Smyth, M.B. (1978). Powerdomain. *Journal of Computer and System Sciences* **16** 23–36.

Sugeno, M. (1974). *Theory of Fuzzy Integrals and Its Applications*, PhD Doctoral Dissertation, Tokyo Institute of Technology.

Tix, R., Keimel, K. and Plotkin, G.D. (2005). Semantic domains for combining probability and non-determinism. *Electronic Notes in Theoretical Computer Science* **129** 1–104.

Warner, M.W. (1990). Fuzzy topology with respect to continuous lattice. *Fuzzy Sets and Systems* **35** 85–91.

Wu, H.Y. and Chen, Y.X. (2012). Semantics of non-deterministic possibility computation. *Fuzzy Sets and Systems* **199** 47–63.

Yao, W. (2010). Quantitative domains via fuzzy sets: Part I: Continuity of fuzzy directed complete posets. *Fuzzy Sets and Systems* **161** 973–987.

Yao, W. (2012). A survey of fuzzifications of frames, the Papert–Papert–Isbell adjunction and sobriety. *Fuzzy Sets and Systems* **190** 63–81.

Yao, W., Li, Y.L. and Wu, G.N. (2013). Crisp posets can not be roughly considered as fuzzy posets. *Journal of Weinan Normal University* **28** 5–7.

Yao, W. and Shi, F.G. (2011). Quantitative domains via fuzzy sets: Part II: Fuzzy Scott topologies on fuzzy dcpos. *Fuzzy Sets and Systems* **173** 60–80.

Yao, W. and Zhao, B. (2014). A duality between fuzzy domains and strongly completely distributive *L*-ordered sets. *Iranian Journal of Fuzzy Systems* **11** 23–43.

https://doi.org/10.1017/S0960129515000201 Published online by Cambridge University Press