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NOTES ON THE *K*-RATIONAL DISTANCE PROBLEM

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Abstract

Let K be an algebraic number field. We investigate the K-rational distance problem and prove that there are infinitely many nonisomorphic cubic number fields and a number field of degree n for every $n \geq 2$ in which there is a point in the plane of a unit square at K-rational distances from the four vertices of the square.

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1. Introduction

There are many intriguing problems in Guy's book [5], including the perfect cuboid problem [5, D.18] and the rational distance problem [5, D.19]. The perfect cuboid problem asks for a cuboid with rational sides and rational diagonals. The rational distance problem asks for points in the plane of a unit square at rational distances from the four corners of the square. These two problems are still open.

We can study these problems in a number field K of degree greater than 1. Bremner [2] studied the K-rational perfect cuboid problem. His solution is ingenious and can be briefly summarised as follows. Bremner starts with Saunderson's parametrisation [7], leading to the equation

$$t^8 + 64t^6 - 122t^4 + 64t^2 + 1 = s^2. \tag{1.1}$$

Solutions to (1.1) in any number field K gives solutions to the K-rational perfect cuboid problem. To find solutions of (1.1) in cubic number fields, Bremner looks for a factorisation [2, (4.1)]:

$$\begin{aligned} &x^8 + 68x^6 - 122x^4 + 68x^2 + 1 - (c_0gx^2 + c_1x + c_0)^2 \\ &\quad = (x^3 + gx^2 + hx - 1)(x^5 - gx^4 + c_2x^3 - c_3x^2 + c_4x + (c_0^2 - 1)). \end{aligned} \tag{1.2}$$

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Factorisations such as (1.2) give solutions to (1.1) in cubic (and quintic) number fields. Solutions for g, h, c_0, c_1, c_2, c_3, c_4 are parametrised by an elliptic curve of rank 1, $y^2 = x^3 + x^2 - x + 15$, which gives infinitely many factorisations in (1.2).

In this paper, in the same spirit, we study the K-rational distance problem and prove the following theorems.

THEOREM 1.1. *For every positive integer $n \geq 1$, there exists a number field K of degree $2n$ where the K-rational distance problem has a positive solution.*

THEOREM 1.2. *There are infinitely many nonisomorphic cubic number fields K where the K-rational distance problem has a positive solution.*

THEOREM 1.3. *For every odd positive integer n, there exists a number field K of degree n where the K-rational distance problem has a positive solution.*

It is worth mentioning here a variant of the rational distance problem. Bremner and Ulas [3] study rectangles of side lengths 1 and a for rational numbers $a \neq 1$ and show that when a is in a certain dense subset of real numbers, points with rational distances from the vertices of the rectangle do exist.

2. K-rational distances to the corners of a unit square

Let S be a square of side length T and let P be a point in the plane of the square at distances X, Y, Z, U from the vertices of the square, respectively, as in Figure 1. Then, solving the rational distance problem is equivalent to finding rational solutions of the following system (see [1, Section 1]):

$$2(Y^4 + T^4) + X^4 + Z^4 = 2(X^2 + Y^2)(Z^2 + T^2), \tag{2.1}$$

$$U^2 + Y^2 = X^2 + Z^2. \tag{2.2}$$

Using geometric tools, Berry [1] found polynomial parametrisations of degree 2, 4, 6, 8 for (2.1). His list is complete for degrees 2 and 4. We make use of the

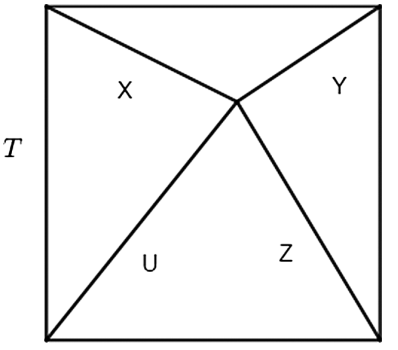


FIGURE 1. Distances to the vertices of a square

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degree 2 parametrisation found by Berry [1, Table 4]:

$$X = t^2 + 2t - 1,$$
$$Y = 1 - t^2,$$
$$Z = t^2 + 1,$$
$$T = 2t.$$

Then (2.2) is equivalent to

$$U^2 = X^2 + Z^2 - Y^2 = t^4 + 4t^3 + 6t^2 - 4t + 1.$$

So solutions to the K-rational distance problem are equivalent to K-rational points (t, U) on the curve

$$U^2 = t^4 + 4t^3 + 6t^2 - 4t + 1, \tag{2.3}$$

with $t \neq 0$. Note that the only rational solutions to (2.3) are $(t, U) = (0, \pm 1)$.

3. Proof of Theorem 1.1

LEMMA 3.1. *The polynomial* $f(x, y) = y^2x^{2n} - x^4 - 4x^3 - 6x^2 + 4x - 1$ *is irreducible in* $\mathbb{Q}[x, y]$.

PROOF. If $f(x, y)$ was reducible over $\mathbb{Q}[x, y]$, then $f(x, y) = 0$ would have a solution $y = p(x)/q(x)$ with $p(x), q(x) \in \mathbb{Q}[x]$. Therefore, (2.3) would have infinitely many rational solutions, which is impossible. Lemma 3.1 is proved. □

PROOF OF THEOREM 1.1. From Lemma 3.1, $f(x, y) = y^2x^{2n} - x^4 - 4x^3 - 6x^2 + 4x - 1$ is irreducible in $\mathbb{Q}[x, y]$. Therefore, by Hilbert's irreducibility theorem [6, Theorem 2.4.1, page 65], there exist infinitely many rational numbers $y \neq 0$ such that $f(x, y)$ is irreducible in $\mathbb{Q}[x]$. Let $K = \mathbb{Q}(\theta)$, where $f(\theta, y) = 0$. Then $|K : \mathbb{Q}| = 2n$ and we have a K-rational solution to (2.1) and (2.2):

$$U = y\theta^n, \quad X = \theta^2 + 2\theta - 1, \quad Y = 1 - \theta^2, \quad Z = 1 + \theta^2, \quad T = 2\theta. \qquad \square$$

4. Proof of Theorem 1.2

Let $U = t^2 + gt + h$, where $g, h \in \mathbb{Q}$. Then

$$U^2 - (t^4 + 4t^3 + 6t^2 - 4t + 1) = (2g - 4)t^3 + (g^2 + 2h - 2)t^2 + 2ght + h^2 - 1.$$

Let

$$F(t)_{g,h} = (2g - 4)t^3 + (g^2 + 2h - 2)t^2 + 2ght + h^2 - 1.$$

If $F(t)_{g,h}$ is reducible in $\mathbb{Q}[t]$, then $F_{g,h}(t) = 0$ has a rational solution $t_0 \in \mathbb{Q}$. Therefore, (2.3) has the rational point $(t_0, t_0^2 + gt_0 + h)$. But the only rational points on (2.3) are $(0, \pm 1)$. Hence $t_0 = 0$. From $F_{g,h}(t_0) = 0$, we have $h^2 = 1$. Hence $h = \pm 1$.

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Let $g = 0$ and $h \neq \pm 1$. Then $F(t)_{0,h} = -4t^3 + (2h-2)t^2 - 1$ is irreducible over $\mathbb{Q}$. Let $K_h = \mathbb{Q}(\lambda)$, where $F(\lambda)_{0,h} = 0$. Then $|K_h : \mathbb{Q}| = 3$. The discriminant of $F(t)_{0,h}$ is

$$G(h) = -32h^5 - 336h^4 - 64h^3 + 800h^2 + 96h - 464.$$

Two cubic number fields are isomorphic if and only if their discriminants are different by a square. But the equation $G(h) = \delta y^2$, where $\delta \in \mathbb{Q}$ is in a finite set, only has a finite number of rational solutions (y, h) by Faltings' theorem [4]. Therefore, there are infinitely many nonisomorphic cubic number fields K_h where the K_h-rational distance problem has a positive solution.

5. Proof of Theorem 1.3

In (2.3), let $t = x^2$ and consider the curve

$$\mathfrak{C}: U^2 = x^8 + 4x^6 + 6x^2 - 4x^2 + 1. \tag{5.1}$$

Let $U = x^4 + 2x^2 + x$. Then

$$x^8 + 4x^6 + 6x^2 - 4x^2 + 1 - U^2 = -2x^5 - 4x^4 - 4x^3 + x^2 + 1,$$

which is an irreducible polynomial in $\mathbb{Q}[x]$. Introduce the field $L = \mathbb{Q}(\alpha)$, where $-2\alpha^5 - 4\alpha^4 - 4\alpha^3 + \alpha^2 + 1 = 0$. Then $(\alpha, \alpha^4 + 2\alpha^2 + \alpha)$ is an L-rational solution to (5.1) and therefore it defines an effective divisor D of degree 5 on $\mathfrak{C}$. For every odd positive integer $n \geq 7$, there exist positive integers k, l such that

$$n = 8k - 5l.$$

Denote by Γ the weighted plane section of the curve $\mathfrak{C}$. Let $F = k\Gamma - lD$. Then F has degree n, so by the Riemann–Roch theorem, F is equivalent to an effective divisor of degree n. Now we can apply the argument in Bremner [2, Theorem 4.2] to prove the existence of a number field K of degree n, such that (5.1) has a solution (x_0, U_0) in K with $x_0 \neq 0$. This gives a solution

$$(X, Y, Z, U, T) = (x_0^4 + 2x_0^2 - 1,\ 1 - x_0^4,\ 1 + x_0^4,\ U_0,\ 2x_0^2)$$

of the K-rational distance problem. The proof is complete.

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