NOTE ON THE SLOW MOTION OF FLUID

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1. In this paper we consider the slow two-dimensional motion of viscous liquid past a sharp edge projecting into and normal to the undisturbed direction of the stream. The liquid is supposed bounded by rigid planes represented by $ABCDE$ in Fig. 1, and, apart from the disturbance caused by the projection, is assumed to be in uniform shearing motion. The stream function is then a biharmonic function that must vanish together with its normal derivative at all points of the boundary, and must be proportional to y^2 at a great distance from the projection.

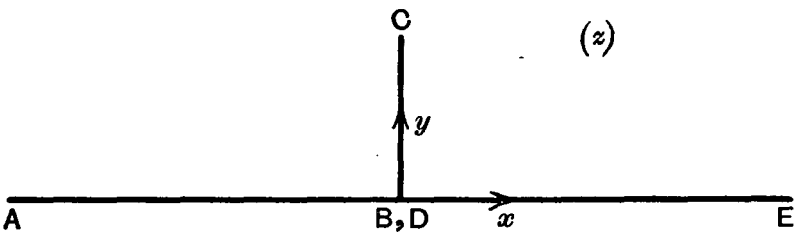


Fig. 1.

A series of biharmonic functions is found, each of which exactly satisfies all the boundary conditions with the exception of the condition that there must be no velocity of slip along the sides of the projection. A linear combination of these functions will also satisfy the conditions with this one exception, and, by choice of the arbitrary constants which it contains, the one remaining boundary condition can in theory be exactly satisfied; at this stage, however, some approximation appears, in practice, to be inevitable. A process of approximation is outlined and carried out until the velocity of slip is everywhere small; this velocity is, in particular, negligible near the end of the projection, because a method was selected that would give its most accurate results at this point, where the fluid motion is naturally of most interest. A consequence of this choice of method is that the approximate stream function is, near the edge, the sum of a number of biharmonic functions which exactly satisfy the boundary conditions on the projection. The form of the first of these functions was given recently by Dr A. Thom; we show how to determine the arbitrary constant in the form, given the undisturbed fluid motion.

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In an earlier paper the writer, considering by quite different means the same fluid motion, calculated the pressure difference between the points B and D, which is of some physical importance; it seemed to be desirable to recalculate this pressure difference, using the improved method of this paper. The earlier result is shown to be of fair accuracy.

2. It is required to find a stream function ψ which satisfies the biharmonic equation

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^2 \psi \equiv \nabla_1^4 \psi = 0, \tag{1}$$

and the boundary conditions

$$\psi = 0, \tag{2}$$

$$\frac{\partial \psi}{\partial \nu} = 0, \tag{3}$$

on $ABCDE$, where $\partial/\partial\nu$ denotes differentiation with respect to distance measured at right angles to the boundary. The stream function must further be such as to give a motion of uniform shear at a great distance from the projection.

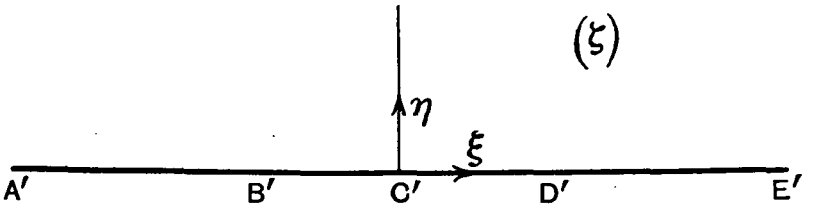


Fig. 2.

The region in the $z\,(x+iy)$ plane bounded by $ABCDE$ can be represented conformally on the upper half of the $\zeta\,(\xi+i\eta)$ plane (Fig. 2) by the transformation

$$z^2 = \zeta^2 - 1 \tag{4}$$

(the projection is at first assumed to be of unit length), and the upper half of the ζ plane can be represented on a semicircle in the $w\,(u+iv)$ plane (Fig. 3) by the transformation*

$$\zeta = -\frac{1}{2}\left(w+\frac{1}{w}\right). \tag{5}$$

Between z and w we have the relation

$$z = \frac{1}{2}\left(w-\frac{1}{w}\right). \tag{6}$$

A uniform shearing motion at great distance from the projection will be obtained if, when $|z|$ is large, the important term in ψ is proportional to y^2, but, since for

* The writer is indebted to Dr A. Weinstein for suggesting the second transformation. Although the functions in § 5 are practically the same as those the writer has used in a recent paper, *Phil. Mag.* 21 (1936), 727, they are much more simply expressed in terms of w than in terms of ζ.

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these values of z the difference between y and η is unimportant, the much more convenient biharmonic function $y\eta$ can be used in place of y^2.

We therefore write $$\psi = y\eta + U + yV, \tag{7}$$

where U and V are harmonic functions (not conjugate); any biharmonic function can be put in this form. With this expression for ψ equation (1) is satisfied, and the first boundary condition (2) is satisfied if

$$U = 0 \quad \text{on } AB,\ DE, \tag{8}$$

$$U + yV = 0 \quad \text{on } BC,\ CD. \tag{9}$$

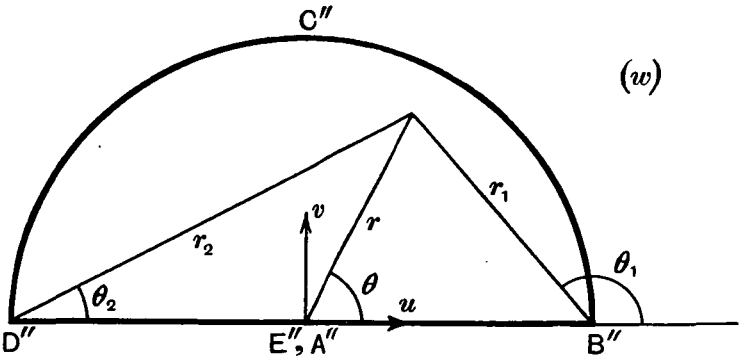


Fig. 3.

At points of AB, DE

$$\frac{\partial}{\partial \nu}(y\eta + U + yV) = \frac{\partial}{\partial y}(y\eta + U + yV) = V + \frac{\partial U}{\partial y},$$

since $y = \eta = 0$, while at points of the projection

$$\frac{\partial}{\partial \nu}(y\eta + U + yV) = y\frac{\partial}{\partial \nu}(\eta + V) + \frac{\partial U}{\partial \nu},$$

since $\eta = \dfrac{\partial y}{\partial \nu} = 0$. The second boundary condition (3) is therefore satisfied if

$$V + \frac{\partial U}{\partial y} = 0 \quad \text{on } AB,\ DE, \tag{10}$$

and

$$y\frac{\partial}{\partial \nu}(\eta + V) + \frac{\partial U}{\partial \nu} = 0 \quad \text{on } BC,\ CD. \tag{11}$$

Now let $$w = r(\cos\theta + i\sin\theta),$$

and express these conditions in terms of r, θ. From (8) and (10) we have

$$U = 0, \quad \theta = 0, \pi, \tag{12}$$

$$V + \frac{2u^2}{1+u^2}\frac{\partial U}{\partial v} = 0, \quad \theta = 0, \pi, \tag{13}$$

since

$$\frac{dw}{dz} = \frac{2w^2}{1+w^2} = \frac{2u^2}{1+u^2}, \quad \theta = 0, \pi,$$

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and since, when $r=1$, $\quad y=\sin\theta, \quad \dfrac{\partial\eta}{\partial r}=-\sin\theta,$

we have from (9) and (11) $\quad U+V\sin\theta=0, \quad r=1, \qquad (14)$

$$\frac{1}{\sin\theta}\frac{\partial U}{\partial r}+\frac{\partial V}{\partial r}=\sin\theta, \quad r=1. \qquad (15)$$

In terms of the new variables the boundary conditions are (12), (13), (14) and (15).

3. We now proceed to find pairs of harmonic functions U, V such that equations (12), (13) and (14) are exactly satisfied by each pair. Any linear combination of these functions will itself satisfy equations (12), (13) and (14) exactly, and we can use the arbitrary constants which it contains to make the combination satisfy the last boundary condition (15) approximately. The solution given below is therefore only an approximation, but we can immediately assess its accuracy by finding to what extent equation (15) is satisfied. It was to obtain this valuable test of the accuracy of the final result that $y\eta$ was used in (7) in place of the obvious expression y^2, for using y^2 we should have (with an approximate method of solution) errors in both equations (14) and (15) and it would have been much more difficult to assess their physical significance. An error in equation (14) implies a velocity of flow through the projection, physically a very different matter from a velocity of slip along the projection which is the consequence of an error in (15). The fluid motion given by this method of solution therefore exactly satisfies all conditions except that there is a velocity of slip along the sides of the projection, and it will be seen in § 7 that this velocity is effectively confined to a small part of the projection (not near the end C) and is everywhere exceedingly small.

4. We write $\quad W=U'+iU=\alpha_1 w+\alpha_3 w^3+\alpha_5 w^5+\ldots, \qquad (16)$

where the constants α_1, α_3, ... are real, U is harmonic and further satisfies the boundary condition (12); it is clear from symmetry that even powers of w are not required in the expansion. Equations (13) and (14) determine, if U is known, the values of V on the arc and on the diameter of the semicircle; the value of V at any internal point follows, and hence the expression for $\dfrac{\partial V}{\partial r}$ when $r=1$. It then remains to adjust the constants α so that equation (15) is satisfied, and here some process of approximation appears inevitable.

Clearly we can consider the terms of (16) separately. First let

$$W_1=U_1'+iU_1=w,$$

and let V_1 be the function corresponding to U_1, the boundary values of V_1 being determined by equations (13) and (14) from those of U_1. Then

$$\frac{\partial U_1}{\partial v}=1,$$

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while, when $r=1$, $U_1=\sin\theta$. Hence

$$V_1=-\frac{2u^2}{1+u^2}, \quad \theta=0,\pi,$$

$$=-1, \quad r=1.$$

If $$V_1=-1+V_1',$$

V_1' is harmonic and is zero when $r=1$, while

$$V_1'=\frac{1-u^2}{1+u^2}, \quad \theta=0,\pi.$$

It is easily verified that we can write

$$V_1'=R\left(\frac{i-w}{i+w}\right),$$

where, here and subsequently, $R(f)$ and $I(f)$ respectively denote the real and imaginary parts of f. V_1' and some similar functions used below can be determined by transforming the semicircle in the w plane into another semicircle so that the arc of either corresponds to the diameter of the other; we have then to find a harmonic function zero on the diameter of a semicircle and expression (16) can again be used. In the case of the simpler functions this process is, of course, unnecessary.

In terms of r and θ, $$V_1'=\frac{1-r^2}{1+2r\sin\theta+r^2},$$

so that $$\frac{1}{\sin\theta}\frac{\partial U_1}{\partial r}+\frac{\partial V_1}{\partial r}=\frac{\sin\theta}{1+\sin\theta}, \quad r=1. \tag{17}$$

5. It was found convenient to group terms arising from the higher powers of w in (16). We therefore write

$$W_3=U_3'+iU_3=\frac{3}{4}\left(w+\frac{w^3}{3}\right),$$

$$W_5=U_5'+iU_5=\frac{5}{8}\left(\frac{w^3}{3}+\frac{w^5}{5}\right)-\frac{1}{4}\left(w+\frac{w^3}{3}\right),$$

and generally

$$W_{2n+1}=U_{2n+1}'+iU_{2n+1}=\frac{2n+1}{4n}\left(\frac{w^{2n-1}}{2n-1}+\frac{w^{2n+1}}{2n+1}\right)-\frac{2n-3}{4(n-1)}\left(\frac{w^{2n-3}}{2n-3}+\frac{w^{2n-1}}{2n-1}\right).$$

We then have

$$\frac{\partial U_3}{\partial v}=R\left(\frac{dW_3}{dw}\right)=R\tfrac{3}{4}(1+w^2)=\tfrac{3}{4}(1+u^2), \quad \theta=0,\pi,$$

and $$\frac{\partial U_{2n+1}}{\partial v}=R\left(\frac{dW_{2n+1}}{dw}\right), \quad n=2,3,\ldots,$$

$$=(1+u^2)\left[\frac{2n+1}{4n}u^{2n-2}-\frac{2n-3}{4(n-1)}u^{2n-4}\right], \quad \theta=0,\pi.$$

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From equations (13) and (14) we have the boundary values of V_3, the harmonic function corresponding to U_3:

$$V_3 = -3u^2/2, \quad \theta = 0, \pi,$$

and

$$V_3 = -1 - \tfrac{1}{2}\cos 2\theta, \quad r = 1.$$

Now let

$$V_3 = -1 - \tfrac{1}{2}r^2 \cos 2\theta + V_3';$$

then V_3' is harmonic, zero when $r = 1$, while

$$V_3' = 1 - u^2, \quad \theta = 0, \pi.$$

It is similarly found in the general case ($n = 2, 3, 4, \ldots$) that

$$V_{2n+1} = \frac{2n-3}{2(n-1)} u^{2n-2} - \frac{2n+1}{2n} u^{2n}, \quad \theta = 0, \pi,$$

$$= -\frac{\cos(2n-2)\theta}{2n-2} - \frac{\cos 2n\theta}{2n}, \quad r = 1,$$

so that if

$$V_{2n+1} = -\frac{r^{2n-2}\cos(2n-2)\theta}{2n-2} - \frac{r^{2n}\cos 2n\theta}{2n} + V_{2n+1}',$$

V_{2n+1}' is harmonic, zero when $r = 1$, while

$$V_{2n+1}' = u^{2n-2} - u^{2n}, \quad \theta = 0, \pi.$$

We now require a harmonic function equal to u^{2n} ($n = 0, 1, 2, \ldots$) on the diameter, and zero on the arc, of the semicircle. Let us denote this function by ϕ_{2n}, and let

$$\phi_{2n} = R\{\chi_{2n}(w)\}.$$

Then it can be verified that

$$\chi_{2n}(w) = -\frac{1}{w^{2n}} + \frac{i}{\pi}\left\{w^{2n} + \frac{1}{w^{2n}}\right\}\{\log(w+1) - \log(w-1)\}$$

$$-\frac{2i}{\pi}\left\{\left(w^{2n-1} + \frac{1}{w^{2n-1}}\right) + \frac{1}{3}\left(w^{2n-3} + \frac{1}{w^{2n-3}}\right) + \ldots\right.$$

$$\left. + \frac{1}{2n-3}\left(w^3 + \frac{1}{w^3}\right) + \frac{1}{2n-1}\left(w + \frac{1}{w}\right)\right\},$$

and, in particular, that

$$\chi_0(w) = -1 + \frac{2i}{\pi}\{\log(w+1) - \log(w-1)\},$$

where (Fig. 3)

$$\log(w+1) = \log r_2 + i\theta_2, \quad 0 \leqslant \theta_2 \leqslant \tfrac{1}{2}\pi,$$

$$\log(w-1) = \log r_1 + i\theta_1, \quad \tfrac{1}{2}\pi \leqslant \theta_1 \leqslant \pi.$$

The functions $\chi_0(w)$, $\chi_2(w)$, ... have, as a consequence of their definition, sin-

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gularities at the points $w = \pm 1$, but not at $w = 0$ nor at any other point inside or on the boundary of the semicircle. We now have

$$V_3 = -1 - \tfrac{1}{2}r^2 \cos 2\theta + \phi_0 - \phi_2,$$

and $$V_{2n+1} = -\frac{r^{2n-2}\cos(2n-2)\theta}{2n-2} - \frac{r^{2n}\cos 2n\theta}{2n} + \phi_{2n-2} - \phi_{2n}, \quad n = 2, 3, \ldots,$$

and hence $$\frac{1}{\sin\theta}\frac{\partial U_3}{\partial r} + \frac{\partial V_3}{\partial r} = \tfrac{3}{2} + \tfrac{1}{2}\cos 2\theta + f_0(\theta) - f_2(\theta), \quad r = 1, \tag{18}$$

$$\frac{1}{\sin\theta}\frac{\partial U_{2n+1}}{\partial r} + \frac{\partial V_{2n+1}}{\partial r} = \frac{2n-1}{2n(n-1)}[1 + 2\cos 2\theta + 2\cos 4\theta + \ldots + 2\cos(2n-2)\theta]$$

$$-\frac{\cos(2n-2)\theta}{2n-2} + \frac{\cos 2n\theta}{2n} + f_{2n-2}(\theta) - f_{2n}(\theta), \quad r = 1, \tag{19}$$

where $$f_{2n}(\theta) = \left(\frac{\partial \phi_{2n}}{\partial r}\right)_{r=1}.$$

The expressions for $f_0(\theta), f_2(\theta), \ldots$ are readily calculated, for

$$\frac{\partial \phi_{2n}}{\partial r} = \cos\theta \frac{\partial \phi_{2n}}{\partial u} + \sin\theta \frac{\partial \phi_{2n}}{\partial v}$$

$$= \cos\theta R\{\chi'_{2n}(w)\} - \sin\theta I\{\chi'_{2n}(w)\},$$

and since, when $r = 1$, $w = \cos\theta + i\sin\theta$,

$$f_{2n}(\theta) = R\{w\chi'_{2n}(w)\}_{r=1}.$$

It is found that

$$f_{2n}(\theta) = \frac{8n}{\pi}\left[\sin(2n-1)\theta + \frac{1}{3}\sin(2n-3)\theta + \ldots + \frac{1}{2n-1}\sin\theta\right.$$

$$\left. + \frac{\pi}{4}\cos 2n\theta + \frac{1}{2}\sin 2n\theta . \log\tan\frac{\theta}{2} - \frac{1}{4n}\operatorname{cosec}\theta\right], \quad n = 1, 2, \ldots, \tag{20}$$

and that $$f_0(\theta) = -\frac{2}{\pi}\operatorname{cosec}\theta. \tag{21}$$

It will be seen that while $f_{2n}(\theta)$ is infinite when $\theta = 0, \pi$, the only combination of these functions which appears above, $f_{2n-2}(\theta) - f_{2n}(\theta)$, is finite for all values of θ.

6. Combining the particular solutions of §§ 4, 5 we have as the expression for the stream function

$$\psi = y\eta + \sum_{n=0}^{\infty} A_{2n+1}(U_{2n+1} + yV_{2n+1}); \tag{22}$$

this is a biharmonic function that, whatever the values of the constants $A_1, A_3, \ldots$, satisfies all the boundary conditions except (15). Consequently in the fluid motion given by (22) there is a velocity of slip along the two sides of the projection, but all other conditions are satisfied.

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To satisfy (15) exactly the constants A must be so chosen that

$$\sum_{n=0}^{\infty} A_{2n+1}\lambda_{2n+1}(\theta) \equiv \sum_{n=0}^{\infty} A_{2n+1}\left[\frac{1}{\sin\theta}\frac{\partial U_{2n+1}}{\partial r}+\frac{\partial V_{2n+1}}{\partial r}\right]_{r=1} = \sin\theta, \qquad (23)$$

for all values of θ between 0 and π, and it is clear that at this stage some process of approximation is necessary. The method used here is to determine the coefficients A so that the expressions on each side of (23) have the same values and the same differential coefficients with regard to θ when $\theta = \frac{1}{2}\pi$, the number of differential coefficients that can be equated depending, of course, on the number of constants used in the expression for ψ. We give below the values of the first six, $A_1, A_3, \ldots, A_{11}$, of the constants; the differential coefficients of odd order of both sides of (23) are zero when $\theta = \frac{1}{2}\pi$, and therefore with six constants at our disposal the two sides can be made to have equal values and equal differential coefficients of the tenth and all lower orders. It will be shown that these six constants are enough to give an approximation of considerable accuracy.

There are, of course, alternative methods of approximation and (in another problem of the same type) some of them were tried, but the method of this paper was found to be the simplest and much the most accurate. What mainly determines the choice is the fact that the motion of the fluid is of most interest in the neighbourhood of the end C of the projection, and C corresponds to the point of the semicircular arc where $\theta = \frac{1}{2}\pi$; we therefore want a process that will give its most accurate results when θ is nearly $\frac{1}{2}\pi$.

That special importance attaches to the neighbourhood of the end of the projection is shown by considering the physical significance of the error of an approximation. Whatever the method of approximation, there will be found, in place of the functions U and V that satisfy (15) exactly, another pair of functions U' and V' which are such that

$$\left[\frac{1}{\sin\theta}\frac{\partial U'}{\partial r}+\frac{\partial V'}{\partial r}\right]_{r=1} = \sin\theta + f(\theta).$$

The corresponding approximate stream function is

$$\psi' = y\eta + U' + yV',$$

and the equation

$$\left(\frac{\partial\psi'}{\partial r}\right)_{r=1} = f(\theta)\sin\theta$$

gives the error in the normal derivative of ψ' at the boundary. The velocity of slip, however, is not $\partial\psi'/\partial r$ but $\partial\psi'/\partial\nu$, and at points of the projection

$$\frac{\partial\psi'}{\partial\nu} = -\frac{\partial\psi}{\partial r}\left|\frac{dw}{dz}\right| = -\frac{\partial\psi}{\partial r}\left|\sec\theta\right|.$$

The result of an error of $f(\theta)$ in the boundary condition (15) is therefore a velocity

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of slip of magnitude $f(\theta)\tan\theta$, which tends to infinity as the end of the projection is approached if $f(\frac{1}{2}\pi)$ is not zero; it is clearly desirable to avoid an error of this kind by the use of some method of approximation which gives an exact result at the end of the projection.

7. The following table gives to six figures the values and successive differential coefficients of $\lambda_1(\theta), \lambda_3(\theta), \ldots$, when $\theta = \frac{1}{2}\pi$; these functions are defined in equation (23) and given in terms of θ by equations (17), (18), (19), (20) and (21).

	$m=1$	$m=3$	$m=5$	$m=7$	$m=9$	$m=11$
λ_m $(\frac{1}{2}\pi)$	0·500	0·453521	−0·0582155	−0·0161511	−0·00649110	−0·00320295
λ_m^{i} $(\frac{1}{2}\pi)$	−0·250	1·63944	−0·150241	−0·0570654	−0·0262456	−0·0138936
λ_m^{iv} $(\frac{1}{2}\pi)$	−0·500	−9·10423	47·0912	−80·4620	111·755	−144·141
λ_m^{vi} $(\frac{1}{2}\pi)$	−2·125	23·6845	−970·405	5433·23	−15684·1	33981·4
λ_m^{viii} $(\frac{1}{2}\pi)$	−15·500	−250·073	15928·3	−242079	1435280	−5273930
λ_m^{x} $(\frac{1}{2}\pi)$	−172·750	−2526·58	−267040	9461390	−109639000	677769000

From equation (23) the simultaneous equations that determine the first six of the constants can now be written down. They are best solved by successively eliminating $A_1, A_3, A_5, \ldots$, in this order; for if this is done it is possible to find the error of the approximation at any stage by comparing the values of the two sides of (23) for values of θ other than $\frac{1}{2}\pi$. Thus after eliminating A_1 and A_3 we have a simple equation for A_5 if we ignore $A_7, A_9, \ldots$, and we can determine the error that is made if only three constants are included in expression (22). The process can be continued until there are enough constants to give a small error. Its utility, of course, depends upon a small error being made with an expression containing only a few constants, for the labour of continuing the process step by step increases very rapidly. It will be seen from the values of the differential coefficients that $\lambda_1(\theta)$ is far better represented over the range $0 \leqslant \theta \leqslant \pi$ by the first six terms of its Taylor's series than is, for instance, $\lambda_{11}(\theta)$, so that a small error depends largely upon the rapid decrease in value of the constants A; however, the error has been shown to be zero, whatever the constants A, when $\theta = 0, \pi$, i.e. at the ends of the range, where the discrepancies between the functions $\lambda(\theta)$ and the first few terms of their Taylor's series would otherwise be most important.

It was found on carrying out the process outlined above that six constants were enough to make the error small throughout the range. To five figures these constants are

$$A_1 = \quad 2{\cdot}2429(5), \qquad A_7 = 0{\cdot}0038936,$$
$$A_3 = -0{\cdot}26800, \qquad A_9 = 0{\cdot}0011261,$$
$$A_5 = -0{\cdot}0023952, \qquad A_{11} = 0{\cdot}00012644.$$

The first column of the following table shows the values of the left side of equation

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(23) with these constants for the values $3\pi/8$, $\pi/4$ and $\pi/8$ of θ, the second column shows the values of the right side, $\sin\theta$, and the third shows the error $f(\theta)$.

		$\sin\theta$	$f(\theta)$
$\theta=\frac{3\pi}{8}$	0·92388	0·92388	0
$\theta=\frac{\pi}{4}$	0·70696	0·70711	−0·00015
$\theta=\frac{\pi}{8}$	0·37487	0·38268	−0·00781

The two sides are exactly equal for $\theta=0$, $\frac{1}{2}\pi$, π, and there is symmetry about the value $\theta=\frac{1}{2}\pi$. The velocity of slip corresponding to the error has been shown to be of magnitude $f(\theta)\tan\theta$; this is largest, for the three values above, when $\theta=\pi/8$, and is then about 0·0032. The physical importance of this error must evidently be exceedingly small.

8. In this paragraph we find the most important part of the stream function at points near the end of the projection, since the fluid motion is naturally of most interest there. Expression (22) with the values of the constants A found in § 7 is an approximate stream function, say ψ', which we write

$$\psi' = y\eta + U' + yV',$$

but it satisfies some of the boundary and other conditions exactly. It is true without approximation that

$$y\eta + U' + yV' = 0$$

when $r=1$ for all values of θ between 0 and π. The boundary condition (15) is satisfied only approximately; it is not therefore exactly true that

$$\left(\frac{1}{\sin\theta}\frac{\partial U'}{\partial r} + \frac{\partial V'}{\partial r} + \frac{\partial \eta}{\partial r}\right)_{r=1}$$

is zero for all θ, but when $\theta=\frac{1}{2}\pi$ the values of the function and of its first and second differential coefficients with regard to θ are exactly zero; differential coefficients of higher order do not at present concern us. Hence the three exact equations

$$\frac{\partial U'}{\partial r} + \frac{\partial V'}{\partial r} + \frac{\partial \eta}{\partial r} = 0,$$

$$\frac{\partial^2 U'}{\partial r\,\partial\theta} + \frac{\partial^2 V'}{\partial r\,\partial\theta} + \frac{\partial^2 \eta}{\partial r\,\partial\theta} = 0,$$

and

$$\frac{\partial^3 U'}{\partial r\,\partial\theta^2} + \frac{\partial U'}{\partial r} + \frac{\partial^3 V'}{\partial r\,\partial\theta^2} + \frac{\partial^3 \eta}{\partial r\,\partial\theta^2} = 0,$$

in which all differential coefficients are to be evaluated for $r=1$, $\theta=\frac{1}{2}\pi$; we can conveniently here and later in the paragraph leave out the suffixes. The second

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equation of the three is an identity, since there is symmetry about the value $\theta = \frac{1}{2}\pi$.

Since ψ' is exactly zero when $r = 1$ for all θ, we have

$$y\eta + U' + yV' = 0,$$

and

$$\frac{\partial^2}{\partial\theta^2}(y\eta + U' + yV') = 0;$$

again all the functions are to be evaluated when $r = 1$, $\theta = \frac{1}{2}\pi$, and the first differential coefficient is identically zero. Since $\eta = 0$ on the boundary, and $y = 1$ at the end of the projection, these two equations can be written

$$U' + V' = 0,$$

$$\frac{\partial^2}{\partial\theta^2}(\eta + U' + V') + V'\frac{\partial^2 y}{\partial\theta^2} = 0.$$

Expressing the four equations in terms of u and v, making use of the fact that y, η, U' and V' are harmonic, and simplifying, we have

$$U' + V' = 0, \tag{24}$$

$$\frac{\partial}{\partial v}(\eta + U' + V') = 0, \tag{25}$$

$$\frac{\partial^2}{\partial v^2}(\eta + U' + V') + V'\left(\frac{\partial y}{\partial v} + \frac{\partial^2 y}{\partial v^2}\right) = 0, \tag{26}$$

and

$$\frac{\partial U'}{\partial v} - \left(3\frac{\partial^2}{\partial v^2} + \frac{\partial^3}{\partial v^3}\right)(\eta + U' + V') = 0, \tag{27}$$

where all the functions and their derivatives are to be evaluated for $u = 0$, $v = 1$.

Now let

$$w = i + w' = i + u' + iv',$$

and find an expression for ψ' valid for small values of w'. Also let

$$z = i + \rho(\cos\phi + i\sin\phi),$$

so that ρ, ϕ are the polar coordinates of any point in the z plane measured from the end of the projection as origin and from a line through it parallel to the boundary as initial line. Then from the relation (6) between z and w it is easy to show that

$$u' = -\rho^{\frac{1}{2}}\left(\cos\frac{\phi}{2} - \sin\frac{\phi}{2}\right) + \rho\cos\phi - \tfrac{1}{4}\rho^{\frac{3}{2}}\left(\cos\frac{3\phi}{2} + \sin\frac{3\phi}{2}\right),$$

$$v' = -\rho^{\frac{1}{2}}\left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\right) + \rho\sin\phi + \tfrac{1}{4}\rho^{\frac{3}{2}}\left(\cos\frac{3\phi}{2} - \sin\frac{3\phi}{2}\right),$$

neglecting terms of order higher than $\frac{3}{2}$ in ρ.

The value of ψ' near the end of the projection is given to terms of order $\rho^{\frac{3}{2}}$ by

$$\left[1 + \left(u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}\right) + \frac{1}{2!}\left(u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}\right)^2 + \frac{1}{3!}\left(u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}\right)^3\right][\eta + U' + V']$$

$$+ \rho\sin\phi\left[1 + \left(u'\frac{\partial}{\partial u} + v'\frac{\partial}{\partial v}\right)\right][\eta + V'];$$

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we use here the relation $y = 1 + \rho \sin \phi$. This expression can be simplified a great deal. Derivatives of odd order with regard to u are zero, since all derivatives are to be evaluated for $u = 0$, $v = 1$, and the three variables η, U' and V' are symmetrical about $u = 0$; these variables are also harmonic so that we can replace derivatives of even order with regard to u by derivatives with regard to v. We then have for the value of ψ'

$$\left[1 + v' \frac{\partial}{\partial v} + \tfrac{1}{2}(v'^2 - u'^2) \frac{\partial^2}{\partial v^2} + \tfrac{1}{6}(v'^3 - 3u'^2 v') \frac{\partial^3}{\partial v^3}\right][\eta + U' + V'] + \rho \sin \phi \left[1 + v' \frac{\partial}{\partial v}\right][\eta + V'],$$

which reduces further by equations (24) and (25) to

$$\left[\tfrac{1}{2}(v'^2 - u'^2) \frac{\partial^2}{\partial v^2} + \tfrac{1}{6}(v'^3 - 3u'^2 v') \frac{\partial^3}{\partial v^3}\right][\eta + U' + V'] + \rho \sin \phi \left[V' + v' \left(\frac{\partial \eta}{\partial v} + \frac{\partial V'}{\partial v}\right)\right].$$

To order $\rho^{\frac{3}{2}}$, $\quad v'^2 - u'^2 = 2\rho \sin \phi + 2\rho^{\frac{3}{2}} \left(\cos \frac{3\phi}{2} - \sin \frac{3\phi}{2}\right)$,

so that the coefficient of $\rho \sin \phi$ in ψ' is

$$\frac{\partial^2}{\partial v^2}(\eta + U' + V') + V';$$

but when $w = i$ $\quad \dfrac{dz}{dw} = 0, \quad \dfrac{d^2 z}{dw^2} = -i,$

whence $\quad \dfrac{\partial y}{\partial v} = 0, \quad \dfrac{\partial^2 y}{\partial v^2} = 1,$

and equation (26) becomes

$$\frac{\partial^2}{\partial v^2}(\eta + U' + V') + V' = 0.$$

The term in $\rho \sin \phi$ accordingly vanishes and the stream function is of order $\rho^{\frac{3}{2}}$ near the end of the projection.

To order $\rho^{\frac{3}{2}}$, $\quad v'^3 - 3u'^2 v' = 2\rho^{\frac{3}{2}} \left(\cos \frac{3\phi}{2} - \sin \frac{3\phi}{2}\right)$,

and ψ' becomes

$$\rho^{\frac{3}{2}} \left(\cos \frac{3\phi}{2} - \sin \frac{3\phi}{2}\right)\left[\frac{\partial^2}{\partial v^2}(\eta + U' + V') + \frac{1}{3} \frac{\partial^3}{\partial v^3}(\eta + U' + V')\right] - \rho^{\frac{3}{2}} \sin \phi \left(\cos \frac{\phi}{2} + \sin \frac{\phi}{2}\right)\left(\frac{\partial \eta}{\partial v} + \frac{\partial V'}{\partial v}\right);$$

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using equations (25) and (27) we have as the final expression for ψ'

$$\frac{\partial U'}{\partial v}\left[\rho^{\frac{3}{2}}\left\{\sin\phi\left(\cos\frac{\phi}{2}+\sin\frac{\phi}{2}\right)+\frac{1}{3}\left(\cos\frac{3\phi}{2}-\sin\frac{3\phi}{2}\right)\right\}\right].$$

When $w=i$, $\quad \dfrac{dW_1}{dw}=1, \quad \dfrac{dW_{2n+1}}{dw}=0, \quad n=1,2,\ldots,$

so that the only contribution to $\dfrac{\partial U'}{\partial v}$ comes from U_1, and $\dfrac{\partial U_1}{\partial v}=1$. Near the end of the projection the most important term in ψ' is accordingly

$$A_1\rho^{\frac{3}{2}}\left[\sin\phi\left(\cos\frac{\phi}{2}+\sin\frac{\phi}{2}\right)+\frac{1}{3}\left(\cos\frac{3\phi}{2}-\sin\frac{3\phi}{2}\right)\right]. \tag{28}$$

Another advantage of the method of approximation selected now appears, for (28) is a biharmonic function that exactly satisfies the boundary conditions

$$\psi=0, \quad \frac{\partial\psi}{\partial\phi}=0,$$

on the projection, on one side of which $\phi=-\frac{1}{2}\pi$ while on the other $\phi=\frac{3}{2}\pi$.

An expression of the same form as (28) has already been found. If we write $\phi=\frac{1}{2}\pi+\phi'$, so that ϕ' is measured from the continuation of the line of the projection and $-\pi\leqslant\phi'\leqslant\pi$, (28) becomes

$$\frac{\sqrt{2}A_1}{6}\rho^{\frac{3}{2}}\left(3\cos\frac{\phi'}{2}+\cos\frac{3\phi'}{2}\right).$$

Now the expression (in the notation of this paper)

$$k\rho^{\frac{3}{2}}\left(3\cos\frac{\phi'}{2}+\cos\frac{3\phi'}{2}\right), \quad k \text{ constant},$$

for the stream function of the slow motion of viscous liquid past a sharp edge has been given by Dr A. Thom* and attributed by him to Dr J. Orr. By means of (28) we can now determine approximately the unknown constant k in this form, given the undisturbed shearing motion of the liquid.

9. It is clear that by continuing the approximation of the preceding paragraph other biharmonic functions which exactly satisfy the conditions on the projection can be found. The next term in ψ' is of order ρ^2, and has been shown to be

$$\rho^2\cos^2\phi\left(\frac{\partial}{\partial v}+\frac{\partial^2}{\partial v^2}\right)(\eta+V'),$$

where the differential coefficients are to be evaluated as usual at the end of the projection. This is the stream function of a uniform shearing motion parallel to

* *Aeronautical Research Committee*, F.M. 101 (1933), 506.

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the line of the projection, this line coinciding with the line of zero velocity of the stream. The third and fourth terms in ψ' are of the forms

$$\rho^{\frac{5}{2}}\left[\cos\frac{5\phi}{2}+\sin\frac{5\phi}{2}-\frac{5}{3}\sin\phi\left(\cos\frac{3\phi}{2}-\sin\frac{3\phi}{2}\right)\right],\quad \rho^3\cos^2\phi\sin\phi,$$

and it is readily verified that these biharmonic functions satisfy the boundary conditions on the projection exactly. There must be slow motions of liquid past a sharp edge in which the stream functions are of these forms near the edge.

10. The velocity of the fluid at great distance from the projection must now be found to show that the motion is ultimately of uniform shear. Let

$$z=R(\cos\alpha+i\sin\alpha),$$

so that R, α are polar coordinates measured now from the foot of the projection as origin; then it is easily shown that when R is large

$$w=-\frac{1}{2z}+\frac{1}{8z^3}+\ldots,$$

and since, from equations (5) and (6), $\zeta=z-w$,

$$y\eta=y^2-\tfrac{1}{2}\sin^2\alpha,$$

ignoring terms of smaller order. The difference between y^2 and the part $y\eta$ of the stream function is therefore independent of R when R is large.

The important part of the other terms $U+yV$ of the stream function is also independent of R when R is large. It is clear that each of $U_1, U_3, \ldots$ tends to zero as R tends to infinity, so that only the term yV has to be examined. When w is small it is easily seen that, approximately,

$$V_1=-\frac{\sin\alpha}{R},$$

$$V_{2n+1}=\frac{8}{(2n-3)(2n-1)(2n+1)}\frac{\sin\alpha}{\pi R},\quad n=1,2,\ldots,$$

so that when R is large yV is proportional to $\sin^2\alpha$.

At a great distance from the projection the whole expression for the stream function accordingly differs from y^2 by some constant times $\sin^2\alpha$ together with terms of smaller order, and the velocity superposed on the uniform shearing motion tends to zero as distance from the projection increases.

11. It is of interest to compare the results of the present paper with those of an earlier paper by the writer* in which by quite different methods the same fluid motion was considered with special reference to the pressure of the fluid.

* *Phil. Mag.* 15 (1933), 929.

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The pressure is determined by the usual equations

$$\frac{\partial p}{\partial x} = \mu \nabla_1^2 u = -\mu \frac{\partial}{\partial y} \nabla_1^2 \psi,$$

$$\frac{\partial p}{\partial y} = \mu \nabla_1^2 v = \mu \frac{\partial}{\partial x} \nabla_1^2 \psi,$$

and, if we write Y for the function conjugate to the harmonic function $\nabla_1^2 \psi$, these equations can be put in the form

$$\frac{\partial}{\partial x}(p - \mu Y) = \frac{\partial}{\partial y}(p - \mu Y) = 0.$$

It follows, if an arbitrary constant is ignored, that

$$p = \mu Y.$$

From expression (7) for ψ we have

$$\nabla_1^2 \psi = 2\left(\frac{\partial \eta}{\partial y} + \frac{\partial V}{\partial y}\right),$$

and hence, finding the conjugate function and expressing V as the real part of $V(w)$, we have

$$p = 2\mu R\left[-i\frac{d\zeta}{dz} + \frac{d}{dz} V(w)\right]$$

$$= 2\mu R \frac{2w^2}{1+w^2}\left[-i\frac{d\zeta}{dw} + V'(w)\right].$$

With the constants of § 7 and the expressions for $V_1, V_3, \ldots$ given in § 5, the pressure can now be written down; it does not, however, seem necessary to give the rather complicated result in full, since most interest attaches to the variation in pressure along the infinite plane boundary and this is given by a much simpler expression. It may, however, be noted that at the end of the projection the pressure is infinite, for the value of w corresponding to this point is i; the vorticity is also infinite, being of order $\rho^{-\frac{1}{2}}$ at points at a small distance ρ from the edge.

The values of w corresponding to points of the plane boundary are real, hence the equation for the pressure reduces at once to

$$p = \frac{4\mu u^2}{1+u^2} R[V'(w)].$$

Now V_{2n+1} is the real part of

$$-\frac{w^{2n-2}}{2n-2} - \frac{w^{2n}}{2n} + \chi_{2n-2}(w) - \chi_{2n}(w),$$

and it is easily seen that when w is real, the real part of $\chi'_{2n}(w)$ is $2nu^{2n-1}$, while the real part of $\chi'_0(w)$ is zero. It follows that the contribution made by the term $A_{2n+1}V_{2n+1}$ of V to the real part of $V'(w)$ is

$$A_{2n+1}[(2n-3)u^{2n-3} - (2n+1)u^{2n-1}], \quad n = 2, 3, \ldots,$$

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while from the terms A_1V_1 and A_3V_3 we have

$$-\frac{4A_1u}{(1+u^2)^2}, \quad -3A_3u.$$

The pressure variation along the plane boundary is therefore given by

$$p=\frac{4\mu u^2}{1+u^2}\left[-\frac{4A_1u}{(1+u^2)^2}-3A_3u+\sum_{n=2}^{\infty}A_{2n+1}\{(2n-3)\,u^{2n-3}-(2n+1)\,u^{2n-1}\}\right],$$

and, in particular, the pressure at D exceeds the pressure at B by

$$4\mu\left[A_1+3A_3+4\sum_{n=2}^{\infty}A_{2n+1}\right].$$

Using the constants given in § 7 we have approximately $5{\cdot}80\mu$ for this pressure difference.

It has been supposed hitherto that the projection is of unit length and that y^2 is the stream function of the undisturbed shearing motion; the undisturbed velocity at unit distance from the plane boundary is in this case 2. If the projection is of height h and V is the velocity of the uniform shearing motion at a distance h from the plane boundary, the pressure difference between D and B is proportional to $\mu V/h$, and the constant follows from the special case we have considered in which $h=1$, $V=2$. We have finally for the pressure difference between points at the foot of the projection and on opposite sides of it

$$2{\cdot}90\,\frac{\mu V}{h}.$$

The result of the earlier paper was

$$3{\cdot}12\,\frac{\mu V}{h},$$

which is in fair agreement with that just found; the present result is much more reliable, for with so small a velocity of slip (there was no corresponding independent check in the earlier paper) the calculated pressure difference should be very accurate.

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