Math. Proc. Camb. Phil. Soc. (1980), **87**, 75
Printed in Great Britain

Transforms which almost vanish at infinity

BY LOUIS PIGNO
Kansas State University, Manhattan, Kansas

(*Received* 3 *October* 1978, *revised* 7 *February* 1979)

Let $\mathbb{T}$ be the circle group, $M(\mathbb{T})$ the set of bounded Borel measures on $\mathbb{T}$ and $\mathbb{Z}$ the additive group of integers. If $\mu \in M(\mathbb{T})$ and $n \in \mathbb{Z}$, define

$$\hat{\mu}(n) = \int_{\mathbb{T}} e^{-in\theta}\, d\mu(\theta).$$

A well-known result of Rajchman states that

$$\limsup_{n\to+\infty} |\hat{\mu}(n)| = 0 \Leftrightarrow \limsup_{n\to-\infty} |\hat{\mu}(n)| = 0.$$

The following quantitative generalization of this result has been given in (2) by K. de Leeuw and Y. Katznelson.

PROPOSITION 1. *For every* $\epsilon > 0$, $\exists \delta(\epsilon) > 0$ *such that, if* $\mu \in M(\mathbb{T})$, $\|\mu\| < 1$ *and*

$$\limsup_{n\to+\infty} |\hat{\mu}(n)| < \delta,$$

then

$$\limsup_{n\to-\infty} |\hat{\mu}(n)| < \epsilon.$$

An infinite set $E \subset \mathbb{Z}$ is said to be *a set of continuity for* $M(\mathbb{T})$ if for every $\epsilon > 0$, $\exists \delta(\epsilon) > 0$ such that, if $\mu \in M(\mathbb{T})$, $\|\mu\| < 1$ and

$$\limsup_{n\notin E} |\hat{\mu}(n)| < \delta,$$

then

$$\limsup_{n\in E} |\hat{\mu}(n)| < \epsilon.$$

Thus Proposition 1 states that $\mathbb{Z}^- = \{n \in \mathbb{Z} : n \leqslant 0\}$ is a set of continuity for $M(\mathbb{T})$. On the other hand, it is easy to see that $E = \mathbb{Z}^- \cup \{2m : m \in \mathbb{Z}\}$ is not a set of continuity for $M(\mathbb{T})$.

In this paper we prove a result which implies among other things that if $m_r \geqslant 0$ and $m_{r+1} - m_r \to \infty$ as $r \to \infty$ then $E = \mathbb{Z}^- \cup \{m_r : r \geqslant 1\}$ is a set of continuity for $M(\mathbb{T})$.

We say that a subset $\mathscr{S}$ of $\mathbb{Z}^+$ satisfies the lacunarity condition $(\mathscr{L}^+)$ if for every infinite sequence $\langle n_j \rangle_1^\infty$, $n_{j+1} > n_j \geqslant 0$,

$$\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} (\mathscr{S} - n_k) \quad \text{is finite.}$$

Notice that if $\mathscr{S} = \{m_1 < m_2 < \ldots\}$ and $m_{j+p} - m_j \to \infty$ as $j \to \infty$ for some fixed $p \geqslant 1$ then $\mathscr{S}$ satisfies $(\mathscr{L}^+)$.

We shall need the following Proposition which is due essentially to P. J. Cohen and H. Davenport(1). In this connexion see the technical lemma in (10).

0305-0041/80/0000-6910 $03.50 © 1980 Cambridge Philosophical Society

https://doi.org/10.1017/S0305004100056541 Published online by Cambridge University Press

PROPOSITION 2. *Let* $r \in \mathbb{Z}^+$, $r \geqslant 3$. *Let* $\lambda \in M(\mathbb{T})$. *Put* $\mathbf{B}(\lambda) = \{n \in \mathbb{Z}: |\hat{\lambda}(n)| \geqslant 1\}$. *Suppose we can find a subset*

$$\{m_0\} \cup \{m_{k,s}: 1 \leqslant k \leqslant r^2, 1 \leqslant s \leqslant r\} \subset \mathbf{B}(\lambda)$$

and from it construct sets $\mathbf{P}_0 = \{m_0\}$ *and*

$$\mathbf{P}_k = \mathbf{P}_{k-1} \cup \{p + m_{k,s} - m_{k,t}: p \in \mathbf{P}_{k-1}, 1 \leqslant s < t \leqslant r\} \cup \{m_{k,s}: 1 \leqslant s \leqslant r\}$$

for $1 \leqslant k \leqslant r^2$ *such that for each* k

$$(\sharp) \qquad |\hat{\lambda}(m)| \leqslant e^{-r} \quad \textit{for all} \quad m \in \mathbf{P}_{k-1} + m_{k,s} - m_{k,t},\ 1 \leqslant s < t \leqslant r.$$

Then $\|\lambda\| \geqslant \frac{1}{4} r^{\frac{1}{2}}$.

Before proving our main result (Theorem 2) we shall next present a simple proof of Proposition 1. The proof uses only Proposition 2 and gives a somewhat nicer relationship of δ to ϵ than that of (2).

THEOREM 1. *Let* $0 < \epsilon < 1$. *Choose* $r \in \mathbb{Z}^+ (r \geqslant 3)$ *such that* $\frac{1}{4} r^{\frac{1}{2}} \geqslant 1/\epsilon$. *Put* $\delta = \epsilon e^{-r}$. *If* $\mu \in M(\mathbb{T})$ *satisfies* $\|\mu\| < 1$ *and*

$$\limsup_{n \to +\infty} |\hat{\mu}(n)| < \delta$$

then

$$\limsup_{n \to -\infty} |\hat{\mu}(n)| \leqslant \epsilon.$$

Proof. Let ϵ and δ be as stated in the hypotheses of the present Theorem. Suppose, by way of contradiction, there is a $\mu \in M(\mathbb{T})$ with $\|\mu\| < 1$ satisfying

$$\limsup_{n \to +\infty} |\hat{\mu}(n)| < \delta \quad \text{and} \quad \limsup_{n \to -\infty} |\hat{\mu}(n)| > \epsilon.$$

Put $\lambda = \mu/\epsilon$ and $E = \{n \in \mathbb{Z}^-: |\hat{\lambda}(n)| \geqslant 1\}$. Inasmuch as $\limsup_{n \to -\infty} |\hat{\mu}(n)| > \epsilon$ it follows that

$$\operatorname{card} E = \infty. \tag{1·1}$$

Set $\mathscr{D} = \{n \in \mathbb{Z}^+: |\hat{\lambda}(n)| \geqslant e^{-r}\}$. As a consequence of $\limsup_{n \to -\infty} |\hat{\mu}(n)| < \delta$ it follows that

$$\operatorname{card} \mathscr{D} \text{ is finite.} \tag{1·2}$$

Select $m_0 \in E$ arbitrarily and put $\mathbf{P}_0 = \{m_0\}$. Suppose that for $1 \leqslant k \leqslant r^2$, $\mathbf{P}_{k-1}$ has been chosen consistent with $(\sharp)$ of Proposition 2. We now choose $\{m_{k,s}\}$ such that

$$(a)\ m_{k,s} \in E,$$

and

$$(b)\ \mathbf{P}_{k-1} + m_{k,s} - m_{k,t} \subset \mathbb{Z}^+ \backslash \mathscr{D} \quad \text{for} \quad 1 \leqslant s < t \leqslant r.$$

Choose $m_{k,1}$ arbitrarily satisfying (a). Let $1 < j \leqslant r$ and suppose the selection of $\{m_{k,i}: 1 \leqslant i < j\}$ consistent with (a) and (b) has been completed. We then choose $m_{k,j}$ satisfying (a) and (b). Clearly such a choice is possible by (1·1) and (1·2). Let

$$m = p + m_{k,i} - m_{k,j},$$

where $p \in \mathbf{P}_{k-1}$ and $1 \leqslant i < j$. As a consequence of (b) we obtain $|\hat{\lambda}(m)| < e^{-r}$. Thus, we may now infer from Proposition 2 that

$$\|\lambda\| \geqslant \tfrac{1}{4} r^{\frac{1}{2}}. \tag{1·3}$$

https://doi.org/10.1017/S0305004100056541 Published online by Cambridge University Press

However, $\|\lambda\| = \frac{1}{\epsilon}\|\mu\|$ so $\|\lambda\| < \frac{1}{\epsilon}$. Since $\frac{1}{4}r^{\frac{1}{2}} \geqslant \frac{1}{\epsilon}$ we have arrived via (1·3) at the following contradiction:

$$\frac{1}{\epsilon} > \|\lambda\| \geqslant \frac{1}{\epsilon}.$$

This completes the proof of Theorem 1.

We now state and prove our main result. Our method of proof is an elaboration on the technique of Theorem 1.

THEOREM 2. *Let $\mathscr{S} \subset \mathbb{Z}^+$ satisfy lacunary condition ($\mathscr{L}^+$). Suppose $0 < \epsilon < 1$. Choose $r \in \mathbb{Z}^+ (r \geqslant 3)$ such that $\frac{1}{4}r^{\frac{1}{2}} \geqslant 1/\epsilon$ and put $\delta = \frac{1}{2}\epsilon e^{-r}$. If $\mu \in M(\mathbb{T})$, $\|\mu\| < 1$ and*

$$\limsup_{n \notin \mathbb{Z}^- \cup \mathscr{S}} |\hat{\mu}(n)| < \delta,$$

then

$$\limsup_{n \in \mathbb{Z}^- \cup \mathscr{S}} |\hat{\mu}(n)| \leqslant \epsilon.$$

Proof. Let ϵ and δ be as stated in the hypotheses of the present Theorem. Suppose, by way of contradiction, that there is a $\mu \in M(\mathbb{T})$ satisfying:

$$\text{(i) } \|\mu\| < 1; \quad \text{(ii) } \limsup_{n \notin \mathbb{Z}^- \cup \mathscr{S}} |\hat{\mu}(n)| < \delta; \quad \text{(iii) } \limsup_{n \in \mathbb{Z}^- \cup \mathscr{S}} |\hat{\mu}(n)| > \epsilon.$$

Put $\lambda = \mu/\epsilon$. As a consequence of (iii) we gather that either $\limsup_{n \in \mathscr{S}} |\hat{\lambda}(n)| > 1$ or $\limsup_{n \in \mathbb{Z}^-} |\hat{\lambda}(n)| > 1$.

We shall first assume that $\limsup_{n \in \mathscr{S}} |\hat{\lambda}(n)| > 1$. It follows that

$$\operatorname{card}\{n \in \mathscr{S} : |\hat{\lambda}(n)| \geqslant 1\} = \infty. \tag{2·1}$$

Let $$\mathscr{B}(\lambda) = \{n \in \mathscr{S} : |\hat{\lambda}(n)| \geqslant 1\} = \{n_1 < n_2 < \ldots\} \subset \mathbf{B}(\lambda).$$

Denote by ν any weak-* cluster point in $M(\mathbb{T})$ of $e^{-in_j\theta}\,d\lambda(\theta)$. Since $\mathscr{S}$ satisfies lacunary condition ($\mathscr{L}^+$) it follows by (ii) that

$$|\hat{\nu}(n)| < \tfrac{1}{2}e^{-r} \tag{2·2}$$

except possibly on a finite set G.

Choose $m_0 \in \mathscr{B}(\lambda)$ arbitrarily. Put $\{m_0\} = \mathbf{P}_0$. Let $1 \leqslant k \leqslant r^2$. Then, since G is finite, we will see that there exists a subset, $\{m_0\} \cup \{m_{k,s} : 1 \leqslant s \leqslant r\}$ of $\mathscr{B}(\lambda)$, satisfying (♯) of Proposition 2. To establish this it suffices to choose $\{m_{k,s}\}$ in a way such that

$$m_{k,s} \in \mathscr{B}(\lambda) \setminus (\mathbf{P}_{k-1} - G) \quad (1 \leqslant s \leqslant r) \tag{2·3}$$

and

$$(\mathbf{P}_{k-1} + m_{k,s} - m_{k,t}) \subset \mathscr{T} \quad (1 \leqslant s < t \leqslant r), \tag{2·4}$$

where

$$\mathscr{T} = \{n \in \mathbb{Z} : |\hat{\lambda}(n)| \leqslant e^{-r}\}.$$

We proceed inductively with our choice of $\{m_{k,s}\}$. Fix k and choose

$$m_{k,r} \in \mathscr{B}(\lambda) \setminus (\mathbf{P}_{k-1} - G)$$

arbitrarily. Suppose for $1 \leqslant j < r$ we have selected $m_{k,i}$ consistent with (2·3) and (2·4) where $j < i \leqslant r$. Choose $m_{k,j}$ to satisfy

$$|(e^{-im_{k,j}\theta}\lambda)^\wedge - \hat{\nu}| < \tfrac{1}{2}e^{-r} \quad \text{on} \quad \bigcup_{i>j} (\mathbf{P}_{k-1} - m_{k,i}).$$

https://doi.org/10.1017/S0305004100056541 Published online by Cambridge University Press

Let $m = p + m_{k,j} - m_{k,i}$ and $p \in \mathbf{P}_{k-1}$. Then for $j < i \leqslant r$,

$$|\hat{\lambda}(m)| \leqslant \tfrac{1}{2}e^{-r} + |\hat{\nu}(p - m_{k,i})| < e^{-r}$$

since $p - m_{k,i} \notin G$. All the above choices are possible since G is finite and ν is a weak-* cluster point of $e^{im_j\theta}\,d\lambda(\theta)$.

Via Proposition 2 and (i)

$$\frac{1}{\epsilon} > \|\lambda\| \geqslant \tfrac{1}{4}r^{\frac{1}{2}} \geqslant \frac{1}{\epsilon};$$

an obvious contradiction.

Now suppose $\limsup_{n \in \mathbb{Z}^-} |\hat{\lambda}(n)| > 1$.

Then $$\operatorname{card}\{n \in \mathbb{Z}^-: |\hat{\lambda}(n)| \geqslant 1\} = \infty.$$

Let $$\mathscr{B}'(\lambda) = \{n \in \mathbb{Z}^-: |\hat{\lambda}(n)| \geqslant 1\} = \{m_1 > m_2 > \ldots\}.$$

Denote by ω any weak-* cluster point in $M(\mathbb{T})$ of $e^{im_j\theta}\,d\lambda(\theta)$. Since $\mathscr{S}$ satisfies lacunary condition ($\mathscr{L}^+$) it follows by (ii) that

$$|\hat{\omega}(n)| < \tfrac{1}{2}e^{-r}$$

except possibly on a finite set G'.

Choose $n_0 \in \mathscr{B}'(\lambda)$ arbitrarily. Put $\{n_0\} = \mathbf{P}_0$. Let $1 \leqslant k \leqslant r^2$. Then, since G' is finite, we will see that there exists a subset, $\{n_0\} \cup \{n_{k,s}: 1 \leqslant s \leqslant r\}$ of $\mathscr{B}'(\lambda)$, satisfying (♯) of Proposition 2. To establish this it suffices to choose $\{n_{k,s}\}$ in a way such that

$$n_{k,s} \in \mathscr{B}'(\lambda) \backslash \{G' - \mathbf{P}_{k-1}\} \quad (1 \leqslant s \leqslant r) \tag{2·5}$$

and $$(\mathbf{P}_{k-1} + n_{k,s} - n_{k,t}) \subset \mathscr{T} \quad (1 \leqslant s < t \leqslant r), \tag{2.6}$$

where, as before, $\mathscr{T} = \{n \in \mathbb{Z}: |\hat{\lambda}(n)| \leqslant e^{-r}\}$.

We proceed inductively with our choice of $\{n_{k,s}\}$. Fix k and choose

$$n_{k,1} \in \mathscr{B}'(\lambda) \backslash \{G' - \mathbf{P}_{k-1}\}$$

arbitrarily. Let $1 < j \leqslant r$ and suppose the selection of $\{n_{k,i}: 1 \leqslant i < j\}$ consistent with (2·5) and (2·6) has been completed. We then choose $n_{k,j}$ satisfying (2·5) and (2·6). The above choices are possible inasmuch as G' is finite and ω is a weak-* cluster point of $e^{im_j\theta}\,d\lambda(\theta)$.

We gather from Proposition 2 and (i) that

$$\frac{1}{\epsilon} > \|\lambda\| \geqslant \tfrac{1}{4}r^{\frac{1}{2}} \geqslant \frac{1}{\epsilon};$$

this contradiction completes the proof.

Comments. (*a*) Theorem 2 provides a quantitative generalization of Theorem 2·20 of (5). In this connexion see also (3).

(*b*) An analogue of Theorem 2 holds for all compact abelian groups with ordered duals.

(*c*) Theorem 2 of the present paper in combination with Theorem 2 of (2) gives a quantitative generalization of H. Helson's semi-idempotent theorem: For any $C > 0$

https://doi.org/10.1017/S0305004100056541 Published online by Cambridge University Press

and $\epsilon > 0$ there is a $\delta = \delta(C, \epsilon) > 0$ satisfying the following: Suppose that $\mu \in M(\mathbb{T})$ has $\|\mu\| < C$ and $\limsup_{n \notin \mathbb{Z}^- \cup \mathscr{S}} |\hat{\mu}(n) - \hat{\mu}(n)^2| < \delta$. Then

$$\mu = \mu_1 + \mu_2,$$

where μ_1 is idempotent with $\hat{\mu}_1$ periodic and

$$\limsup_{|n| \to \infty} |\hat{\mu}_2(n)| \leqslant \epsilon.$$

In this connexion the reader is also referred to (8) and (9).

(*d*) Related results may be found in (4), (6), (7) and (11).

REFERENCES

(1) Davenport, H. On a theorem of P. J. Cohen. *Mathematika* **7** (1960), 93–97.

(2) de Leeuw, K. and Katznelson, Y. The two sides of a Fourier–Stieltjes transform and almost idempotent measures. *Israel J. Math.* **8** (1970), 213–229.

(3) Gardner, P. and Pigno, L. The two sides of a Fourier–Stieltjes transform. *Archiv der Mathematik* **32** (1979), 75–78.

(4) Graham, C. C. Non-Sidon sets in the support of a Fourier–Stieltjes transform. *Colloq. Math.* **36** (1976), 269–273.

(5) López, J. M. and Ross, K. A. *Sidon Sets* (New York, Marcel Dekker).

(6) Pigno, L. Fourier–Stieltjes transforms which vanish at infinity off certain sets. *Glasgow Math. J.* **19** (1978), 49–56.

(7) Pigno, L. Small sets in the support of a Fourier–Stieltjes transform. *Colloq. Math.* (to appear).

(8) Pigno, L. and Smith, B. Almost idempotent measures on compact abelian groups. *J. Functional Analysis* (to appear).

(9) Pigno, L. and Smith, B. Semi-idempotent and semi-strongly continuous measures. (Submitted.)

(10) Ramsey, L. T. and Wells, B. B. Fourier–Stieltjes transforms of strongly continuous measures. *Michigan Math. J.* **24** (1977), 13–19.

(11) Wallen, L. J. Fourier–Stieltjes transforms tending to zero. *Proc. Amer. Math. Soc.* **24** (1970), 651–652.

https://doi.org/10.1017/S0305004100056541 Published online by Cambridge University Press