The Euler numbers and their last digits

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1. *Introduction*

The Euler numbers are the coefficients E_n in the expansion

$$\sec x + \tan x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}. \tag{1}$$

Since $\sec x$ is an even function and $\tan x$ an odd one, this decouples into

$$\sec x = \sum_{n=0}^{\infty} E_{2n} \frac{x^{2n}}{(2n)!} \tag{2}$$

and

$$\tan x = \sum_{n=0}^{\infty} E_{2n+1} \frac{x^{2n+1}}{(2n+1)!}, \tag{3}$$

where all three series converge if $|x| < \frac{\pi}{2}$. E_{2n} and E_{2n+1} are sometimes called the *secant numbers* (or secant coefficients) and *tangent numbers* (or tangent coefficients) respectively.

There is some controversy about this definition, which can be traced back to the Bernoulli numbers. These are the coefficients B_n in the expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!},$$

where the series converges for $|z| < 2\pi$. It is easy to see that

$$\frac{z}{2} + \frac{z}{e^z - 1} = \frac{z}{2} \frac{\cosh \frac{z}{2}}{\sinh \frac{z}{2}},$$

which is an even function, so all the Bernoulli numbers with odd subscripts are zero except $B_1 = -\frac{1}{2}$, and we have

$$\frac{z}{2} \coth \frac{z}{2} = \sum_{n=0}^{\infty} B_{2n} \frac{z^{2n}}{(2n)!}.$$

Since $B_0 = 1$, we can rewrite this as

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} (-1)^{n-1} B_{2n} 2^{2n} \frac{x^{2n-1}}{(2n)!}$$

after setting $z = 2ix$. We can now get expansions of $\csc x$ and $\tan x$ in terms of Bernoulli numbers by using

$$\csc x = \cot \frac{x}{2} - \cot x \qquad \text{and} \qquad \tan x = \cot x - 2 \cot 2x$$

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respectively. All of this development is due to Euler, who gives

$$\tan x = \sum_{n=1}^{\infty} (-1)^{n-1} B_{2n} 2^{2n} \left(2^{2n} - 1\right) \frac{x^{2n-1}}{(2n)!} \tag{4}$$

in [1, Section 222]. He remarks that there is no similar trick for the expansion of $\sec x$, because it is even while $\cot x$ is odd, and he therefore makes the definition (2) in [1, Section 224].

For this reason, the name 'Euler numbers' is frequently reserved for what we have called the secant numbers, but there are several indications that the secant and tangent numbers should be treated together. One knows *a priori* that the Taylor series coefficients of $\tan x$ are nonnegative integers, since they are derivatives of $\tan x$ at $x = 0$, which suggests that (3) may be preferable to (4). Both kinds of Euler numbers appear in several places in Euler's work, and in one important instance they occur together: the first fourteen are in [2] in connection with the formula

$$\sum_{k=0}^{\infty} \frac{(-1)^{kn}}{(2k+1)^n} = \frac{E_{n-1}}{(n-1)!} \frac{\pi^n}{2^{n+1}} \text{ for } n \geqslant 1.$$

We have $E_0 = 1$ since $\sec 0 = 1$, and $E_1 = 1$ since $\left(\tan x\right)'\big|_{x=0} = \sec^2 0 = 1$. The next twelve are

$$\begin{array}{llll} E_2 = 1 & E_3 = 2 & E_4 = 5 & E_5 = 16 \\ E_6 = 61 & E_7 = 272 & E_8 = 1385 & E_9 = 7936 \\ E_{10} = 50521 & E_{11} = 353792 & E_{12} = 2702765 & E_{13} = 22368256 \end{array}$$

Perhaps the best reason to treat the secant and tangent numbers together comes from combinatorics, and we will discuss this in the next section. After that, we give a simple proof that the last digits repeat the pattern 1, 2, 5, 6 starting with E_2, as the data above suggests. We also fill in a gap left by H. G. Forder when he took up this subject in *The Mathematical Gazette* in 1928 [3].

2. *Up-down permutations*

The numbers E_n defined by (1) satisfy several recurrences. The one I like best comes from

$$\frac{d}{dx}(\sec x + \tan x) = \sec x (\sec x + \tan x).$$

As a series identity, this says

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n+1} \frac{x^n}{n!} &= \left(\sum_{j=0}^{\infty} E_{2j} \frac{x^{2j}}{(2j)!}\right)\left(\sum_{k=0}^{\infty} E_{2k} \frac{x^k}{k!}\right) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \sum_{k+2j=n} \frac{n!}{k!\,(2j)!} E_k E_{2j}, \end{aligned}$$

so that

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$$E_{n+1} = \sum_{j=2}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j} E_{2j} E_{n-2j}. \tag{5}$$

This is well suited for hand calculation because it includes only half as many terms as the other most common recurrences.

An *up-down permutation* of $\{1, 2, \ldots, n\}$ is a list of these numbers that alternately rises and falls, starting with a rise, so that 3615472 is an up-down permutation of {1, 2, 3, 4, 5, 6, 7}. A *down-up* permutation is the same thing only beginning with a fall, so that 5273416 is a down-up permutation of {1, 2, 3, 4, 5, 6, 7}. André [4, 5] showed that E_n is the number of up-down permutations of $\{1, 2, \ldots, n\}$. It is also the number of down-up permutations, because we can subtract every element in one kind of permutation from $n + 1$ to get one of the other kind, as with 3615472 and 5273416 where $n + 1 = 8$. André's term *permutations alternées* is still in common use, for example in the beautiful survey [6] by Stanley, but Carlitz thought it suggested a spurious connection with the alternating group, and used 'up-down' instead in a series of papers in the 1970s [7, 8, 9, 10]. I prefer 'up-down' because its companion is the natural "down-up" rather than the awkward 'reverse alternating'.

Both the empty permutation of the empty set and the permutation 1 of {1} are considered to be up-down and down-up, consistent with $E_0 = 1 = E_1$. For $n = 2$, 12 is up-down and 21 is down-up, consistent with $E_2 = 1$. For $n = 3$, the up-down permutations are 132 and 231, and the down-up permutations are 312 and 213, consistent with $E_3 = 2$.

We can prove André's theorem in general by induction from (5). Assume it is true up to n, and look for the element 1 in all the up-down permutations of $\{1, 2, \ldots, n+1\}$. The odd positions are local minima, so 1 must be preceded by an even number $2j$ of the elements $\{2,3, \ldots, n+1\}$, which must be in an up-down pattern. There are $\binom{n}{2j}$ ways to choose these elements, and by induction there are E_{2j} ways to arrange them. The $n - 2j$ elements that come after 1 must be in a down-up pattern, so there are E_{n-2j} ways to arrange them. Since j is a nonnegative integer that cannot exceed $\frac{n}{2}$, the number of up-down permutations of $\{1, 2, \ldots, n+1\}$ is given by the right side of (5), which is André's theorem.

3. *The last digit property*

Finally we will prove the last digit property: the last digits of the Euler numbers repeat the cycle 1, 2, 5, 6 starting with $E_2 = 1$. If $f(x) = \sec x + \tan x$, then since $f'(x) = f(x) \sec x$ we have

$$f''(x) = f'(x)\sec x + f(x)\sec x \tan x = f'(x)(\sec x + \tan x) = f(x)f'(x).$$

It follows that

$$f'''(x) = (f'(x))^2 + f(x)f''(x),$$

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and

$$f''''(x) = 4f'(x)f''(x) + (f(x))^2 f''(x).$$

The fifth derivative can be written as

$$f^{(5)}(x) = 11(f''(x))^2 + 4f'(x)f''(x)\sec x + (f(x))^4 f'(x). \tag{6}$$

Now observe that

$$\begin{aligned}[f(x)]^4 - 1 &= \left([f(x)]^2 + 1\right)\left([f(x)]^2 - 1\right) \\ &= \left(2\sec x(\sec x + \tan x)\right)\left(2\tan x(\sec x + \tan x)\right) \\ &= 4f'(x)f(x)\tan x = 4f''(x)\tan x.\end{aligned}$$

Using this with (6), we obtain

$$\begin{aligned}f^{(5)}(x) - f'(x) &= 11(f''(x))^2 + 4f'(x)f''(x)\sec x + \left[(f(x))^4 - 1\right]f'(x) \\ &= 11(f''(x))^2 + 4f'(x)f''(x)(\sec x + \tan x) \\ &= 11(f''(x))^2 + 4f(x)f'(x)f''(x) = 15(f''(x))^2\end{aligned}$$

and hence

$$f^{(6)}(x) - f''(x) = 30f''(x)f''''(x). \tag{7}$$

As a series identity, (7) reads

$$\sum_{n=0}^{\infty}\left(E_{n+6} - E_{n+2}\right)\frac{x^n}{n!} = 30\left(\sum_{j=0}^{\infty}E_{j+2}\frac{x^j}{j!}\right)\left(\sum_{j=0}^{\infty}E_{k+3}\frac{x^k}{k!}\right).$$

and therefore we have

$$\frac{E_{n+6} - E_{n+2}}{30} = \sum_{j=0}^{n}\binom{n}{j}E_{j+2}E_{n-j+3}. \tag{8}$$

Since the right side is a positive integer, $E_{n+6} - E_{n+2}$ must be divisible by 30. Using the values of E_2 through E_5, it follows that

$$\begin{aligned}E_{4n+2} &\equiv 1 \pmod{30} \qquad E_{4n+4} \equiv 5 \pmod{30} \\ E_{4n+3} &\equiv 2 \pmod{30} \qquad E_{4n+5} \equiv 16 \pmod{30}\end{aligned} \tag{9}$$

for $n \geqslant 0$, which proves the last digit property. Forder gives an argument like this for the secant numbers in [3] (though naturally I think mine is nicer), and says that the tangent numbers can be treated similarly. He laments that his approach does not seem to give the best possible congruences of this type for the tangent numbers, but we just have to do a little more with (8). If n is even, say $n = 2m$, then one of $j + 2$ and $2m - j + 3$ is even and one is odd, so $E_{j+2}E_{2m-j+3}$ is an odd number times an even number for each j. Therefore every term on the right side of (8) is even, so $E_{2m+6} - E_{2m+2}$ is actually divisible by 60 for $m \geqslant 0$. We can't do better than this, since it equals 60 when $m = 0$. If n is odd, say

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$n = 2m + 1$, then the right side of (8) is

$$\sum_{j=0}^{2m+1} \binom{2m+1}{j} E_{j+2}E_{2m-j+4} = \sum_{k=0}^{m} \binom{2m+1}{k} E_{k+2}E_{2m-k+4} + \sum_{j=m+1}^{2m+1} \binom{2m+1}{j} E_{j+2}E_{2m-j+4}.$$

Setting $j = 2m + 1 - k$ in the last sum, (8) becomes

$$\frac{E_{2m+7} - E_{2m+3}}{30} = \sum_{k=0}^{m} \binom{2m+1}{k} \left(E_{k+2}E_{2m-k+4} + E_{k+3}E_{2m-k+3}\right). \tag{10}$$

Suppose $r, s \geqslant 2$ are integers with $r + s \equiv 2 \pmod 4$. If r and s are both even, then one is divisible by 4 and one isn't, so (9) tells us that $E_rE_s \equiv 1 \cdot 5 \equiv 5 \pmod{30}$. If r and s are both odd, then both are congruent either to 1 or to 3 (mod 4). In the former case (9) says that $E_rE_s \equiv 16^2 \equiv 16 \equiv -14 \pmod{30}$, and in the latter case it says $E_rE_s \equiv 2^2 \equiv 4 \pmod{30}$. It follows that

$$E_{k+2}E_{2m-k+4} + E_{k+3}E_{2m-k+3} \equiv \pm 9 \pmod{30}$$

for each k, so the right side of (10) is divisible by 3. Therefore $E_{2m+7} - E_{2m+3}$ is actually divisible by 90 for $m \geqslant 0$. We can't do better than this, since $E_7 - E_3 = 90 \times 3$ and $E_9 - E_5 = 90 \times 88$. Thus we finally have

$$\begin{aligned} E_{4n-2} &\equiv 1 \pmod{60} & E_{4n} &\equiv 5 \pmod{60} \\ E_{4n-1} &\equiv 2 \pmod{90} & E_{4n+1} &\equiv 16 \pmod{90} \end{aligned} \tag{11}$$

for $n \geqslant 1$.

Scherk [11] was the first to prove the last digit property for the secant numbers. Forder refers to Stern [12] for it, who in turn refers to [11]. By a more sophisticated analysis than our own, [13] and [14] show that the secant numbers are periodic not only mod 60 but for any odd prime modulus. (I thank the referee for pointing out the latter reference.) The tangent numbers have received much less attention for the reasons outlined in the introduction.

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