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TAIL BEHAVIOR OF STOPPED LÉVY PROCESSES WITH MARKOV MODULATION—CORRIGENDUM

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Theorem A.2 in Beare, Seo, and Toda (2022), henceforth BST, is not correct. It was not proved there, but was claimed to be an amalgamation of results proved in Steinberg (1968), Howland (1971) and Schumacher (1986). We misunderstood the contribution of Howland (1971) and consequently the part of Theorem A.2 attributed to that article is incorrect. Specifically, the second of the three conditions claimed to be equivalent in Theorem A.2 is not in fact equivalent to the first and third conditions. We also incorrectly wrote r (the matrix rank) in place of $N-r$ (the matrix nullity) in the statement of the third condition, though we applied Theorem A.2 elsewhere as though we had correctly written $N-r$. We provide here the following corrected statement of Theorem A.2.

THEOREM 1 (Correction to Theorem A.2 in BST). *Let $A(z)$ be an $N \times N$ complex matrix-valued holomorphic function of $z \in \Omega$, where Ω is an open and connected subset of the complex plane. Suppose that $A(z)$ is invertible for some $z \in \Omega$. Then $A(z)$ has a meromorphic inverse $A(z)^{-1}$ on Ω, with poles at the points of noninvertibility of $A(z)$. If $A(z_0)$ has rank $r < N$ for some $z_0 \in \Omega$, so that z_0 is a pole of $A(z)^{-1}$, then the following two conditions are equivalent.*

(i) *z_0 is a simple pole of $A(z)^{-1}$.*
(ii) *There exist $N \times (N-r)$ complex matrices x and y of full column rank such that $A(z_0)x = 0$, $y^\top A(z_0) = 0$, and $y^\top A'(z_0)x$ is invertible.*

Under either of these equivalent conditions, the residue of $A(z)^{-1}$ at the simple pole z_0 is equal to $x(y^\top A'(z_0)x)^{-1}y^\top$.

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The part of Theorem 1 asserting meromorphicity of $A(z)^{-1}$ is supplied by Steinberg (1968, Thm. 1). The equivalence of conditions (i) and (ii), and the residue formula, are supplied by Schumacher (1986, Thm. 4.1 and Remark 2); see also Schumacher (1991, pp. 562–563). The relevant results in Schumacher (1986) constitute a version of Keldysh's theorem, introduced in Keldysh (1951, 1971). See Schumacher (2024) for further discussion of the connection to Keldysh's theorem. Also note that the equivalence of conditions (i) and (ii) in Theorem 1 follows from Schumacher (2024, Prop. 7.3), and that the residue formula is given by Schumacher (2024, Cor. 7.5).

Theorem A.2 in BST includes a third condition claimed to be equivalent to (i) and (ii): that the zero eigenvalue of $A(z_0)$ has equal geometric and algebraic multiplicities. In fact, if one considers the very simple example in which $\Omega = \mathbb{C}$, $N = 1$, $A(z) = z^2$ and $z_0 = 0$, then it is clear that the zero eigenvalue of the 1×1 matrix $A(z_0) = 0$ has geometric and algebraic multiplicities equal to one, and yet z_0 is a pole of order two of $A(z)^{-1} = z^{-2}$. Thus in this example (i) is not satisfied despite the eigenvalue multiplicity condition being satisfied.

Theorem A.2 in BST is used to prove Theorem 3.1 in BST and Theorem 3 in Beare and Toda (2022), henceforth BT. In particular, the eigenvalue multiplicity condition is used to verify that a pole is simple. Consequently the proofs are not correct. We show in this corrigendum how to repair the proofs.

The central concern in BST and BT is the upper tail of a probability distribution on the real line with Laplace transform of the form $\varphi(z) = -v^\top A(z)^{-1} w$. Here, v and w are semipositive $N \times 1$ vectors, and $A(z)$ is an $N \times N$ complex matrix-valued holomorphic function of $z \in \mathbb{C}$. The objective is to show that the upper tail probabilities decay exponentially, and to characterize the rate of exponential decay. This can be accomplished by showing that the right abscissa of convergence $\alpha > 0$ of the Laplace transform $\varphi(z)$ is a simple pole of $A(z)^{-1}$, and that the corresponding residue is not annihilated when one constructs the bilinear form $v^\top A(z)^{-1} w$, so that α is also a simple pole of $\varphi(z)$. The exponential decay rate of upper tail probabilities is then given by α. Unfortunately, the method used in BST and BT to verify that α is a simple pole of $A(z)^{-1}$ relies on showing that the zero eigenvalue of $A(\alpha)$ has equal geometric and algebraic multiplicities. As already discussed, this condition is not sufficient for α to be a simple pole of $A(z)^{-1}$.

The proofs of Theorem 3.1 in BST and Theorem 3 in BT may be repaired by using the following lemma to establish that α is a simple pole of $A(z)^{-1}$. In its statement we write $\zeta(\cdot)$ for the spectral abscissa of a square complex matrix; that is, for the maximum real part of all eigenvalues of that matrix. See BST for the definition of an irreducible Metzler matrix.

LEMMA 1. *Let $A(z)$ be an $N \times N$ complex matrix-valued holomorphic function of $z \in \Omega$, where Ω is an open and connected subset of the complex plane containing a real number z_0. Assume:*

(a) *$A(z)$ is irreducible Metzler for all real $z \in \Omega$.*
(b) *$\zeta(A(z_0)) = 0$.*
(c) *$\zeta(A(z))$ has strictly positive left-derivative at z_0 as a function of real $z \in \Omega$.*

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Then $A(z_0)$ has rank $r = N - 1$, and conditions (i) and (ii) in Theorem 1 are satisfied.

Proof. Condition (a) implies that $A(z_0)$ is irreducible Metzler. Therefore, Theorem A.3 in BST, i.e., the Perron–Frobenius theorem, implies that $\zeta(A(z_0))$ is an algebraically simple eigenvalue of $A(z_0)$. Since $\zeta(A(z_0)) = 0$ under condition (b), we deduce that the nullity of $A(z_0)$ is one. Thus, by the rank-nullity theorem, $A(z_0)$ has rank $r = N - 1$. The Perron–Frobenius theorem further implies that the one-dimensional null spaces of $A(z_0)$ and $A(z_0)^\top$ are respectively spanned by unique real $N \times 1$ vectors x_0 and y_0 with strictly positive entries summing to one.

We now verify condition (ii) in Theorem 1. Since the null spaces of $A(z_0)$ and $A(z_0)^\top$ are respectively spanned by x_0 and y_0, it suffices to show that $y_0^\top A'(z_0)x_0 \neq 0$. Let $z_1, z_2, \ldots$ be a strictly increasing sequence of real numbers in Ω which converges to z_0. Such a sequence exists because Ω is open. For each $n \in \mathbb{N}$, $A(z_n)$ is irreducible Metzler under condition (a). Thus the Perron–Frobenius theorem implies that $\zeta(A(z_n))$ is an algebraically simple eigenvalue of $A(z_n)$, and that to this eigenvalue there correspond unique right- and left-eigenvectors x_n and y_n with strictly positive entries summing to one. Since $\zeta(A(z_0))$ is an algebraically simple eigenvalue of $A(z_0)$ and $A(z_n) \to A(z_0)$ as $n \to \infty$, we must have $x_n \to x_0$ as $n \to \infty$; see e.g., Ortega (1972, p. 45).

For each $n \in \mathbb{N}$, observe that

$$\begin{aligned}[A(z_n) - A(z_0)]x_0 &= A(z_n)x_n - A(z_n)(x_n - x_0) - A(z_0)x_0 \\ &= \zeta(A(z_n))x_n - A(z_n)(x_n - x_0) \\ &= [\zeta(A(z_n)) - \zeta(A(z_0))]x_n - A(z_n)(x_n - x_0), \end{aligned} \tag{1}$$

where we use $A(z_n)x_n = \zeta(A(z_n))x_n$ and $A(z_0)x_0 = 0$ to obtain the second equality, and condition (b) to obtain the third. Premultiplying both sides of (1) by $y_0^\top$, dividing by $z_n - z_0$, and using the fact that $A(z_0)^\top y_0 = 0$, we obtain

$$y_0^\top \frac{A(z_n) - A(z_0)}{z_n - z_0} x_0 = \frac{\zeta(A(z_n)) - \zeta(A(z_0))}{z_n - z_0} y_0^\top x_n - y_0^\top \frac{A(z_n) - A(z_0)}{z_n - z_0}(x_n - x_0). \tag{2}$$

Since $A(z)$ is differentiable, we have $[A(z_n) - A(z_0)]/(z_n - z_0) \to A'(z_0)$ as $n \to \infty$. And, as observed earlier, $x_n \to x_0$ as $n \to \infty$. Thus, by letting $n \to \infty$ in (2), we obtain

$$y_0^\top A'(z_0)x_0 = \left(\lim_{n \to \infty} \frac{\zeta(A(z_n)) - \zeta(A(z_0))}{z_n - z_0} \right) y_0^\top x_0. \tag{3}$$

The limit in parentheses is strictly positive under condition (c), and we have $y_0^\top x_0 > 0$ because x_0 and y_0 have strictly positive entries. Thus $y_0^\top A'(z_0)x_0 > 0$. This verifies condition (ii) in Theorem 1.

Condition (c) implies that there exists $n \in \mathbb{N}$ such that $\zeta(A(z_n)) < \zeta(A(z_0))$. (Indeed, this must be true for all sufficiently large n.) Condition (b) therefore implies that $\zeta(A(z_n)) < 0$. Thus $A(z_n)$ is invertible. The requirement in Theorem 1 that $A(z)$ is invertible for some $z \in \Omega$ is therefore satisfied. Consequently,

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Theorem 1 asserts the equivalence of conditions (i) and (ii), and our verification of condition (ii) serves also to verify condition (i). □

The proof of Theorem 3.1 in BST may be repaired as follows. There are two appeals to the incorrect Theorem A.2. The first appeal is not problematic because it does not involve the troublesome eigenvalue multiplicity condition. The second appeal to Theorem A.2 should be replaced with an appeal to Theorem 1 and Lemma 1 here. Specifically, it should be observed that $A(z)$ satisfies conditions (a) and (b) of Lemma 1 by assumption or construction, and satisfies condition (c) because $\zeta(A(z))$ has been shown to be convex on a real interval containing zero and containing α as an interior point, and because $\zeta(A(0)) < 0$ and $\zeta(A(\alpha)) = 0$. Thus Lemma 1 implies that conditions (i) and (ii) in Theorem 1 are satisfied, and Theorem 1 supplies the desired residue formula.

The proof of Theorem 3 in BT may be repaired in similar fashion. Once again, there are two appeals to Theorem A.2 in BST, but only the second is problematic, and should be replaced with an appeal to Theorem 1 and Lemma 1 here. Note that the notation $A(z)$ as used here should be understood to correspond to $A(z) - I$ in the notation used in BT. With the latter notation, $A(z)$ is irreducible and nonnegative for real z, and thus $A(z) - I$ is irreducible Metzler with spectral abscissa equal to the spectral radius of $A(z)$ minus one.

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