*Macroeconomic Dynamics*, **1**, 1997, 485–512. Printed in the United States of America.

# CAPM RISK ADJUSTMENT FOR EXACT AGGREGATION OVER FINANCIAL ASSETS

**WILLIAM A. BARNETT**
*Washington University in St. Louis*

**YI LIU**
*University of South Alabama*

**MARK JENSEN**
*University of Missouri*

Barnett originated the Divisia monetary aggregates, which in continuous time exactly track any monetary aggregator function under perfect certainty. With user costs measuring the prices of the services of components, Barnett's aggregates are based on Francois Divisia's derivation of the Divisia line integral from the first-order conditions for optimizing behavior by economic agents under perfect certainty. We derive an extended Divisia index from the first-order conditions (Euler equations) that apply under risk. Our extended Divisia index is the first extension of index number theory into the domain of decision making under risk and thereby produces a route for the extension of all index number theory to permit non-risk-neutrality. We generate simulated data from a modeled rational consumer and investigate the tracking accuracy of the extended Divisia index to the consumer's exact aggregator function.

**Keywords:** Aggregation, Index Number Theory, Risk, Capital Asset Pricing

> The economic statistics that the government issues every week should come with a warning sticker: User beware. In the midst of the greatest information explosion in history, the government is pumping out a stream of statistics that are nothing but myths and misinformation.
>
> ———Michael J. Mandel, "The Real Truth About the Economy: Are Government Statistics So Much Pulp Fiction? Take a Look," *Business Week*, cover story, November 7, 1994, pp. 110–118.

Barnett's research was partially supported by NSF grant SES 9223557; Jensen's research was supported under that grant as research assistant. Address correspondence to: William A. Barnett, Department of Economics, Washington University, Campus Box 1208, One Brookings Drive, St. Louis, MO 63130–4899, USA; e-mail: barnett@wuecon.wustl.edu.

© 1997 Cambridge University Press 1365-1005/97 $9.00 + .10 

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

## 1. INTRODUCTION

In the case of perfect certainty, the Divisia index exactly tracks any aggregator function. This follows from the fact that the Divisia line integral is directly derivable from the first-order conditions for optimizing behavior. This result is especially well known in the case of consumer behavior, in which the Divisia index is derived directly from the total differential of the demand function, after substitution of the first-order conditions for maximizing utility subject to a budget constraint. However, the exact tracking property of the Divisia index also applies to the demand for monetary services by firms and the supply of produced monetary services by financial intermediaries. See Barnett (1987).

Risk aversion is another story. The first-order conditions in the case of risk aversion are Euler equations. Because those are not the first-order conditions used in deriving the Divisia index under perfect certainty, the tracking ability of the unadjusted Divisia index is compromised. The degree to which the tracking ability degrades is a function of the degree of risk aversion and the amount of risk. In principle, this problem could be solved by estimating the Euler equations by generalized method of moments and producing the estimated exact rational expectations monetary aggregator function. This inference procedure is in accordance with the one widely advocated as the solution to the Lucas critique and, more recently, also advocated as the solution to what Chrystal and MacDonald (1994, p. 76) have called the Barnett critique.[1] However, estimation of aggregator functions, although in strict accordance with the principles of microeconomic aggregation theory, produces results that depend on the parametric specification of the aggregator function and the choice of econometric estimator for estimating the parameters of the aggregator.

Index number thoery exists precisely for the purpose of permitting specification-free, nonparametric tracking ability. The Divisia index is such a parameter-free index number and hence depends only on data. Although the Divisia index number is known to permit exact tracking for any economic aggregator function under perfect certainty [see Hulten (1973)], that index has never been extended to a statistical index number that will track exactly under risk aversion. In fact, to our knowledge, no nonparametric, statistical index numbers have ever previously been derived directly from Euler equations in a manner that retains tracking ability under risk. In this paper, we derive a statistical index number directly from the Euler equations.[2] The resulting index number turns out to be an extension of the original Divisia index derived by Francois Divisia (1925) under perfect certainty, such that our extended Divisia index remains exact under risk aversion and reduces to the usual Divisia index in the special case of perfect certainty. The derivation is analogous to that for the usual Divisia index, but our extended Divisia index is derived from the Euler equations that are the correct first-order conditions produced from rational behavior of economic agents under risk.

If additional assumptions are imposed, we find that the resulting generalized Divisia index has a direct connection with the capital asset pricing model (CAPM)

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

in finance. In a sense, our theory is a simultaneous generalization of both the CAPM and economic index number theory, because our theory contains both as nested special cases. In particular, CAPM deals with a two-dimensional trade-off between expected return and risk, whereas the Divisia index deals with the two-dimensional trade-off between investment return and liquidity. Our generalized theory includes the three-dimensional trade-off among mean return, risk, and liquidity. The two well-known special cases are based on two-dimensional sections, which are orthogonal to each other, through the relevant three-dimensional space.

A particularly productive area of possible application of this new index number is monetary aggregation, because money-market assets are characterized by substantially different degrees of each of the three characteristics: mean rate of return, risk, and liquidity, especially when the collection of money-market assets includes those subject to prepayment penalties, such as Series EE bonds and nonnegotiable certificates of deposit, and those subject to regulated low rates of return, such as currency. When central banks first produced monetary aggregates, all of the components over which they aggregated yielded no interest. Hence, there was perfect certainty about the rate of return on each component. In addition, because that rate of return was exactly zero for each component, the user costs were known to be the same for each component. Under those circumstances, it is well known in aggregation theory that the correct method of aggregation is simple summation, but monetary assets no longer yield the same rates of return and cannot be viewed as perfect substitutes. In addition, the interest yield is not a monetary service, so that the interest yield's capitalized value, although embedded in the value of the stock of such assets, is not part of the economic monetary stock. The capitalized value of the monetary service flow, net of that interest yield, is the economy's economic monetary stock. Furthermore, because interest is not paid in advance, there is some degree of uncertainty about that rate of return, which is needed to compute the foregone interest (user cost) of any interest-yielding monetary asset. These observations indicate that the ability to track a nonlinear aggregator function under risk is needed to be able to measure the economy's monetary service flow.

In the case of the current monetary aggregates, the component assets yield rates of return having low variance and low correlation with consumption. As a result, the ordinary Divisia index produced from perfect-certainty first-order conditions may be adequate to track the service flows of those collections of assets, but there is growing research interest in the possibility of incorporating into monetary aggregates some assets that have substantial risk, such as common stock and bond funds. See, e.g., Barnett and Zhou (1994) and Feldstein and Stock (1994). With such potential component assets, the perfect-certainty first-order conditions are not suitable and hence the ordinary Divisia monetary aggregates may not track well.

In its most general interpretation, the objective of this paper is to extend index number theory to apply to aggregation over goods for which contemporaneous prices are not known with certainty. Because the user cost prices of financial assets depend upon risky contemporaneous rates of return, aggregation over financial

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

assets seems like an especially suitable area for application of risk-extended index number theory.

### 1.1. Money in the Utility Function

A large and growing literature seeks to explain why rate-of-return-dominated money exists (i.e., has positive value) in equilibrium. This issue is important and merits much research. Nevertheless, it is well known in general equilibrium theory that if money has positive value in equilibrium, then a derived utility function containing money exists such that behavior can be described by maximizing that derived utility subject to a budget constraint. See, e.g., Samuelson (1948), Quirk and Saposnik (1968), Arrow and Hahn (1971), Phlips and Spinnewyn (1982), Feenstra (1986), Poterba and Rotemberg (1987), and Croushore (1993). The same result is available for firms. See, e.g., Fischer (1974). In fact, the resulting derived utility or production function has indeed been derived for many of the explicit mechanisms producing positive value for money in equilibrium.

The converse of the theorem is especially challenging in its implications. If a model containing an explicit motive for holding money cannot generate a derived utility function from its original utility function and constraints, then money cannot have positive value in equilibrium. Only two possible conclusions then can be reached. Either the model fails to produce a positive-equilibrium quantity of money, and money is driven out of existence in equilibrium as a result of its rate dominance, or the asset being modeled as "money" is in fact not money and need not exist in equilibrium.

Empirically, that well-known theorem and its converse imply that behavior under explicit motives for holding money induces rational (transitive, consistent) behavior within the space of goods and monetary assets, and indeed many tests of the axioms of revealed preference have accepted those axioms in that space. However, knowledge of the induced preference preordering over that Cartesian product space is not sufficient to produce a unique inverse mapping back to the original decision. Hence, the properties of the derived utility function do not uniquely reveal the explicit motive for holding money. The nonuniqueness of the inverse mapping is a constructive reason for interest in models containing an explicit motive for holding money, but, for some purposes, we do not need to know that motive.

Macroeconomists, monetary economists, and central bankers have many reasons for wanting an explanation of the fact that positive quantities of money are held in equililbrium, despite the fact that many monetary assets are rate dominated as investments. However, in aggregation theory we have no need for such an explanation. Monetary assets are indeed held in positive quantities, and not all monetary assets yield the same investment rate of return. Within the derived utility function over the Cartesian product space of goods and monetary assets, we know that monetary assets having different own rates of return are most certainly not perfect substitutes. That fact alone is sufficient to permit us to deduce that simple sum aggregation over currency and any interest-yielding monetary asset makes no

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

sense whatsoever, and any empirical research using such a simple sum aggregate over such assets is contaminated by those data. We need no further information about the explicit motive for holding money to deduce that result in microeconomic aggregation theory.

In this paper, we display the derivation of the CAPM-extended Divisia monetary index based upon the derived utility function containing money. However, it should be observed that exactly the same result would be produced from any explicit motive for holding money, such as a model having transactions technology constraint. Even the case of perfect substitution between components is a nested special case of the index derived below.

## 2. CONSUMER DEMAND FOR MONETARY ASSETS

### 2.1. The Decision

In this section we formulate a representative consumer's stochastic decision problem over consumer goods and monetary assets. The consumer's decisions are made in discrete time over an infinite planning horizon for the time intervals, $t, t+1, \ldots, s, \ldots$, where $t$ is the current time period. The variables used in defining the consumer's decision are as follows: $\boldsymbol{x}_s = n$-dimensional vector of real consumption of goods and services during period $s$, $\boldsymbol{p}_s = n$-dimensional vector of goods and services prices and of durable-goods rental prices during period $s$, $\boldsymbol{a}_s = k$-dimensional vector of real balances of monetary assets during period $s$, $\boldsymbol{\rho}_s = k$-dimensional vector of nominal holding-period yields of monetary assets, $A_s =$ holdings of the benchmark asset during period $s$, $R_s =$ the one-period holding yield on the benchmark asset during period $s$, $I_s =$ the sum of all other sources of income during period $s$, and $p_s^* = p_s^*(\boldsymbol{p}_s) =$ the true cost-of-living index.

Define $Y$ to be the consumer's survival set, assumed to be a compact subset of the $n+k+2$ dimensional nonnegative orthant. The consumer's consumption possibility set, $S(s)$ for period $s$, is

$$
\begin{aligned}
S(s) = \Bigg\{ & (\boldsymbol{a}_s, \boldsymbol{x}_s, A_s) \in Y : \sum_{i=1}^{n} p_{is} x_{is} \\
& = \sum_{i=1}^{k} \left[ (1+\rho_{i,s-1}) p_{s-1}^* a_{i,s-1} - p_s^* a_{is} \right] \\
& \quad + (1+R_{s-1}) p_{s-1}^* A_{s-1} - p_s^* A_s + I_s \Bigg\}.
\end{aligned}
$$

The benchmark asset $A_s$ provides no services other than its yield $R_s$. As a result, the benchmark asset does not enter the consumer's contemporaneous utility function. The asset is held only as a means of accumulating wealth. The consumer's

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

subjective rate of time preference, $\xi$, is assumed to be constant.[3] The single-period utility function, $u(\boldsymbol{a}_t, \boldsymbol{x}_t)$, is assumed to be increasing and strictly quasi-concave.

The consumer's decision problem is the following.

Problem 1. Choose the deterministic point $(\boldsymbol{a}_t, \boldsymbol{x}_t, A_t)$ and the stochastic process $(\boldsymbol{a}_s, \boldsymbol{x}_s, A_s)$, $s = t+1, \ldots, \infty$, to maximize

$$u(\boldsymbol{a}_t, \boldsymbol{x}_t) + E_t \left[ \sum_{s=t+1}^{\infty} \left( \frac{1}{1+\xi} \right)^{s-t} u(\boldsymbol{a}_s, \boldsymbol{x}_s) \right] \tag{1}$$

subject to $(\boldsymbol{a}_s, \boldsymbol{x}_s, A_s) \in S(s)$ for $s, t$, and also subject to the transversality condition

$$\lim_{s \to \infty} E_t \left( \frac{1}{1+\xi} \right)^{s-t} A_s = 0.$$

The transversality condition rules out perpetual borrowing at the benchmark rate, $R_t$.

## 2.2. Existence of a Monetary Aggregate for the Consumer

To ensure the existence of a monetary aggregate for the consumer, we partition the vector of monetary asset quantities, $\boldsymbol{a}_s$, such that $\boldsymbol{a}_s = (\boldsymbol{m}_s, \boldsymbol{h}_s)$. We correspondingly partition the vector of interest rates of those assets, $\boldsymbol{\rho}_s$, such that $\boldsymbol{\rho}_s = (\boldsymbol{r}_s, \boldsymbol{i}_s)$. We then assume that the utility function, $u$, is blockwise weakly separable in $\boldsymbol{m}_s$ and in $\boldsymbol{x}_s$ for some such partition of $\boldsymbol{a}_s$ and blockwise strongly separable in $\boldsymbol{h}_s$.[4] Hence, there exist a monetary aggregator ("category utility") function, $M$, a consumer goods aggregator function, $X$, and utility functions, $F$ and $H$, such that

$$u(\boldsymbol{m}_s, \boldsymbol{h}_s, \boldsymbol{x}_s) = F[M(\boldsymbol{m}_s), X(\boldsymbol{x}_s)] + H(\boldsymbol{h}_s), \tag{2}$$

and we define the implied utility function $V(\boldsymbol{m}_s, c_s)$ by $V(\boldsymbol{m}_s, c_s) = F[M(\boldsymbol{m}_s), c_s]$, where aggregate consumption of goods is defined by $c_s = X(\boldsymbol{x}_s)$. Then, it follows that the exact monetary aggregate is

$$M_s = M(\boldsymbol{m}_s). \tag{3}$$

We define the dimension of $\boldsymbol{m}_s$ to be $k_1$, and the dimension of $\boldsymbol{h}_s$ to be $k_2$, so that $k = k_1 + k_2$. The fact that blockwise weak separability is a necessary condition for exact aggregation is well known in the perfect-certainty case. In fact, if the resulting aggregator function also is linearly homogeneous, the theory of two-stage budgeting can be used to prove that the consumer behaves as if the exact aggregate were an elementary good. Because two-stage budgeting theory is not applicable under risk, we provide in the Appendix an aggregation theorem proving that $M(\boldsymbol{m}_s)$ can be treated as a quantity aggregate, in a well-defined sense, even under risk.

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

The Euler equations that will be of the most use to us below are those for monetary assets. Those Euler equations are

$$E_s\left[\frac{\partial V}{\partial m_{is}} - \rho\frac{p_s^*(R_s - r_{is})}{p_{s+1}^*}\frac{\partial V}{\partial c_{s+1}}\right] = 0 \tag{4a}$$

for $s$, $t$, and $i = 1, \ldots, k_1$, where $\rho = 1/(1+\xi)$ and where $p_s^*$ is the exact price aggregate that is dual to the consumer goods quantity aggregate $c_s$.[5] Similarly, we can acquire the Euler equation for the consumer goods aggregate $c_s$, rather than for each of its components. The resulting Euler equation for $c_s$ is

$$E_s\left[\frac{\partial V}{\partial c_s} - \rho\frac{p_s^*(1+R_s)}{p_{s+1}^*}\frac{\partial V}{\partial c_{s+1}}\right] = 0. \tag{4b}$$

For the formal derivation of (4a) and (4b), using Bellman's method, see Barnett (1995, Sec. 2.3).

## 3. THE PERFECT-CERTAINTY CASE

In the perfect-certainty case, nonparametric index number theory is highly developed and is applicable to monetary aggregation. In the perfect-certainty case, Barnett (1978, 1980) proved that the contemporaneous real user cost of the services of $m_{it}$ is $\pi_{it}$, where

$$\pi_{it} = \frac{R_t - r_{it}}{1 + R_t}. \tag{5}$$

The corresponding nominal user cost is $p_t^*\pi_{it}$. It can be shown that the solution value of the exact monetary aggregate $M(\boldsymbol{m}_t)$ can be tracked without error in continuous time [see, e.g., Barnett (1980)] by the Divisia index:

$$d\log M_t = \sum_{i=1}^{k_1} s_{it}\, d\log m_{it}, \tag{6}$$

where the user-cost-evaluated expenditure shares are

$$s_{it} = \pi_{it}m_{it} \Big/ \sum_{j=1}^{k_1}\pi_{jt}m_{jt}.$$

The flawless tracking ability of the index in the perfect-certainty case holds regardless of the form of the unknown aggregator function, $M$. In discrete time, the Simpson's rule discrete time approximation to the Divisia index is called the Törnqvist index and is exact for any aggregator function up to a third-order remainder term in the changes. That remainder term usually is less than the roundoff error in the component data and typically is negligible for data that are available with at least annual frequency. However, under risk, the ability of equation (6) to

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

track $M(\boldsymbol{m}_t)$ is compromised in continuous time and hence also in discrete time at all data sampling frequencies.

## 4. NEW GENERALIZED DIVISIA INDEX

### 4.1. User Cost of Money Under Risk Aversion

We now return to the Euler equations for optimal behavior of consumers under risk. Those Euler equations are displayed in equation (4a) for monetary assets and equation (4b) for consumer goods. Our objective is to find the formula for the user cost of monetary services in a form that is applicable to our model of decision under risk. The following definition for the contemporaneous user cost simply states that the real user cost price of a monetary asset is the marginal rate of substitution between those assets and consumer goods.

DEFINITION 1. *The contemporaneous risk-adjusted real user cost price of the services of monetary asset $i$ is $\Pi_{it}$, defined such that*

$$\Pi_{it} = \frac{\partial V}{\partial m_{it}} \Big/ \frac{\partial V}{\partial c_t}.$$

No expectations operators appear in that definition, because the marginal utilities at $t$ are known with certainty in period $t$. Nevertheless, formula (5), which applies under perfect certainty, cannot be correct under risk because the interest rates in equation (5) are not known contemporaneously, and so, the right-hand side of equation (5) is stochastic, whereas Definition 1 defines $\Pi_{it}$ to be deterministic. In this section, we derive the correct deterministic formula for the user cost defined by Definition 1.

For notational convenience, we sometimes convert the nominal rates of return, $r_{it}$ and $R_t$, to real total rates of return, $1 + r_{it}^*$ and $1 + R_t^*$, such that

$$1 + r_{it}^* = \frac{p_t^*(1 + r_{it})}{p_{t+1}^*} \quad \text{and} \quad 1 + R_t^* = \frac{p_t^*(1 + R_t)}{p_{t+1}^*}, \tag{7}$$

where $r_{it}^*$ and $R_t^*$ defined in that manner are called the real rates of excess return. Under this change of variables and observing that current-period marginal utilities are known with certainty, Euler equations (4a) and (4b) become

$$\frac{\partial V}{\partial m_{it}} - \rho E_t \left[ \left( R_t^* - r_{it}^* \right) \frac{\partial V}{\partial c_{t+1}} \right] = 0 \tag{8}$$

and

$$\frac{\partial V}{\partial c_t} - \rho E_t \left[ \left( 1 + R_t^* \right) \frac{\partial V}{\partial c_{t+1}} \right] = 0. \tag{9}$$

We now can prove our user cost theorem under risk.

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

THEOREM 1. *The risk adjusted user cost of the services of monetary asset i under risk is* $\Pi_{it} = \pi_{it} + \psi_{it}$, *where*

$$\pi_{it} = \frac{E_t R_t - E_t r_{it}}{1 + E_t R_t} \tag{10}$$

*and*

$$\psi_{it} = \rho(1 - \pi_{it}) \frac{\operatorname{Cov}\left(R_t^*, \dfrac{\partial V}{\partial c_{t+1}}\right)}{\dfrac{\partial V}{\partial c_t}} - \rho \frac{\operatorname{Cov}\left(r_{it}^*, \dfrac{\partial V}{\partial c_{t+1}}\right)}{\dfrac{\partial V}{\partial c_t}}. \tag{11}$$

Proof. Equation (8) can be rewritten for current period $t$ to be

$$\frac{\partial V}{\partial m_{it}} = \rho E_t \left[ \left(R_t^* - r_{it}^*\right) \frac{\partial V}{\partial c_{t+1}} \right]. \tag{12}$$

If the marginal utility and the interest rates in the expectation on the right-hand side of (12) were uncorrelated, we could write the expectation of the product as the product of the expectations. However, under our assumption of weak separability in monetary assets, $\boldsymbol{m}_t$, the utility function $V$ can be written in the form $V(\boldsymbol{m}_t, c_t) = F[M(\boldsymbol{m}_t), c_t]$, where the consumer is risk neutral if and only if $F$ is linear in $M_t = M(\boldsymbol{m}_t)$ and in $c_t$. Hence, under risk neutrality, $V$ must be linear in $c_t$, so that the marginal utility of consumption must be a constant. Without risk neutrality and the resulting constancy of the marginal utility of consumption, we have no reason to expect the interest rates and marginal utility on the right-hand side of (27) to be uncorrelated. The result is that (12) becomes

$$\frac{\partial V}{\partial m_{it}} = \rho E_t \left[ \frac{\partial V}{\partial c_{t+1}} \right] \left(E_t R_t^* - E_t r_{it}^*\right) + \rho \operatorname{Cov}\left(R_t^*, \frac{\partial V}{\partial c_{t+1}}\right) - \rho \operatorname{Cov}\left(r_{it}^*, \frac{\partial V}{\partial c_{t+1}}\right), \tag{13}$$

where the covariances would become zero if we were to assume risk neutrality. Similarly, without risk neutrality, equation (9) becomes

$$\frac{\partial V}{\partial c_t} = \rho E_t \left[ \frac{\partial V}{\partial c_{t+1}} \right] + \rho E_t \left[R_t^*\right] E_t \left[ \frac{\partial V}{\partial c_{t+1}} \right] + \rho \operatorname{Cov}\left(R_t^*, \frac{\partial V}{\partial c_{t+1}}\right). \tag{14}$$

By eliminating $\rho E_t[\partial V / \partial c_{t+1}]$ between equations (13) and (14), we get

$$\frac{\partial V}{\partial m_{it}} = (\pi_{it} + \psi_{it}) \frac{\partial V}{\partial c_t}, \tag{15}$$

where

$$\pi_{it} = \frac{E_t R_t^* - E_t r_{it}^*}{1 + E_t R_t^*} \tag{16}$$

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

and

$$\psi_{it} = \rho(1-\pi_{it})\frac{\text{Cov}\left(R_t^*, \dfrac{\partial V}{\partial c_{t+1}}\right)}{\dfrac{\partial V}{\partial c_t}} - \rho\frac{\text{Cov}\left(r_{it}^*, \dfrac{\partial V}{\partial c_{t+1}}\right)}{\dfrac{\partial V}{\partial c_t}}. \tag{17}$$

Using equation (7) to convert the real rates in equation (16) back to nominal rates, equation (16) becomes (10), whereas equation (17) is immediately identical to equation (11). Solving equation (15) for $\Pi_{it} = \pi_{it} + \psi_{it}$, Theorem 1 follows from Definition 1. ■

Under risk neutrality, the covariances in (17) would all be zero because the utility function would be linear in consumption. Hence, the user cost would reduce to $\pi_{it}$, as defined in equation (10). The following corollary is immediate.

COROLLARY 1 TO THEOREM 1. *Under risk neutrality, the user cost formula is the same as equation* (5) *in the perfect-certainty case, but with all interest rates replaced by their expectations.*

However, under risk aversion the utility function is strictly concave in consumption, so that marginal utility is inversely related to consumption. In principle, it is possible for the interest rate on a slightly risky investment to reduce the risk in the consumer's consumption stream if that interest rate and consumption are negatively correlated. Because of the inverse relationship between consumption and marginal utility, we conclude that risk is decreased by an investment if the rate of return is positively correlated with marginal utility. For monetary assets, with little or no principle risk and low volatility, the riskiness of the asset is likely to contribute relatively little to the riskiness of the household's consumption stream, and hence the sign of the covariance between the asset's rate of return and of the consumption stream is not easy to predict a priori. But with a very risky asset, such as common stock, it is far more likely that holding such a risky investment will increase risk rather than decrease it. That occurs if the rate of return on the asset is positively correlated with consumption and thereby negatively correlated with marginal utility. This phenomenon is central to the consumption-based capital asset pricing model (CCAPM).

Consider the interpretation of equation (11), which defines the adjustment for risk under risk aversion. Suppose we normalize relative to $\partial V/\partial c_t$, so that we need not consider the denominator of equation (11). Now consider first the second term on the right-hand side of equation (11). Suppose that the own rate of return on monetary asset $i$ is positively correlated with the marginal utility of consumption of goods, so that holding that monetary asset decreases risk. Because holding the asset decreases the consumer's consumption risk, we should expect that the risk-adjusted user cost price $\Pi_{it} = \pi_{it} + \psi_{it}$ that the consumer would have to "pay" to hold that asset would be decreased as that positive covariance increases, and that is precisely what the second term of equation (11) would do in that case. Conversely, if the covariance between the own rate and the marginal utility of consumption of

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

goods is negative, so that holding the asset increases the risk of the consumer's consumption stream, the second term in equation (11) introduces a positive term into the risk-adjusted user cost $\Pi_{it} = \pi_{it} + \psi_{it}$ to reflect the increased cost of holding the asset as that covariance increases the consumer's risk. If the central bank were to introduce common stock or bond funds into monetary aggregates or other assets having substantial principal risk, we should expect to find the latter case would apply to those assets.

Now consider the first term on the right-hand side of equation (11). The benchmark rate is the interest rate foregone by not holding the benchmark asset. If the benchmark rate decreases consumption risk through a positive covariance between the benchmark rate and the marginal utility of consumption of goods, then the opportunity cost of foregoing the benchmark asset by instead holding monetary asset $i$ is increased. Hence, we should expect that such a positive covariance should increase the risk-adjusted user cost $\Pi_{it}$, as indeed is the effect of the first term of equation (11). Conversely, if that covariance is negative, so that holding the benchmark asset increases the consumer's risk, then foregoing the benchmark asset in favor of monetary asset $i$ decreases risk and hence results in a subtraction from the risk-adjusted user cost, $\Pi_{it}$, of holding asset $i$.

### 4.2. Generalized Divisia Index Under Risk Aversion

The ordinary Divisia index was derived by Francois Divisia from the first-order conditons for rational consumer behavior under perfect certainty. In the case of risk aversion, the first-order conditions are Euler equations, and we have found that those Euler equations for monetary assets demanded by consumers can be put into the form (8), which we now use to derive a generalized Divisia index, as follows.

THEOREM 2. *In the share equations* $s_{it} = \pi_{it} m_{it} / \sum_{j=1}^{k_1} \pi_{jt} m_{jt}$ *replace the unadjusted user costs* $\pi_{it}$, *defined by* (5), *by the risk-adjusted user costs* $\Pi_{it}$, *defined by Definition* 1, *to produce the adjusted shares* $S_{it} = \Pi_{it} m_{it} / \sum_{j=1}^{k_1} \Pi_{jt} m_{jt}$. *Under our weak-separability assumption,* $V(\boldsymbol{m}_t, c_t) = F[M(\boldsymbol{m}_t), c_t]$, *and our assumption that the monetary aggregator function* $M$ *is linearly homogeneous, the following generalized Divisia index is true under risk:*

$$d \log M_t = \sum S_{it}\, d \log m_{it}. \tag{18}$$

Proof. Under our weak-separability assumption, $V(\boldsymbol{m}_t, c_t) = F[M(\boldsymbol{m}_t), c_t]$, we have that

$$\frac{\partial V}{\partial m_{it}} = \frac{\partial F}{\partial M_t} \frac{\partial M_t}{\partial m_{it}}. \tag{19}$$

Substituting (15) into (19), we acquire

$$\frac{\partial M_t}{\partial m_{it}} = (\pi_{it} + \psi_{it}) \left( \frac{\partial V}{\partial c_t} \Big/ \frac{\partial F}{\partial M_t} \right). \tag{20}$$

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

Because the total differential of $M_t = M(\boldsymbol{m}_t)$ is

$$dM_t = \sum_i \frac{\partial M}{\partial m_{it}} dm_{it}, \tag{21}$$

we can substitute (20) into (21) to get

$$dM_t = \left(\frac{\partial V}{\partial c_t} \Big/ \frac{\partial F}{\partial M_t}\right) \sum (\pi_{it} + \psi_{it})\, dm_{it}. \tag{22}$$

Using the linear homogeneity of $M$, we have from Euler's theorem for homogeneous functions that

$$M_t = \sum_i \frac{\partial M}{\partial m_{it}} m_{it}. \tag{23}$$

Substituting (20) into (23), we acquire

$$M_t = \left(\frac{\partial V}{\partial c_t} \Big/ \frac{\partial F}{\partial M_t}\right) \sum (\pi_{it} + \psi_{it}) m_{it}. \tag{24}$$

Dividing (22) by (24), we get equation (18). ■

Hence, we see that the exact tracking of the Divisia monetary index is not compromised by risk aversion, as long as the adjusted user costs $\pi_{it} + \psi_{it}$ are used in computing the index. As we have observed, the adjusted user costs reduce to the usual user costs in the case of perfect certainty and our generalized Divisia index (18) reduces to the usual Divisia index (6). Similarly, the risk-neutral case is acquired as the special case with $\psi_{it} = 0$, so that equation (16) serves as the user cost. In short, our generalized Divisia index (18) is a true generalization in the sense that the risk-neutral and perfect-certainty cases are strictly nested special cases. Formally, that conclusion is the following.

COROLLARY 1 TO THEOREM 2. *Under risk neutrality, the generalized Divisia index* (18) *reduces to* (6)*, where the user costs in the formula are defined by* (16)*. Under perfect certainty, the user costs reduce to* (5)*.*

The need for the generalization can be explained as follows. The consumer has a three-dimensional decision, in terms of asset characteristics. The monetary assets having nonzero own rates of return produce investment returns, contribute to risk, and provide liquidity services. Our objective is to track the nested utility function, $M(\boldsymbol{m}_t)$, which measures only liquidity and is the true economic monetary aggregate. To do so, we must remove the other two motives: investment yield and risk aversion. Although those two motives are relevant to savings and intertemporal substitution, we seek to track the liquidity flow alone. The ordinary Divisia monetary aggregate removes the investment motive and would track the liquidity services if there were no risk. The generalized Divisia index removes both the investment motive and and the aversion-to-risk motive to extract the liquidity service flow, when the data are produced by consumers who in fact are making decisions

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

that involve a three-way trade-off among mean investment return, risk aversion, and liquidity service consumption.

## 5. CCAPM SPECIAL CASE

As a means of illustrating the nature of the risk adjustment $\psi_{it}$, we consider a special case, based on the usual assumptions in CAPM theory of either quadratic utility or Gaussian stochastic processes. Direct empirical use of Theorems 1 and 2, without any CAPM simplifications, would require availability of prior econometric estimates of the parameters of the utility function $V$ and of the subjective rate of time discount. Under the usual CAPM assumptions, we show in this section that empirical use of Theorems 1 and 2 would require prior estimation of only one property of the utility function: the degree of risk aversion, on which a large body of published information is available.

Consider first the following case of utility that is quadratic in consumption of goods, conditionally on the level of monetary asset service consumption.

Assumption 1. Let $V$ have the form

$$V(\boldsymbol{m}_t, c_t) = F[M(\boldsymbol{m}_t), c_t] = A[M(\boldsymbol{m}_t)]c_t - \tfrac{1}{2}B[M(\boldsymbol{m}_t)]c_t^2, \tag{25}$$

where $A$ is a positive, increasing, concave function and $B$ is a nonnegative, decreasing, convex function.

The alternative assumption is Guassianity, as follows:

Assumption 2. Let $(r_{it}^*, c_{t+1})$ be a bivariate Gaussian process for each asset $i = 1, \ldots, k_1$.

We also make the following conventional CAPM assumption[6]:

Assumption 3. The benchmark rate process is deterministic or already risk-adjusted, so that $R_t^*$ is a risk-free rate.

Under this assumption, it follows that

$$\operatorname{Cov}\left(R_t^*, \frac{\partial V}{\partial c_{t+1}}\right) = 0.$$

We define $H_{t+1} = H(M_{t+1}, c_{t+1})$ to be the well-known Arrow–Pratt measure of absolute risk aversion,

$$H(M_{t+1}, c_{t+1}) = \frac{-E_t[V'']}{E_t[V']}, \tag{26}$$

where $V' = \partial V(\boldsymbol{m}_{t+1}, c_{t+1})/\partial c_{t+1}$ and $V'' = \partial^2 V(\boldsymbol{m}_{t+1}, c_{t+1})/\partial c_{t+1}^2$. In this definition, risk aversion is measured relative to consumption risk, conditionally upon the level of monetary services produced by $M_{t+1} = M(\boldsymbol{m}_t)$. Under risk aversion,

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

$H_{t+1}$ is positive and increases as the degree of absolute risk aversion increases. The following lemma is central to our Theorem 3.

LEMMA 1. *Under Assumption* 3 *and either Assumption* 1 *or Assumption* 2, *the user-cost risk adjustment,* $\psi_{it}$, *defined by equation* (11) *reduces to*

$$\psi_{it} = \frac{1}{1+R_t^*} H_{t+1} \operatorname{Cov}\left(r_{it}^*, c_{t+1}\right). \tag{27}$$

Proof. Assuming that $R_s^*$ is a risk-free rate, equation (9) simplifies to

$$\frac{\partial V}{\partial c_t} = \rho\left(1+R_t^*\right) E_t\left[\frac{\partial V}{\partial c_{t+1}}\right], \tag{28}$$

and the risk-adjustment term (11) simplifies to

$$\psi_{it} = -\rho\left[\operatorname{Cov}\left(r_{it}^*, \frac{\partial V}{\partial c_{t+1}}\right) \Big/ \frac{\partial V}{\partial c_t}\right]. \tag{29}$$

Substituting (28) into (29), we acquire

$$\psi_{it} = -\frac{1}{1+R_t^*}\left[\operatorname{Cov}\left(r_{it}^*, \frac{\partial V}{\partial c_{t+1}}\right) \Big/ E_t\left[\frac{\partial V}{\partial c_{t+1}}\right]\right]. \tag{30}$$

Consider first the case in which we accept Assumption 1. Substituting the quadratic specification (25) into (30), we get

$$\psi_{it} = \frac{1}{1+R_t^*}\left[\frac{-EV''}{EV'}\right] \operatorname{Cov}\left(r_{it}^*, c_{t+1}\right), \tag{31}$$

which, under our definition of $H_{t+1}$, is identical to (27).

Now, consider the alternative possibility of accepting Assumption 2 instead of Assumption 1. Applying Stein's lemma for bivariate normal distributions to equation (30), we again acquire (31) and thereby (27).[7] ■

Observe that equation (27) provides a CCAPM-type result, because the risk adjustment term, $\psi_{it}$, is very much like the risk premium on a risky asset in CCAPM. In CCAPM, as in our model, compensation for risk is proportional to the covariance of the asset's return with consumption through the factor $\operatorname{Cov}(r_{it}, c_{t+1})$ in (27) and also to the degree of risk aversion $H_{t+1}$ in (27).[8]

In effect, what the adjustment does for very risky rates is to remove the risk premium from $E_t r_{it}$ so that the adjusted user cost becomes positive. To see this more clearly, define $Z_t = H_{t+1} c_t$, where $Z_t$ is a modified (time shifted) Arrow–Pratt relative risk aversion measure. Our theorem now follows immediately.

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

THEOREM 3. *Under the assumptions of Lemma* 1, *we have*

$$\Pi_{it} = \frac{E_t R_t^* - \left(E_t r_{it}^* - \phi_{it}\right)}{1 + E_t R_t^*}, \tag{32}$$

*where*

$$\phi_{it} = Z_t \mathrm{Cov}\left(r_{it}^*, \frac{c_{t+1}}{c_t}\right). \tag{33}$$

Proof. Substitute (27) and (10) into $\Pi_{it} = \pi_{it} + \psi_{it}$ and substitute $Z_t = H_{t+1} c_t$. ■

As is evident from this theorem, the risk-premium adjustment is $\phi_{it}$, where $c_{t+1}/c_t$ is a measure of the consumption growth rate. Hence, the risk adjustment depends on relative risk aversion and the covariance between the consumption growth path $c_{t+1}/c_t$ and the real rates of excess return $r_{it}^*$. We see that the adjusted user cost $\Pi_{it} = \pi_{it} + \psi_{it}$ can be written in the same form as the unadjusted user cost (10), if the benchmark rate is defined to be risk free and if the risk-premium adjustment $\phi_{it}$ is subtracted out of the expected value of the real rates of excess return $r_{it}^*$. As we have observed, that adjustment should be expected to decrease the expected own rate of return, if the asset is very risky and thereby contibutes positively to consumption risk.

## 6. MAGNITUDE OF THE ADJUSTMENT

In accordance with the large and growing literature on the equity premium puzzle, we should expect that $\phi_{it}$ is small for most $i$. In fact, our initial computations of that risk-premium adjustment term for the components of the existing monetary aggregates produces very small correction terms, such that the usual Divisia monetary aggregate and the extended Divisia monetary aggregate are nearly identical to within the roundoff error of the data. There recently has been growing interest in the inclusion of common stock and bond mutual funds in monetary aggregates, but no such change has yet been made officially by the Federal Reserve. In addition, our confidence in the available data on those additional potential components is limited at the present time. Because the rates of return on those assets are subject to much more risk than the rates of return on the existing components of the monetary aggregates, the gain in moving from the usual Divisia index to the extended Divisia index could be greater, if those assets are incorporated into monetary aggregates. Nevertheless, our initial computations with data including those risky assets produce small, although no longer trivial, differences between the extended and unextended Divisia monetary aggregates. The equity premium puzzle issues with CCAPM do not go away with incorporation of assets with substantial rate-of-return risk.

One possible explanation of the surprisingly small risk adjustment terms, even with risky assets, may be aggregation over economic agents. In some sense, the risk adjustments may tend to cancel each other out across economic agents, and

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

our initial computations, mentioned above, are produced from data aggregated over economic agents. To explore this possibility further, we now turn to the use of simulated data generated from a single modeled economic agent. Generation of those data requires numerical solution of that consumer's Euler equations. Having generated the simulated data, we compare the exact aggregator function implied by the weakly separable structure of the utility function with the unextended Divisia monetary aggregate to isolate the effect of the needed risk adjustment, because the correctly computed adjusted Divisia aggregate should track the exact aggregator function accurately.

The loss in tracking ability of the unextended Divisia index under risk has been investigated previously by Barnett et al. (1995), but with an aggregator function that was estimated using actual data aggregated over consumers. Hence there was no control in that study for the possible decrease in the risk adjustment that may have been produced by aggregation over consumers or for possible specification error in the estimated parametric model. In particular, there is no way to know the degree to which the tracking error of the unadjusted Divisia index was produced by the missing risk adjustment or alternatively by the specification error in the estimated utility function of the representative consumer. By the use of simulated data below, we know the exact aggregator function for the simulated single consumer and hence we control for the possible downward bias that may have been produced by earlier studies using macroeconomic data and we eliminate any possible specification error effect, because our simulated data are exactly consistent with rational behavior of our modeled consumer.

## 7. GENERATION OF SIMULATED DATA

### 7.1. Introduction

To show how well the unadjusted Divisia monetary aggregate tracks the "true" theoretical aggregate for one consumer, we numerically solve for the individual monetary assets from the Euler equations associated with a dynamic optimization problem. We use the calculated rational-expectation equilibrium to determine both the Divisia and the theoretical monetary aggregate, and we compare how well the unadjusted Divisia index tracks the movement of the theoretical aggregate. The magnitude of the effect of the missing risk adjustment in the unadjusted Divisia index is measured by the gap between our two simulated aggregates.

Although there are a number of numerical methods that solve for the rational expectation equilibrium of such problems, for this paper we have selected the approach advocated by Den Haan and Marcet (1990).[9] The Den Haan and Marcet approach has been shown to provide good results for a number of complex optimization problems.[10] Furthermore, the complexity of the dynamic programming problem found in this paper is an important factor in choosing the Den Haan and Marcet algorithm, because this solution method does not require discretizing

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

the state space—a formidable task for any computer, when the number of state variables is as plentiful as in our model.

### 7.2. Parameterized Model of Preferences

In this section we formulate a parameterized special case of the model presented in Section 2. The decision problem again is a discrete-time-period optimization problem for a consumer, under the assumption that monetary assets are weakly separable from consumer goods, so that an exact aggregate exists over consumer goods and another exact aggregate exists over monetary assets. Because our model of tastes will be parameterized, the model now is a special case of the model that was used above to derive the extended Divisia index, which therefore can be assumed to track the exact aggregate over monetary assets accurately. In our model, there are three component monetary assets within the exact monetary aggregate. We make that choice so that we can use the empirical results found by Barnett and Yue (1991) to set the unknown parameters of the model. In addition, we have tried to use the same notation that Barnett and Yue (1991) used in an attempt to keep some consistency in that literature.

We assume that the consumer's intertemporal utility, (1), has the form

$$F[M(\boldsymbol{m}_t), c_t] + E_t\left[\sum_{s=t+1}^{\infty} \rho^{s-t} F[M(\boldsymbol{m}_s), c_s]\right], \tag{34}$$

where $F$ is the constant relative-risk-aversion utility function

$$F(M_s, c_s) = \frac{1}{\sigma}\left[c_s^{\beta} M_s^{1-\beta}\right]^{\sigma}$$

with $\sigma \in (-\infty, 0) \cup (0, 1)$, $\rho \in (0, 1)$, and $\beta \in (0, 1)$, and where $c_s$ and $M_s = M(\boldsymbol{m}_s)$ are, respectively, the consumption good and the monetary asset quantity aggregates.[11] The three-dimensional vector $\boldsymbol{m}_s$ contains the three component assets. As $\sigma \to 0$, the consumer's utility function becomes $F(M_s, c_s) = \log(c_s^{\beta} M_s^{1-\beta})$. This is the risk-neutral case, because the degree of relative risk aversion is equal to $1 - \sigma$.

We assume that the exact monetary aggregator function, $M(\boldsymbol{m}_s)$, is the CES function

$$M(\boldsymbol{m}_s) = \left(\sum_{i=1}^{3} \delta_i m_{is}^{\alpha}\right)^{1/\alpha} \quad \text{with} \quad \sum_{i=1}^{3} \delta_i = 1 \quad \text{and} \quad \alpha \in (0, 1].$$

The simple-sum and Cobb–Douglas aggregates are nested special cases, because the monetary aggregator function becomes Cobb–Douglas if $\alpha \to 0$ and simple sum if $\alpha = 1$. Substitutability among components declines as $\alpha$ declines below 1.

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

### 7.3. Euler Equations

Following Barnett and Yue (1991) and Poterba and Rotemberg (1987), the Euler equations from the consumer's decision in Problem 1 with the above specification for the objective function are

$$c_s^{\sigma\beta-1} M_s^{\sigma(\beta-1)} = E_s\left\{\left[\rho \frac{p_s^*}{p_{s+1}^*}(1+R_s)\right]^{-1} c_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right\},$$

$$c_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{1s}^{1-\alpha} = E_s\left[\left(\frac{1}{\delta_1}\frac{\beta\rho}{1-\beta}\frac{p_s^* R_s}{p_{s+1}^*}\right)^{-1} c_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right],$$

$$c_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{2s}^{1-\alpha} = E_s\left\{\left[\frac{1}{\delta_2}\frac{\beta\rho}{1-\beta}\frac{p_s^*(R_s-r_{2s})}{p_{s+1}^*}\right]^{-1} c_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right\},$$

$$c_s^{-\sigma\beta} M_s^{\sigma(\beta-1)+\alpha} m_{3s}^{1-\alpha} = E_s\left\{\left[\frac{1}{\delta_3}\frac{\beta\rho}{1-\beta}\frac{p_s^*(R_s-r_{3s})}{p_{s+1}^*}\right]^{-1} c_{s+1}^{1-\sigma\beta} M_{s+1}^{\sigma(\beta-1)}\right\}.$$

In the dynamic programming Problem 1, the consumer faces the exogenous stochastic state vector

$$\phi_s = \left(R_{s-1}, r_{2,s-1}, r_{3,s-1}, \frac{p_s^*}{p_{s-1}^*}, \frac{I_s}{I_{s-1}}\right),$$

along with the endogenously evolving state vector $\boldsymbol{m}_{s-1}$. Together, they define the state vector $\boldsymbol{\sigma}_s' = (\boldsymbol{m}_{s-1}', \phi_s')$, whereas the consumer's control vector for the optimization problem is $\boldsymbol{z}_s' = (\boldsymbol{m}_s', c_s)$.

We assume that the stochastic process $\{\phi_s\}$ behaves as two independent Markovian processes such that

$$\phi_{1s} = \boldsymbol{a}_1 + \Lambda_1 \phi_{1,s-1} + \boldsymbol{u}_{1s}$$

and

$$\ln(\phi_{2s}) = \boldsymbol{a}_2 + \Lambda_2 \ln(\phi_{2,s-1}) + \boldsymbol{u}_{2s},$$

where $\phi_{1s} = (R_{s-1}, r_{2,s-1}, r_{3,s-1})'$ and $\phi_{2s} = (p_s^*/p_{s-1}^*, I_s/I_{s-1})$.[12] The disturbances $\boldsymbol{u}_{1s}$ and $\boldsymbol{u}_{2s}$ are both distributed i.i.d. $N(\boldsymbol{0}, \Omega_i)$ for $i = 1, 2$, with $\Lambda_1$ and $\Omega_1$ being $3 \times 3$ matrices, whereas $\Lambda_2$ and $\Omega_2$ are $2 \times 2$ matrices. The processes' intercept terms $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are $3 \times 1$ and $2 \times 1$ vectors, respectively.

Each Markovian process is a first-order vector autoregressive process. The natural logarithmic transformation of $\phi_{2s}$ ensures that the process is stationary. Because $\phi_{1s}$ is comprised only of interest rates, there is no reason to transform this process. Regarding the method of computing the benchmark asset's rate of return, the sources of exogenous data, and the algorithms used in solving the system, see Jensen (1997).

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

If we define the economy's parameter vector as $\lambda = (\rho, \sigma, \alpha, \beta, \delta_1, \delta_2, \delta_3, \boldsymbol{a}_1', \boldsymbol{a}_2', \text{vec}(\boldsymbol{\Omega}_1)', \text{vec}(\boldsymbol{\Omega}_2)')$, then for each value of $\lambda$, the above optimization problem defines a nonlinear mapping from the exogenous state process $\{\phi_s\}$ to $\{\boldsymbol{z}_s\}$. The theoretical monetary aggregate, $\{M_s\}$, and the Divisia monetary aggregate

$$Q_s = Q_{s-1} \prod_{i=1}^{3} \left( \frac{m_{is}}{m_{i,s-1}} \right)^{s_{is}^*}$$

then can be calculated from $\{\phi_s\}$ and $\{\boldsymbol{z}_s\}$, where

$$s_{is}^* = \tfrac{1}{2}(s_{is} + s_{i,s-1}),$$

$$s_{is} = \frac{\pi_{is} m_{is}}{\sum_{j=1}^{3} \pi_{js} m_{js}},$$

and the user costs $\pi_{is}$ are as in equation (5).

### 7.4. Solving the Euler Equations

To explore the implications of risk aversion and of substitutability among component assets for the tracking ability of the unadjusted Divisia index, we use the solution for $\{\boldsymbol{z}_s\}$ provided by Jensen (1997) for various settings of the degree of risk aversion and of substitutability. On the basis of the simulation results in his paper, we generate the plots of the unadjusted Divisia index and of the exact parametric aggregate. Because the procedures used by Jensen (1997) are provided in detail in his paper, we outline the approach only briefly.

A rapidly growing literature exists on numerical methods for solving dynamic optimization problems.[13] Of those algorithms currently available, the Parameterized Expectation Approach (PEA) of Den Haan and Marcet (1990) has performed well in head-to-head tests with other algorithms. In addition, the PEA has been applied to a number of different economic areas, including growth models [Den Haan and Marcet (1990)], asset markets with heterogenous agents [Marcet and Singleton (1990)], and monetary economies [Den Haan (1990a,b)], Marshall (1992), Bansal et al. (1994, 1995)].

The PEA approximates the expectation operators found in the Euler equations by parameterizing them with a basis function that spans the set of expectation operators in Hilbert space. The basis function is a globally flexible functional form having arguments that are the state variables $\boldsymbol{\sigma}_s$.[14] We have chosen the first-order polynomial function, as in Jensen (1997). One then iterates over the parameters of these flexible functional forms until a convergence criterion is met.

Intuitively, each iteration of the PEA can be viewed as nonlinear least-squares learning behavior by the consumer.[15] Once learning is no longer occurring, the

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

agent selects that stochastic solution $\{z_s\}$ that satisfies the Euler equations, given the learned prediction of the expectation operators. Hence, it can be argued that the numerical solution found with the PEA is an equilibrium for an agent restricted to the learning associated with a specific globally flexible functional form.

In applying the PEA Jensen (1997) used the convergence criterion recommended by Bansal et al. (1994, 1995), except for cases in which the degree of risk aversion was set at high levels. In those cases, Jensen (1997) used the convergence criterion advocated by Marshall (1992) to determine the stopping point for the PEA algorithm.[16] Marshall argues that the PEA algorithm converges to an approximate equilibrium, if the coefficient vectors, $\boldsymbol{v}_i$ for $i = 1, 2, 3, 4$, are the optimal prediction vectors.

To determine whether the numerical solution is close to the true solution, Jensen (1997) employed the Den Haan and Marcet test statistic (DHM-stat) [see Den Haan and Marcet (1994) and Taylor and Uhlig (1990)]. The DHM-stat provides a test of the theoretical martingale property $E[\nu_s \otimes \boldsymbol{h}_{sij}] = 0$, where $\nu_s$ is the $4 \times 1$ residual vector from the Euler equations and $\boldsymbol{h}_{sij} = \sigma^i_{s-j}$. If Hansen's (1982) regularity conditions hold and the approximation is an exact solution satisfying the above martingale property, then the DHM-stat $\mathrm{TBA}^{-1}\mathrm{B}$ will be distributed as $\chi^2$ with 20 degrees of freedom, where

$$B = (1/T)\sum_{s=t}^{t+T}[\nu_s \otimes \boldsymbol{h}_{sij}] \quad \text{and} \quad A = (1/T)\sum_{s=t}^{t+T}[\nu_s \otimes \boldsymbol{h}_{sij}][\nu_s \otimes \boldsymbol{h}_{sij}]'.$$

To ensure that our solution reflects the empirical world, Jensen (1997) set the parameters in the Euler equations equal to the generalized method of moment estimates of those parameters found by Barnett and Yue (1991). The other parameters associated with the two Markovian processes were set equal to the estimated parameters for the two VAR(1), using monthly data from January 1960 to December 1990.

The size of the simulation is equal to 500, i.e., $T = 500$. We know from Corollary 1 to Theorem 1 that the unadjusted Divisia index will track well under risk neutrality. Hence our interest is in other cases. Conditionally basing Jensen's (1997) solutions of the Euler equations on Barnett and Yue's (1991) parameter estimates, we computed the mean squared errors of the fit of the Divisia growth rates to the numerically solved exact aggregate's growth rates. The mean squared errors are supplied in Table 1 for each of the settings used in the numerical solutions.

For our constant relative-risk-aversion specification, the degree of relative risk aversion is equal to $1 - \sigma$. Clearly, the tracking ability of the unadjusted Divisia index is not independent of the degree of risk aversion, and tracking accuracy declines as relative risk aversion increases. In addition, at any setting of the degree of risk aversion, the tracking ability depends on substitutability among the components of the aggregator function. In our CES specification of the aggregator function, component assets are perfectly substitutable at $\alpha = 1$, and substitutability

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

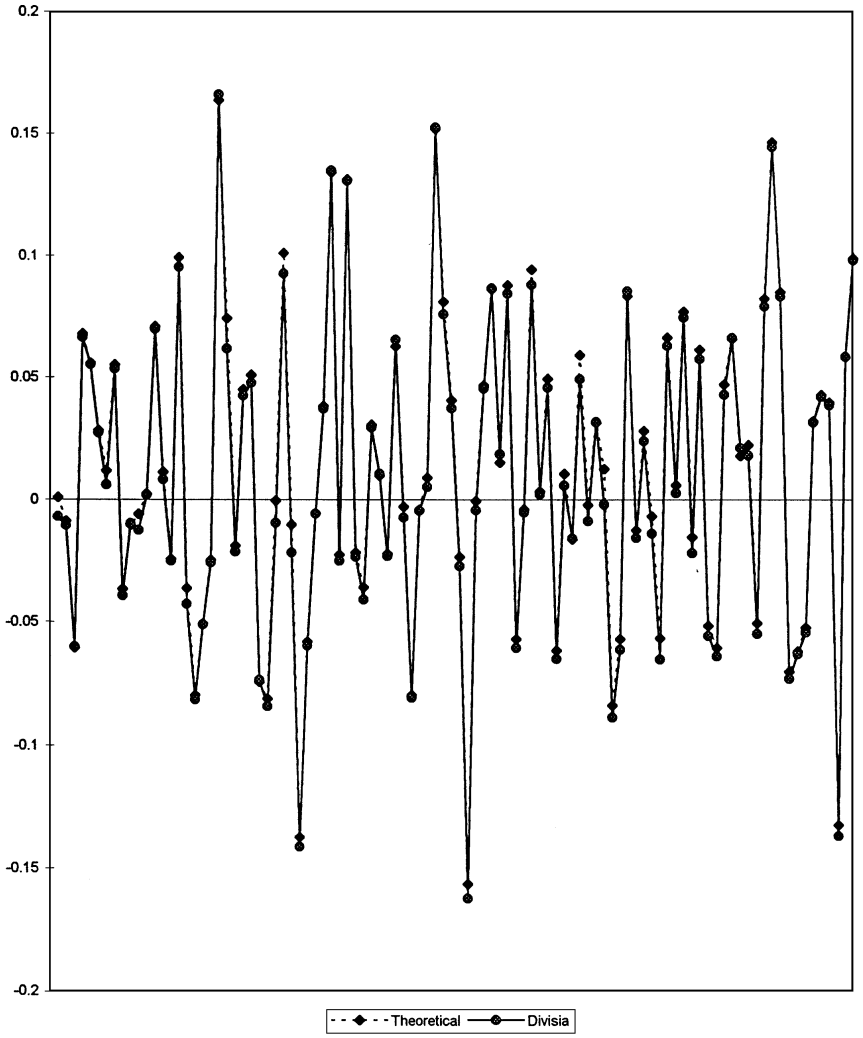



**FIGURE 1.** Divisia and theoretical growth: $\alpha \to 0$ and $\sigma \to 0$ (centered moving average over 5 obs).

among components declines as $\alpha$ declines below 1. As is evident from Table 1, tracking error grows as substitutability increases, but as can be expected from the theory, the dependency of tracking error upon substitutability is small relative to the dependency upon relative risk aversion. In fact, just how small is graphically displayed in two cases in Figures 1 and 2 for two of the risk-neutral cases summarized in Table 1. Because 500 points produce a dense and unclear plot for the growth rates, we produced Figures 1 and 2 from centered moving averages with window of size 5.

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

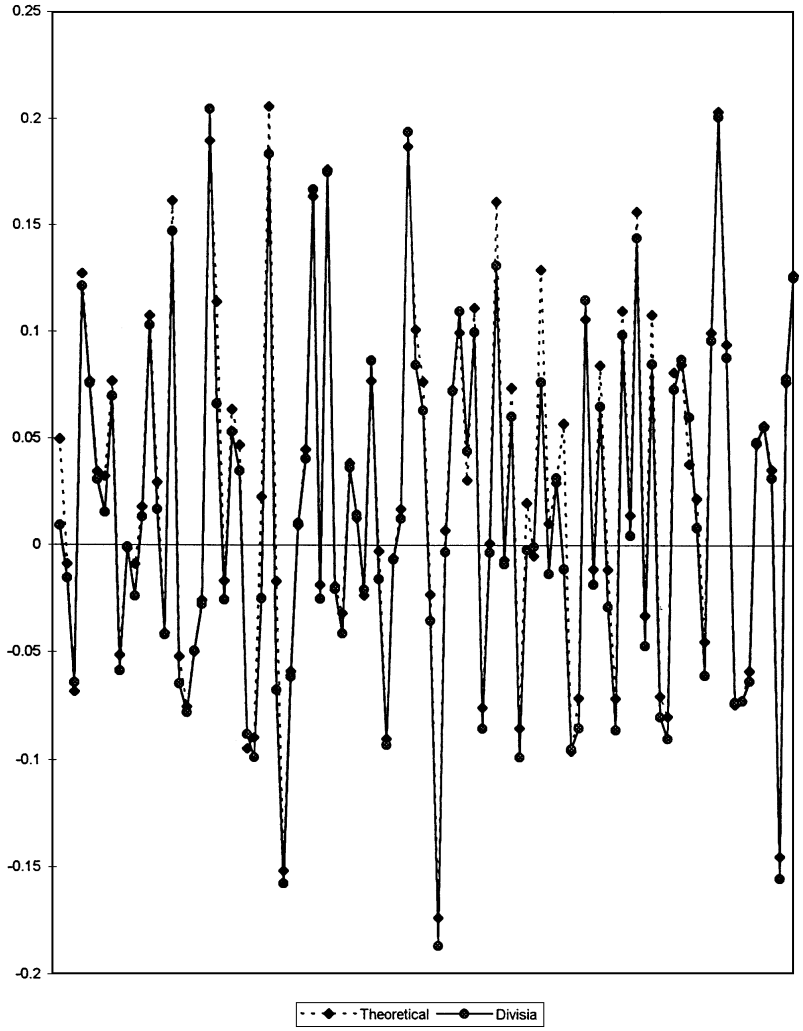



**FIGURE 2.** Divisia and theoretical growth: $\alpha = 0.6$ and $\sigma \to 0$ (centered moving average over 5 obs).

Hence the risk-adjusted Divisia index and the associated risk-adjusted user costs are likely to be useful, in some non-risk-neutral cases, although in many cases with moderate risk aversion, the gain may be slight. It also appears that aggregation over economic agents is a relevant factor, because we find greater loss in tracking ability with simulated data from one consumer than with the economy's aggregate per-capita data.[17]

For details about the parameter estimation and data, see Barnett and Yue (1991), and for details of the solution for the endogenous variables conditional upon the

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

**TABLE 1.** Divisia index's growth-rate tracking of the simulated exact aggregate, mean squared errors

| $\alpha$ | $\sigma$ | MSE |
|---|---|---|
| 0 | 0 | 0.000070 |
| 0.1 | 0 | 0.000097 |
| 0.2 | 0 | 0.000137 |
| 0.3 | 0 | 0.000208 |
| 0.4 | 0 | 0.000336 |
| 0.5 | 0 | 0.000570 |
| 0.6 | 0 | 0.001063 |
| 0 | −0.1 | 0.011939 |
| 0 | −0.2 | 0.011961 |
| 0 | −0.3 | 0.012425 |
| 0 | −0.4 | 0.012917 |
| 0 | −0.5 | 0.013413 |
| 0 | −0.6 | 0.013881 |

exogenous processes, see Jensen (1997). At this point, we are not comfortable about reaching stronger conclusions in this regard, because competing approaches to solving rational-expectations models may be preferable in some of the cases considered above, and hence we cannot be certain of the degree to which our conclusions have been affected by inaccuracies in the solution method used. It would be useful to repeat these experiments with the alternative solution methods proposed by Baxter et al. (1990), Coleman (1990, 1991), or Baxter (1991).

## 8. CONCLUSION

In earlier research on aggregation theoretic foundations for monetary modeling, the primary issue has been to explore the degree of the tracking error of the unadjusted Divisia monetary index, derived from the first-order conditions under perfect certainty, to the degree of risk aversion. In this paper, we develop a CCAPM adjustment to user costs that permits the Divisia index to be derived directly from the Euler equations under risk, so that the tracking error produced from risk aversion disappears, because the risk is internalized within the resulting extended Divisia index. Further details regarding that derivation can be found in the closely related working paper by Barnett and Liu (1995).

Using the components of the usual monetary aggregates, we find that the CCAPM adjustment to user costs is very small, and the gain from moving from the unadjusted Divisia index to the extended index seems slight, at least with those relatively low-risk components. These results are very similar to related results on the equity premium puzzle, because the small adjustments in both cases result

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

from the very low covariance between rates of return and the consumption stream. We then explore the possibility that aggregation over economic agents may have smoothed that covariance, or the possibility that earlier results on the tracking error of the unadjusted Divisia index may have been produced by specification error in the parametric model of the utility function. We do so by producing simulated data from a modeled rational consumer. The procedure used to solve the Euler equations is that of Jensen (1997). We find that the tracking error of the unadjusted index is nontrivial under risk aversion, and depends upon the degree of risk aversion. Hence, the CCAPM adjustment to user costs and the risk-adjusted Divisia index can be expected to be needed, when modeling the behavior of one economic agent under risk aversion. How to progress further in the direction of aggregation over economic agents is not explored in this paper.

Although the CCAPM approach to risk adjustment is a natural extension of the Divisia index, we hasten to add that as better approaches to risk adjustment become available, they should be used in this literature. Because economic index number theory and aggregation theory are directly derived from microeconomic theory, the Divisia index always can be extended to incorporate new developments. Considering the established empirical problems of CCAPM, especially in the equity-premium-puzzle literature, we have no doubt that better or perhaps generalized methods will become available in the future. We no more advocate unwavering commitment to the CCAPM risk adjustment than unwavering commitment to perfect-certainty modeling. We do wish to indicate the relevancy to financial aggregation theory of developments in finance and decision making under risk—including future developments not yet in existence.

## NOTES

1. According to the Barnett critique, the appearance of structural shift can be produced from an inconsistency between the aggregator function tracked by the index number used to produce the data and the aggregator function that is implied by the structural model within which the index number is used. The use of simple sum monetary aggregates as variables within models is an extreme example. See Barnett et al. (1992), Chrystal and MacDonald (1994), and Belongia (1996).

2. Many of the results in this paper are based on those in the recent working paper by Barnett and Liu (1995).

3. Although money may not exist in the elementary utility function, there exists a derived utility function that contains money, so long as money has positive value in equilibrium. See, e.g., Quirk and Saposnik (1968), Arrow and Hahn (1971), Phlips and Spinnewyn (1982), Samuelson (1984), and Feenstra (1986). We implicitly are using that derived utility function.

4. The strong separability assumption is largely for expository convenience. Weak separability would be sufficient.

5. Assuming that $X$ is linearly homogeneous, the exact price aggregator function is the unit cost function.

6. It amounts to the assumption that the risk premium already has been extracted from the benchmark rate. In practice, this assumption is harmless because the risk premia adjustments are applied to all component assets before the benchmark rate is computed—usually as an upper envelope of the component rates. If other asset paths also are included among those used to produce the upper envelope, then this assumption requires that the same risk premia adjustments also be applied to those paths

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

before the upper envelope is generated. Because the risk premia already have been extracted at the time that the envelope is produced, the benchmark rate automatically is risk adjusted.

7. For Stein's lemma, see Rubinstein (1976).

8. If own interest rates are positively correlated with consumption, (27) is positive because $H_{t+1}$ would be positive under risk aversion. Alternatively, if the asset's return is not sufficiently risky to dominate the direction of the net shocks to consumption from risk, the opposite could happen. The asset's rate of return could correlate negatively with the consumption stream in a manner tending to decrease the household's consumption risk, and hence (27) would be negative. In the CCAPM theory of finance, beta of a very risky asset is usually positive, where beta is defined to be $\beta_{ic} = \text{Cov}(r^*_{it}, c_{t+1}/\text{Var}(c_{t+1})$. The subscript $c$ in $\beta_{ic}$ designates "consumption-based" beta, and the lack of a time subscript in the notation $\beta_{ic}$ results from the assumption of stationarity of the interest rate and consumption bivariate process. Clearly, the usual finance view of positive $\beta_{ic}$ can hold if and only if $\text{Cov}(r_{it}, c_{t+1})$ is positive.

This conclusion about the sign of the adjustment term, $\psi_{it}$, in the adjusted user cost $\pi_{it} + \psi_{it}$ of very risky assets is especially revealing, when the benchmark rate is defined to be riskless, as we have just done. Consider the definition of the unadjusted user cost in equation (10). Because we now are assuming that the benchmark rate is defined to be the maximum available rate of return on a risk-free asset, we can conclude that the benchmark asset has no embedded liquidity premium and cannot be less than the own rate of return on any risk-free monetary asset $i$. Hence, (10) is nonnegative if monetary asset $i$ is risk free. But suppose that consumers are risk averse and that monetary asset $i$ is not risk free. Then, $E_t r_{it}$ will contain a risk premium, despite the fact that $R_t$ does not (or its risk premium has been removed). Hence, the unadjusted user cost (10) could be negative. But as we have just observed, $\psi_{it}$ in this case will be positive, and we would expect that, in fact, it will be sufficiently positive to offset the possible negativity of the unadjusted user cost to produce positive value of the adjusted user cost $\pi_{it} + \psi_{it}$.

9. See Taylor and Uhligh (1990) for a list and comparison of the known methodologies for solving dynamic optimization problems.

10. See Den Haan and Marcet (1990, 1994), Marshall (1992), and Bansal et al. (1994, 1995).

11. In much of this literature, a finite-period planning horizon is assumed, as in Jensen (1997), but the resulting exact aggregate is the same in the finite and infinite planning cases. See Barnett (1995) regarding the equivalency.

12. More complex stochastic processes for the exogenous state process could be used if the economist so desires. We here define the log of a vector to be the vcctor of the logarithms.

13. See Taylor and Uhlig (1990) for an excellet source on the available methods and a comparison of their performance record.

14. See Gallant (1982), Barnett and Jonas (1983), and Barnett and Yue (1988) for the properties and examples of globally flexible functions.

15. See Marcet and Sargent (1989a,b) for examples of a linear least-squares learning model that have a locally stable equilibrium.

16. We initially used the convergence criterion of Bensal et al. (1994, 1995) but the PEA algorithm failed to converge within 40,000 iterations.

17. Regarding the results with aggregate per-capita data, see Barnett et al. (1995) and Barnett and Liu (in press).

## *REFERENCES*

Arrow, K.J. & F. Hahn (1971) *General Competitive Analysis.* San Francisco: Holden-Day.

Bansal, R., A.R. Gallant, R. Hussey & G. Tauchen (1994) Computational aspects of nonparametric simulation estimation. In David Belsley (ed.), *Computational Techniques for Econometrics and Economic Analysis*, pp. 3–22. Boston: Kluwer Academic Publishers.

Bansal, R., A.R. Gallant, R. Hussey & G. Tauchen (1995) Nonparametric estimation of structural models for high-frequency currency market data. *Journal of Econometrics* 66, 251–287.

Barnett, W.A. (1978) The user cost of money. *Economic Letters* 1, 145–149.

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

Barnett. W.A. (1980) Economic monetary aggregates: An application of index number and aggregation theory. *Journal of Econometrics* 14, 11–48.

Barnett, W.A. (1987) The microeconomic theory of monetary aggregation. In W. Barnett and K. Singleton (eds.), *New Approaches to Monetary Economics*, pp. 115–168. New York: Cambridge University Press.

Barnett, W.A. (1995) Exact aggregation under risk. In W. Barnett, H. Moulin, M. Salles & N. Schofield (eds.), *Social Choice, Welfare, and Ethics*, pp. 353–374. New York: Cambridge University Press.

Barnett, W.A. & A.B. Jonas (1983) The Müntz-Szatz demand system: An application of a globally well behaved series expansion. *Economics Letters* 11, 337–342.

Barnett, W.A. & Y. Liu (1995) The CAPM-Extended Divisia Monetary Aggregate with Exact Tracking Under Risk. Working paper 194, Washington University at St. Louis.

Barnett, W.A. & Y. Liu (in press) Beyond the risk neutral utility function. In Michael Belongia (ed.), *Divisia Monetary Aggregates: Correct in Theory, Useful in Practice?* New York: Macmillan.

Barnett, W.A., & P. Yue (1988) Semiparametric estimation of the asymptotically ideal model (AIM): The AIM demand system. In G. Rhodes and T. Formby (eds.), *Nonparametric and Robust Inference*, Advances in Econometrics 7, pp. 229–252. Greenwich, CT: JAI Press.

Barnett, W.A. & P. Yue (1991) Exact Monetary Aggregation Under Risk. Working paper 163, Washington University at St. Louis.

Barnett, W.A. & G. Zhou (1994) Commentary. *Federal Reserve Bank of St. Louis Review* 76, 53–62.

Barnett, W.A., D. Fisher & S. Serletis (1992) Consumer theory and the demand for money. *Journal of Economic Literature* 92, 2086–2119.

Barnett, W.A., M. Kirova & M. Pasupathy (1995) Estimating policy-invariant deep parameters in the financial sector when risk and growth matter. *Journal of Money, Credit, and Banking* 27, Pt. 2, 1402–1430.

Baxter, M. (1991) Approximating suboptimal dynamic equilibria: An Euler equation approach. *Journal of Monetary Economics* 27, 173–200.

Baxter, M., M.J. Crucini & K.G. Rouwenhorst (1990) Solving the stochastic growth model by a discrete-state-space, Euler-equation approach. *Journal of Business and Economic Statistics* 8, 19–21.

Belongia, M.T. (1996) Measurement matters: Recent results from monetary economics re-examined. *Journal of Political Economy* 104, 1065–1083.

Chrystal, K.A. & R. MacDonald (1994) Empirical evidence on the recent behavior and usefulness of simple-sum and weighted measures of the money stock. *Federal Reserve Bank of St. Louis Review* 76, 73–109.

Coleman, W.J., II (1990) Solving the stochastic growth model by policy-function iteration. *Journal of Business and Economic Statistics* 8, 27–29.

Coleman, W.J., II (1991) Equilibrium in a production economy with an income tax. *Econometrica* 59, 1091–1104.

Croushore, D. (1993) Money in the utility function: Functional equivalence to a shopping-time model. *Journal of Macroeconomics* 15, 175–182.

Den Haan, W.J. (1990a) The optimal inflation path in a Sidrauski-type model with uncertainty. *Journal of Monetary Economics* 25, 389–409.

Den Haan, W.J. (1990b) The Term Structure of Interest Rates in Real and Monetary Production Economies. Working paper, Carnegie-Mellon University.

Den Haan, W.J. & A. Marcet (1990) Solving the stochastic growth model by parameterizing expectations. *Journal of Business and Economic Statistics* 8, 31–34.

Den Haan, W.J. & A. Marcet (1994) Accuracy in simulations. *Review of Economic Studies* 61, 3–17.

Divisia, F. (1925) L'indice monétaire et la théorie de la monnaie. *Revue d'Economie Politique* 39, 980–1008.

Feenstra, R.C. (1986) Functional equivalence between liquidity costs and the utility of money. *Journal of Monetary Economics* 17, 271–291.

Feldstein, M. & J.H. Stock (1994) Measuring Money Growth When Financial Markets Are Changing. NBER working paper 4888, National Bureau of Economic Research.

Fischer, S. (1974) Money and the production function. *Economic Inquiry* 12, 517–533.

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

Gallant, A.R. (1982) Unbiased determination of production technologies. *Journal of Econometrics* 20, 285–323.
Hansen, L.P. (1982) Large sample properties of generalized method of moments estimators. *Econometrica* 50, 1029–1054.
Hulten, C.R. (1973) Divisia index numbers. *Econometrica* 63, 1017–1026.
Jensen, M. (1997) A homotopy approach to solving nonlinear rational expectations problems. *Computational Economics* 10, 47–65.
Marcet, A. & T.J. Sargent (1989a) Convergence of least squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic Theory* 48, 337–368.
Marcet, A. & T.J. Sargent (1989b) Convergence of least-squares learning in environments with hidden state variables and private information. *Journal of Political Economy* 97, 1306–1322.
Marcet, A. & K. Singleton (1990) Simulation Analysis of Dynamic Stochastic Models: Applications to Theory and Estimation. Mimeo, Stanford University.
Marshall, D.A. (1992) Inflation and asset returns in a monetary economy. *Journal of Finance* 47, 1315–1342.
Phlips, L. & F. Spinnewyn (1982) Rationality versus myopia in dynamic demand systems. In R.L. Basmann and G.F. Rhodes (eds.), *Advances in Econometrics*, 229–252. Greenwich, CT: JAI Press.
Poterba, J.M. & J.J. Rotemberg (1987) Money in the utility function: An empirical implementation. In W. Barnett and K. Singleton (eds.), *New Approaches to Monetary Economics*, pp. 219–240. New York: Cambridge University Press.
Quirk, J. & R. Saposnik (1968) *Introduction to General Equilibrium Theory and Welfare Economics*. New York: McGraw–Hill.
Rubinstein, M. (1976) The valuation of uncertain income streams and the pricing of options. *Bell Journal of Economics* 7, 407–425.
Samuelson, P. (1948) *Foundations of Economic Analysis.* Cambridge, MA: Harvard University Press.
Taylor, J.B. & H. Uhlig (1990) Solving nonlinear stochastic growth models: A comparison of alternative solution methods. *Journal of Business and Economic Statistics* 8, 1–17.

# APPENDIX: AGGREGATION THEOREM

It is clear that equation (3) does define the exact monetary aggregate in the welfare sense, because $M_s$ measures the consumer's subjective evaluation of the services that he or she receives from holding $\boldsymbol{m}_s$. However, it also can be shown that equation (6) defines the exact monetary aggregate in the aggregation theoretic sense. In particular, the stochastic process $M_s$ contains all of the information about $\boldsymbol{m}_s$ that is needed by the consumer to solve the rest of his or her decision problem. This conclusion is based on the following theorem, which we call the consumer's aggregation theorem:

Let

$$D_s = I_s + \sum_{i=1}^{k_1} \left[(1 + r_{i,s-1}) p^*_{s-1} m_{i,s-1} - p^*_s m_{is}\right],$$

and let

$$D(s) = \left\{ (\boldsymbol{h}_s, \boldsymbol{x}_s, A_s) \in Y : \sum_{i=1}^{n} p_{is} x_{is} = \sum_{i=1}^{k_2} \left[(1 + i_{i,s-1}) p^*_{s-1} h_{i,s-1} - p^*_s h_{is}\right] \right.$$

$$\left. + (1 + R_{s-1}) p^*_{s-1} A_{s-1} - p^*_s A_s + D_s \right\}. \qquad \textbf{(A.1)}$$

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press

Let $(\boldsymbol{a}_s^*, \boldsymbol{x}_s^*, A_s^*)$ solve Problem 1, and assume that the utility function $u(\boldsymbol{m}_s, \boldsymbol{h}_s, \boldsymbol{x}_s)$ is weakly separable in $\boldsymbol{m}_s$, so that there exists aggregator function $M$ and utility function $U$ such that $U(M(\boldsymbol{m}_s), \boldsymbol{h}_s, \boldsymbol{x}_s) = u(\boldsymbol{m}_s, \boldsymbol{h}_s, \boldsymbol{x}_s)$. Consider the following decision problem, which is conditional upon prior knowledge of the aggregate process $M_s^* = M(m_s^*)$, although not upon the component processes $\boldsymbol{m}_s^*$.

**Problem A.1.** Choose the deterministic point $(\boldsymbol{h}_t, \boldsymbol{x}_t, A_t)$ and the stochastic process $(\boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$, $s = t+1, \ldots, \infty$, to maximize

$$U\left(M_t^*, \boldsymbol{h}_t, \boldsymbol{x}_t\right) + E_t\left[\sum_{s=t+1}^{\infty}\left(\frac{1}{1+\xi}\right)^{s-t} U\left(M_s^*, \boldsymbol{h}_s, \boldsymbol{x}_s\right)\right] \tag{A.2}$$

subject to $(\boldsymbol{h}_s, \boldsymbol{x}_s, A_s) \in D(s)$ for $s - t$, and also subject to

$$\lim_{s\to\infty} E_t\left(\frac{1}{1+\xi}\right)^{s-t} A_s = 0,$$

with the process $M_s^*$ given for $s - t$.

THEOREM A. 1 (Consumer's Aggregation Theorem). *Let the deterministic point* $(\boldsymbol{m}_t, \boldsymbol{h}_t, \boldsymbol{x}_t, A_t)$ *and the stochastic process* $(\boldsymbol{m}_s, \boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$, $s = t+1, \ldots, t+T$ *solve Problem* 1. *Then, the deterministic point* $(\boldsymbol{h}_t, \boldsymbol{x}_t, A_t)$ *and the stochastic process* $(\boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$, $s = t+1, \ldots, t+T$, *will solve Problem* A.1 *conditionally upon* $M_s^* = M(\boldsymbol{m}_s)$ *for* $s = t, \ldots, t+T$.

**Proof**. Let $(\boldsymbol{m}_s, \boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$ solve Problem 1, but let $(\boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$ not solve Problem A.1 conditionally upon the process $M_s^* = M(\boldsymbol{m}_s)$ given for $s = t, \ldots, t+T$. Then there exist $(\tilde{\boldsymbol{h}}_s, \tilde{\boldsymbol{x}}_s, \tilde{\boldsymbol{A}}_s) \in D(s)$ satisfying the transversality condition, such that equation (A.2) evaluated at $(\tilde{\boldsymbol{h}}_s, \tilde{\boldsymbol{x}}_s, \tilde{\boldsymbol{A}}_s)$ is strictly greater than equation (A.2) evaluated at $(\boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$ conditionally upon $M_s^* = M(\boldsymbol{m}_s)$.

Hence, equation (1) evaluated at $(\boldsymbol{m}_s, \tilde{\boldsymbol{h}}_s, \tilde{\boldsymbol{x}}_s, \tilde{\boldsymbol{A}}_s)$ is strictly greater than (1) evaluated at $(\boldsymbol{m}_s, \boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$. However, because $(\tilde{\boldsymbol{h}}_s, \tilde{\boldsymbol{x}}_s, \tilde{\boldsymbol{A}}_s)$ is feasible for Problem A.1 conditionally upon $M_s = M(\boldsymbol{m}_s)$, it follows that $(\boldsymbol{m}_s, \tilde{\boldsymbol{h}}_s, \tilde{\boldsymbol{x}}_s, \tilde{\boldsymbol{A}}_s)$ is feasible for Problem 1. Our assumption that $(\boldsymbol{m}_s, \boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$ solves Problem 1 is contradicted. ■

Clearly, this proof by contradiction applies not only when $M_s$ is produced by voluntary behavior, but also when the $M_s$ process is exogenously imposed upon the consumer, as through a perfectly inelastic supply function for $M_s$ imposed by central bank policy. In that case, Problem A.1 describes optimal behavior by the consumer in the remaining variables. Clearly, the information about $M_s$ is needed in the solution of Problem A.1 for the processes $(\boldsymbol{h}_s, \boldsymbol{x}_s, A_s)$. Alternatively, information about the usual simple-sum monetary aggregate over the components of $\boldsymbol{m}_s$ is of no use in solving either Problem 1 or A.1 unless the monetary aggregator function $M$ happens to be a simple sum. In other words, the simple-sum aggregate contains useful information about behavior only if the components of $\boldsymbol{m}_s$ are perfect substitutes in identical ratios (linear aggregation with *equal* coefficients).

https://doi.org/10.1017/S1365100597003088 Published online by Cambridge University Press