Probability in the Engineering and Informational Sciences, **33**, 2019, 471–486.
doi:10.1017/S0269964818000244

ON EXTROPY PROPERTIES OF MIXED SYSTEMS

GUOXIN QIU
Department of Business Administration
Xinhua University of Anhui
Hefei 230088, China
and
Department of Statistics and Finance, School of Management School of Business
University of Science and Technology of China
Hefei 230026, China
E-mail: qiugx02@ustc.edu.cn

LECHEN WANG
Department of Statistics and Finance, School of Management School of Business
University of Science and Technology of China
Hefei 230026, China
E-mail: lcwang@mail.ustc.edu.cn

XINGYU WANG
Department of Statistics and Finance, School of Management School of Business
University of Science and Technology of China
Hefei 230026, China
E-mail: ustcwxy@mail.ustc.edu.cn

An expression of the extropy of a mixed system's lifetime was given firstly. Based on this expression, two mixed systems with same signature but with different components were compared. It was shown that the extropy of lifetime of a mixed system equals to that of its dual system if the lifetimes of the components have symmetric probability density function. Moreover, some bounds of the extropy of lifetimes of mixed systems were obtained and the concept of Jensen–extropy (JE) divergence of mixed systems was proposed. The JE divergence is non-negative and it can be used as an alternative information criteria for comparing mixed systems with homogeneous components. To illustrate the applications of JE divergence, some examples are addressed at the end of this paper.

Keywords: extropy, mixed system, signature, stochastic orders, symmetric distribution

1. INTRODUCTION

The r-out-of-n systems are widely used in reliability engineering which function if and only of at least r of the components function. As an extension of the r-out-of-n systems, a system is said to be coherent if it does not have any irrelevant components and its structure function is monotone, see for example Barlow and Proschan [5]. A mixed system

© Cambridge University Press 2018 0269-9648/18 $25.00

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

is a stochastic mixture of coherent systems. Hence, any r-out-of-n system is a degenerated mixed system (Samaniego [21]). Let T denote the lifetime of a mixed system consisting of n independent and identically distributed (iid) components with lifetimes $X_1, X_2, \ldots, X_n$ having an absolutely continuous cumulative distribution function (cdf) F, survival function $\bar{F} = 1 - F$, and probability density function (pdf) f. Boland and Samaniego [6] showed that T has the pdf given by

$$f_{_T}(x) = \sum_{i=1}^{n} s_i f_{i:n}(x), \tag{1.1}$$

where

$$f_{i:n}(x) = \frac{F^{i-1}(x)\bar{F}^{n-i}(x)f(x)}{B(i, n-i+1)}, \quad i = 1, \ldots, n, \ x \geq 0, \tag{1.2}$$

is the pdf of the ordered component lifetime $X_{i:n}$, that is, the lifetime of $(n-i+1)$-out-of-n system. $B(i, n-i+1) = (i-1)!(n-i)!/n!$ is the beta function with parameters i and $(n-i+1)$. The vector of weights $\mathbf{s} = (s_1, \ldots, s_n)$ in (1.1) is called the signature of the mixed system which are defined as $s_i = \mathrm{P}(T = X_{i:n}), i = 1, \ldots, n$. Note that $s_1, \ldots, s_n$ are non-negative real numbers which do not depend on the common cdf F and such that $\sum_{i=1}^{n} s_i = 1$. If a mixed system has uniform signature, that is, $\mathbf{s} = (1/n, \ldots, 1/n)$, one can see from (1.1) that $f_{_T}(x) = f(x)$. In this case, the mixed system can be viewed as the uniform mixture of r-out-of-n systems. For more details on system signatures, one may refer to D'andera and De Sanctis [9], Samaniego and Navarro [22], and Samaniego [21].

Based on system signatures, many authors showed their interests in studying the information properties of mixed systems. For example, Toomaj and Doostparast [27,28] obtained an expression for the mixed system's entropy and then the bounds for the entropy of the system's lifetime were obtained. They also provided expressions for the Kullback–Leibler discrimination information of a mixed system and component lifetimes. Asadi, Ebrahimi, Soofi, and Zohrevand [3] proposed the Jensen–Shannon (JS) information criteria for comparison of mixed systems, which is a scalar function of the signature and ranks systems based on their designs. They proved that the JS information is non-negative and its minimum is attained by r-out-of-n systems. Most recently, stochastic comparisons of Rènyi entropy and cumulative residual entropy of mixed system are discussed respectively by Toomaj [26] and Toomaj, Sunoj, and Navarro [29] provided that both systems have the same signature.

The objective of this paper is to explore the information properties of mixed systems from the perspective of extropy. For random variable X with pdf f, its extropy was defined by Lad, Sanfilippo, and Agrò [14] as follows,

$$J(X) = -\frac{1}{2}\int_{-\infty}^{\infty} f^2(x)\, dx = -\frac{1}{2}\int_0^1 f(F^{-1}(v))\, dv, \tag{1.3}$$

where $F^{-1}(u) = \inf\{t : F(t) \geq u\}$, $0 < u < 1$, is the quantile function of X. One of the statistical applications of extropy is to score the forecasting distributions using the total log scoring rule. See Gneiting and Raftery [11], Agrò, Lad, and Sanfilippo [1], Capotorti, Regoli, and Vattari [7] and Lad, Sanfilippo, and Agrò [13] for a greater detail. Another statistical application of extropy is to compare the uncertainties of two random variables. For example, the uncertainty in X may be measured by the difference in the outcomes from two independently conducted experiments under identical conditions. Let X_1 and X_2 be two such outcomes. Then $X_1 - X_2$ measures the uncertainty in X. If the pdf of X is denoted

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

by f, then the pdf of $X_1 - X_2$ is given by

$$g(u) = \int_{-\infty}^{\infty} f(x) f(x-u)\, dx.$$

This implies the probability of $X_1 = X_2$ approximately equals to $g(0) = -2J(X)$. Therefore, if the extropy of X is less than another random variable, say Y, that is, $J(X) \leq J(Y)$, then X has more uncertainty than Y. Recently, several authors paid their attention on extropy for its fruitful applications mentioned above. Qiu [17] explored some characterization results, monotone properties, and lower bounds of extropy of order statistics and record values. The residual extropy of order statistics was studied by Qiu and Jia [18]. Qiu and Jia [19] discussed the extropy estimators with applications in testing uniformity.

The rest of the paper is organized as follows. Section 2 gives an expression of the extropy of a mixed system's lifetime in terms of the pdfs of beta distributions. Based on this expression, we compare two mixed systems with the same signature but with different components. In this section, we also show that the extropy of a mixed system's lifetime equals to that of its dual system if the lifetimes of the components have symmetric pdfs. Some bounds of the extropy of lifetimes of mixed systems were obtained in Section 3. Moreover, we prove that the extropy of a sum of independent variables is larger than that of each one. The concept of JE divergence of mixed systems was proposed in Section 4. It was shown that the JE divergence is non-negative and it can be used as alternative information criteria for comparing mixed systems with homogeneous components.

2. EXTROPY OF MIXED SYSTEMS

In this section, we first provide an expression for the extropy of a mixed system's lifetime with given signature $\mathbf{s} = (s_1, \ldots, s_n)$. To this end, substituting (1.1) and (1.2) into (1.3), we see that the extropy of T is given by

$$\begin{aligned} J(T) &= -\frac{1}{2}\int_0^{\infty} f_T^2(x)\, dx \\ &= -\frac{1}{2}\int_0^{\infty} \left(\sum_{i=1}^{n} s_i f_{i:n}(x)\right)^2 dx \\ &= -\frac{1}{2}\int_0^{\infty} \left(\sum_{i=1}^{n} s_i \frac{F^{i-1}(x)\bar{F}^{n-i}(x)}{B(i, n-i+1)}\right)^2 f(x)\, dF(x) \\ &= -\frac{1}{2}\int_0^{1} \left(\sum_{i=1}^{n} s_i \frac{v^{i-1}(1-v)^{n-i}}{B(i, n-i+1)}\right)^2 f(F^{-1}(v))\, dv. \end{aligned} \tag{2.1}$$

For $i = 1, \ldots, n$, if we use the probability integral transformation $U_i = F(X_i)$, then the associated order statistics $W_i = F(X_{i:n})$ has a beta distribution with parameters i and $(n - i + 1)$ and with the following pdf

$$g_i(v) = \frac{v^{i-1}(1-v)^{n-i}}{B(i, n-i+1)}, \quad 0 \leq v \leq 1. \tag{2.2}$$

The pdf of $V = F(T)$ is given by

$$g_V(v) = \sum_{i=1}^{n} s_i g_i(v), \quad 0 \leq v \leq 1. \tag{2.3}$$

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

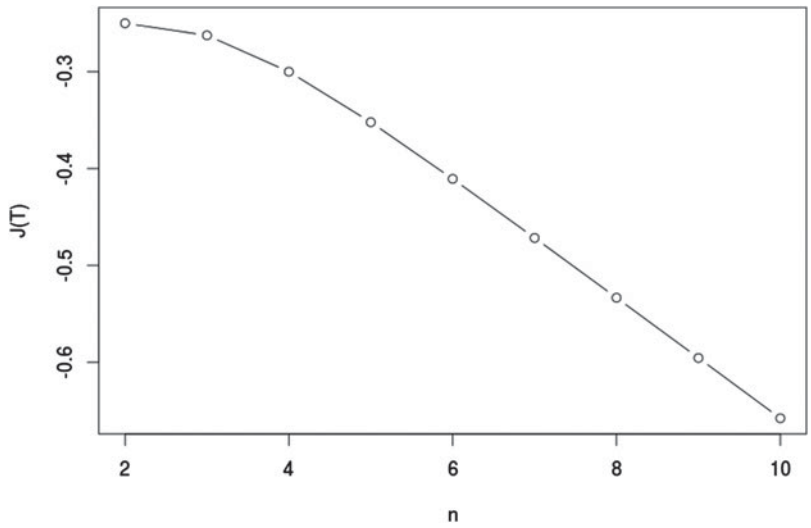


FIGURE **1.** Exact values of $J(T)$ for $n = 2 : 10$.

From (2.1)–(2.3), we obtain an expression for the extropy of T by using the aforementioned transformations. The expression is stated in the following Proposition 2.1.

PROPOSITION 2.1: *The extropy of T can be expressed as follows:*

$$J(T) = -\frac{1}{2}\int_0^1 g_V^2(v) f(F^{-1}(v))\, dv, \tag{2.4}$$

where $g_V(v)$ is given by (2.3).

Example 2.2: Consider the uniform mixture of series and parallel systems with n components. The signatures of these systems are in the following form:

$$\mathbf{s} = \left(\frac{1}{2}, 0, \ldots, 0, \frac{1}{2}\right).$$

The pdf of the probability integral transformed lifetime of the system is given by $g_V(v) = n[(1-v)^{n-1} + v^{n-1}]/2$, $0 \le v \le 1$. If the lifetimes of components have standard exponential distribution, then the extropy of these systems is given by:

$$J(T) = -\frac{n^2}{8}\int_0^1 [(1-v)^{n-1} + v^{n-1}]^2 (1-v)\, dv.$$

Figure 1 gives the plot of $J(T)$ as n increases from 2 to 10. We see from Figure 1 that the value of $J(T)$ is decreasing. This means that the uncertainty of these systems is increasing as n increases from 2 to 10.

Let Y be another random variable with cdf G, survival function $\bar{G} = 1 - G$, and pdf g. X is said to be smaller than Y in the disperse order, denoted by $X \le_{\text{disp}} Y$, if $f(F^{-1}(v)) \ge g(G^{-1}(v))$ for all $0 < v < 1$. X is said to be smaller than Y in the hazard rate order, denoted by $X \le_{\text{hr}} Y$, if $\bar{G}(x)/\bar{F}(x)$ is increasing in $x > 0$. X is said to have a decreasing failure rate (DFR) if $f(x)/\bar{F}(x)$ is decreasing in $x > 0$. For properties and applications of these concepts,

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

one may refer to Shaked and Shanthikumar [23], or Müller and Stoyan [15]. From (2.4), the following Proposition 2.3 is obvious.

PROPOSITION 2.3: *Let T^X and T^Y be the lifetimes of two mixed systems with the same signature consisting n iid components having cdfs F and G, respectively. If $X \leq_{\text{disp}} Y$, then $J(T^X) \leq J(T^Y)$.*

Remark 2.4: Proposition 2.3 is a generalization of Proposition 2.4 in Qiu [17] which is applicable for r-out-of-n systems as the special cases of mixed systems.

Bagai and Kochar [4] stated that $X \leq_{\text{disp}} Y$ if $X \leq_{\text{hr}} Y$ and either X or Y is DFR. From this fact and Proposition 2.3, it is easy to prove the following Corollary 2.5.

COROLLARY 2.5: *Under the assumptions of Proposition 2.3, if $X \leq_{\text{hr}} Y$ and either X or Y is* DFR*, then $J(T^X) \leq J(T^Y)$.*

Theorem 4.5(b) in Qiu [17] proved that the first order statistic achieves the minimal extropy among order statistics associated with a DFR sample. The following Corollary 2.6 generalizes this result to mixed systems. That is, the series system has the smallest extropy among mixed systems with components having DFR properties.

COROLLARY 2.6: *Let T be the lifetime of a mixed system consisting of n iid components which are* DFR*. Then $J(X_{1:n}) \leq J(T)$.*

PROOF: It is easy to see that $X_{1:n} \leq_{\text{hr}} T$ by Theorem 4.4 in Samaniego [21]. Since $X_{1:n}$ is DFR provided that X is DFR, $J(X_{1:n}) \leq J(T)$ by Corollary 2.5. ■

Lemma 2.3 in Qiu [17] shows that $X \leq_{\text{disp}} Y$ implies $J(X) \leq J(Y)$. One may wonder whether the condition $X \leq_{\text{disp}} Y$ in Proposition 2.3 can be relaxed to be $J(X) \leq J(Y)$. The following theorem gives a positive answer.

THEOREM 2.7: *Let* $\Delta(v) = f(F^{-1}(v)) - g(G^{-1}(v))$,

$$A_1 = \{0 \leq v \leq 1 | \Delta(v) > 0\}, \quad A_2 = \{0 \leq v \leq 1 | \Delta(v) \leq 0\}.$$

If $\inf_{A_1} g_V(v) \geq \sup_{A_2} g_V(v)$, *and if* $J(X) \leq J(Y)$, *then* $J(T^X) \leq J(T^Y)$.

PROOF: Since $J(X) \leq J(Y)$, we have from (1.3) that

$$\int_0^1 \Delta(v)\, dv = \int_0^1 [f(F^{-1}(v)) - g(G^{-1}(v))]\, dv \geq 0.$$

It follows from (2.4) that

$$\begin{aligned} -2[J(T^X) - J(T^Y)] &= \int_0^1 g_V^2(v)\Delta(v)\, dv \\ &= \int_{A_1} g_V^2(v)\Delta(v)\, dv + \int_{A_2} g_V^2(v)\Delta(v)\, dv \\ &\geq (\inf_{A_1} g_V(v))^2 \int_{A_1} \Delta(v)\, dv + (\sup_{A_2} g_V(v))^2 \int_{A_2} \Delta(v)\, dv \end{aligned}$$

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

$$\begin{aligned}
&\geq (\sup_{A_2} g_V(v))^2 \int_{A_1} \Delta(v)\, dv + (\sup_{A_2} g_V(v))^2 \int_{A_2} \Delta(v)\, dv \\
&= (\sup_{A_2} g_V(v))^2 \int_0^1 \Delta(v)\, dv \\
&\geq 0,
\end{aligned}$$

where the second inequality is obtained from the assumption that $\inf_{A_1} g_V(v) \geq \sup_{A_2} g_V(v)$, and the proof is completed. ■

Example 2.8: Let U be an uniformly distributed random variable on $(0,1)$, and Z be an exponential random variable with rate parameter $\lambda = 1$. It is easy to verify that $J(U) = -0.5 < -0.25 = J(Z)$. On the other hand, one can see that $A_1 = (0,1]$, $A_2 = \{0\}$, and hence $\inf_{A_1} g_V(v) = \sup_{A_2} g_V(v) = 0$ if the mixed system has uniform signature $\mathbf{s} = (1/n, \ldots, 1/n)$. Therefore, Theorem 2.7 implies $J(T^U) \leq J(T^Z)$.

If a mixed system with lifetime T has a signature $\mathbf{s} = (s_1, \ldots, s_n)$, then its dual system with lifetime T^D has signature $\mathbf{s}^D = (s_n, \ldots, s_1)$. Therefore, the amount of calculation of signature can be cut roughly in half, as the signature of a dual system can be obtained from the system's signature via symmetry arguments. The following Theorem 2.11 shows that the concept of the duality can also be used to reduce the calculation of the extropy of mixed systems. To prove Theorem 2.11, we need the forthcoming Lemmas 2.9 and 2.10. Lemma 2.9 is borrowed from Fashandi and Ahmadi [10]. Lemma 2.10 is a corollary of Stone–Weierstrass Theorem (Aliprantis and Burkinshaw [2]), which will be used to prove the necessity of Theorem 2.11.

LEMMA 2.9: *Let X be a random variable with cdf F, pdf f, and finite mean μ. Then $f(\mu + x) = f(\mu - x)$ for all $x \geq 0$ if and only if $f(F^{-1}(v)) = f(F^{-1}(1-v))$ for all $v \in (0,1)$.*

LEMMA 2.10: *If η is a continuous function on $[0,1]$ such that $\int_0^1 x^n \eta(x)\, dx = 0$ for all $n \geq 0$, then $\eta(x) = 0$ for any $x \in [0,1]$.*

THEOREM 2.11: *Let T be the lifetime of a mixed system with signature $\mathbf{s}$ consisting of n iid components. Then $J(T) = J(T^D)$ for all $\mathbf{s}$ and all n if and only if the pdf f of lifetimes of components satisfies $f(\mu + x) = f(\mu - x)$ for all $x \geq 0$, where μ is the mean of lifetimes of the components.*

PROOF: For the sufficiency, assume that $f(\mu + x) = f(\mu - x)$ for all $x \geq 0$. It follows from Lemma 2.9 that $f(F^{-1}(v)) = f(F^{-1}(1-v))$ for all $v \in (0,1)$. Note that $g_i(1-v) = g_{n-i+1}(v)$ for all $i = 1, \ldots, n$ and $v \in (0,1)$. Thus, we have from (2.4) that

$$\begin{aligned}
J(T^D) &= -\frac{1}{2} \int_0^1 \left(\sum_{i=1}^n s_{n-i+1} g_i(v))^2 f(F^{-1}(v) \right) dv \\
&= -\frac{1}{2} \int_0^1 \left(\sum_{r=1}^n s_r g_{n-r+1}(v))^2 f(F^{-1}(v) \right) dv
\end{aligned}$$

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

$$= -\frac{1}{2}\int_0^1 \left(\sum_{r=1}^{n} s_r g_r(1-v))^2 f(F^{-1}(1-v)\right) dv$$
$$= -\frac{1}{2}\int_0^1 \left(\sum_{r=1}^{n} s_r g_r(u)\right)^2 f(F^{-1}(u))\, du$$
$$= J(T).$$

For the necessity, since equation $J(T) = J(T^D)$ holds for all $\mathbf{s}$ and all n. Particularly, let $\mathbf{s} = (1, 0, \ldots, 0)$, then it follows from (2.4) that the assumption $J(T) = J(T^D)$ is equivalent to

$$\int_0^1 g_1^2(v) f(F^{-1}(v))\, dv = \int_0^1 g_n^2(v) f(F^{-1}(v))\, dv.$$

Or, equivalently,

$$\int_0^1 g_n^2(1-v) f(F^{-1}(v))\, dv = \int_0^1 g_n^2(v) f(F^{-1}(v))\, dv.$$

Putting $u = 1 - v$ in the left side of the above equation leads to

$$\int_0^1 g_n^2(u) f(F^{-1}(1-u))\, du = \int_0^1 g_n^2(v) f(F^{-1}(v))\, dv.$$

Thus, we have

$$\int_0^1 g_n^2(u)[f(F^{-1}(1-u)) - f(F^{-1}(v))]\, du = 0.$$

By (2.2), we have

$$\int_0^1 u^{n-1} \left[f(F^{-1}(1-u)) - f(F^{-1}(v))\right]\, du = 0,$$

and hence $f(F^{-1}(1-u)) = f(F^{-1}(v))$ by Lemma 2.10. This completes the proof by Lemma 2.9. ■

3. BOUNDS OF EXTROPY OF MIXED SYSTEMS

In practical situations, the number of components of the system is very large and the structure of system is very complicated, so it is not easy to compute the extropy of the system's lifetime. Hence, in such situations it is important to have bounds to approximate the behavior of the extropy of the system's lifetime. The bounds for the extropy of r-out-of-n system were built by Qiu [17]. Toomaj and Doostparast [27] obtained bounds for the entropy of mixed system's lifetime. This problem was also considered by Toomaj [26] for the Rènyi entropy of mixed system's lifetime. Hereafter, we derive some bounds for the extropy of mixed system's lifetime. In the forthcoming theorem, we obtain bounds for the extropy of mixed system's lifetime which depend on the extropy of components.

THEOREM 3.1: *Let T be the lifetime of a mixed system with signature $\mathbf{s}$ consisting of n iid components having lifetimes X. Then*

$$M^2 J(X) \le J(T) \le m^2 J(X),$$

where $m = \inf_{v\in(0,1)} g_V(v)$, $M = \sup_{v\in(0,1)} g_V(v)$.

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

PROOF: The upper bound can be derived from (2.4) and (1.3) as follows:

$$J(T) = -\frac{1}{2}\int_0^1 g_V^2(v) f(F^{-1}(v))\,dv \le -\frac{1}{2}m^2\int_0^1 f(F^{-1}(v))\,dv = m^2 J(X).$$

The lower bound can be obtained in a similar way. ■

Different from Theorem 3.1, next we will derive another lower bound for the extropy of mixed system's lifetime in terms of the extropy of r-out-of-n system's lifetimes.

THEOREM 3.2: *Let T be the lifetime of a mixed system with signature* $\mathbf{s}$ *consisting of n iid components having lifetimes X. Then $J(T) \ge \sum_{i=1}^n s_i J(X_{i:n})$. The equality in above inequality holds for r-out-of-n systems.*

PROOF: It follows from (2.4) that

$$\begin{aligned} J(T) &= -\frac{1}{2}\int_0^1 \left(\sum_{i=1}^n s_i g_i(v)\right)^2 f(F^{-1}(v))\,dv \\ &\ge -\frac{1}{2}\int_0^1 \sum_{i=1}^n s_i g_i^2(v) f(F^{-1}(v))\,dv \\ &= -\frac{1}{2}\sum_{i=1}^n s_i \int_0^1 g_i^2(v) f(F^{-1}(v))\,dv \\ &= \sum_{i=1}^n s_i J(X_{i:n}), \end{aligned}$$

where the inequality comes from the Jensen's inequality. For an r-out-of-n system, its signature satisfies $s_i = 0$ for all $i \ne n-r+1$ and $s_i = 1$ for $i = n-r+1$. Hence, $J(T) = J(X_{(n-r+1):n})$ and the lower bound in Theorem 3.2 is achieved. ■

Including Murthy and Jiang [16], Jiang, Zuo, and Li [12], and Castet and Saleh [8], many authors studied the behaviors of a mixed system with components having Weibull distributions. To compare the lower bounds obtained in Theorems 3.1 and 3.2, we also consider a mixed system with Weibull distributed components in the following example.

Example 3.3: Consider a mixed system having three iid components with lifetime T and with signature $\mathbf{s} = (\frac{1}{3}, \frac{2}{3}, 0)$. Denote the lifetime of each component by X and let X be Weibull distributed random variable with pdf given by

$$f(x) = kx^{k-1}e^{-x^k}, \quad x \ge 0,\ k > 0.$$

It is easy to verify that

$$m = \inf_{v\in(0,1)} g_V(v) = 0, \quad M = \sup_{v\in(0,1)} g_V(v) = 1.333.$$

Thus, the lower bounds for $J(T)$ obtained from Theorems 3.1 and 3.2 are given by $A_1 = 1.333^2 J(X)$ and $A_2 = [J(X_{1:3}) + 2J(X_{2:3})]/3$, respectively. Figure 2 shows that A_2 performs well than A_1. We can also see from Figure 2 that both A_1 and A_2 are decreasing as k increases from 1 to 10.

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

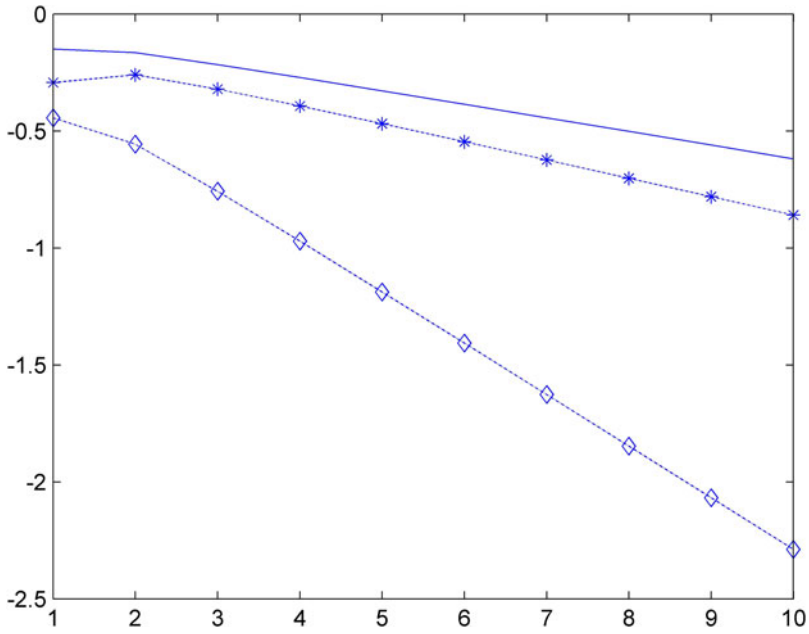


FIGURE **2.** Exact value of $J(T)$ (solid line) as well as A_1 (dashed line with rhombus) and A_2 (dashed line with star) for Weibull distribution with common scale parameter $\lambda = 1$ and shape parameters $k = 1 : 10$.

To end this section, we rewrite $J(X)$ by $J(f)$, where f is the pdf of X. The proof of Theorem 3.2 shows that $J(\sum_{i=1}^n s_i f_i) \geq \sum_{i=1}^n s_i J(f_i)$ for all $s_i \geq 0$ and $\sum_{i=1}^n s_i = 1$, where f_i is the pdf of X_i, $i = 1, \ldots, n$. That is, the extropy $J(f)$ is a concave functional with respect to pdf f. It is still an open question whether inequality $J(\sum_{i=1}^n s_i X_i) \geq \sum_{i=1}^n s_i J(X_i)$ holds or not for all $s_i \geq 0$ and $\sum_{i=1}^n s_i = 1$, even if $X_1, \ldots, X_n$ are independent. However, we can prove that the extropy of the sum of two independent variables is larger than that of each one. The following Theorem 3.4 states this fact. From an information-theoretic vantage point, Theorem 3.4 reflects a fact that the sum of independent variables will increase the uncertainty. It should be noted that similar results were proved for Shannon's entropy and cumulative residual entropy, see Shannon [25] and Rao et al. [20].

THEOREM 3.4: *Let X and Y be two non-negative and independent rv's. Then*

$$J(X + Y) \geq \max\{J(X),\, J(Y)\}.$$

PROOF: Since X and Y are independent, the pdf of $X + Y$ is given by

$$h(x) = \int_0^x f(x - t)\, dG(t),$$

where f and G are pdf and cdf of X and Y, respectively. Using Jensen's inequality, we have

$$\begin{aligned} h^2(x) &= \left[\int_0^\infty f(x - t) I(x \geq t)\, dG(t)\right]^2 \\ &\leq \int_0^\infty f^2(x - t) I(x \geq t)\, dG(t), \end{aligned}$$

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

where $I(x \geq t)$ is the indicator function of $\{x \geq t\}$. Integrating each side of the above inequality with respect to x from 0 to ∞, we have

$$\begin{aligned}\int_0^\infty h^2(x)\,dx &\leq \int_0^\infty dx \int_0^\infty f^2(x-t)I(x \geq t)\,dG(t)\\ &= \int_0^\infty dG(t) \int_0^\infty f^2(x-t)I(x \geq t)\,dx\\ &= \int_t^\infty f^2(x-t)\,dx\\ &= \int_0^\infty f^2(x)\,dx.\end{aligned}$$

Therefore,

$$J(X+Y) = -\frac{1}{2}\int_0^\infty h^2(x)\,dx \geq -\frac{1}{2}\int_0^\infty f^2(x)\,dx = J(X).$$

Similarly, $J(X+Y) \geq J(Y)$. This completes the proof. ■

Remark 3.5: From the proof of Theorem 3.4, we see that $J(X+Y) \geq J(Y)$. On the other hand, $J(X)$ is always non-positive by (1.3). Thus, it holds that $J(X+Y) \geq J(Y) + J(X)$. This implies that the extropy $J(X)$ is a super-additive functional.

4. JE DIVERGENCE OF MIXED SYSTEM

The entropy of a random variable X with pdf f is defined as

$$H(X) = -\int_{-\infty}^{\infty} f(x)\log(x)\,dx$$

provided that the integral exists (Shannon [25]). Based on this notion, Asadi, Ebrahimi, Soofi, and Zohrevand [3] proposed the JS divergence between T with pdf f_T given in (1.1) and $X_{1:n}, \ldots, X_{n:n}$ as

$$\mathrm{JS}(T : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}) = H(T) - \sum_{i=1}^{n} s_i H(X_{i:n}).$$

They proved that JS divergence is non-negative and its minimum is attained by r-out-of-n systems, which are the least complex mixed systems. This property offers JS divergence as a measure of complexity of a system.

By analogy to JS divergence, we define the JE divergence between T with pdf f_T given in (1.1) and $X_{1:n}, \ldots, X_{n:n}$ as

$$\mathrm{JE}(T : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}) = J(T) - \sum_{i=1}^{n} s_i J(X_{i:n}). \tag{4.1}$$

Using (2.4) and the probability integral transformation $U = F(X)$ again, we see that JE divergence in (4.1) can be rewritten as

$$\mathrm{JE}(T : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}) = -\frac{1}{2}\int_0^1 [g_V^2(v) - \sum_{i=1}^{n} s_i g_i^2(v)] f(F^{-1}(v))\,dv, \tag{4.2}$$

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

where $g_V(v)$ and $g_i(v), i = 1, \ldots, n$, are defined in (2.3) and (2.2), respectively. Thus, the JE divergence of a mixed system is a functional of its signature $\mathbf{s}$ and the common cdf F of lifetimes of its components. Additionally, Theorem 3.2 shows that the JE divergence is non-negative and the minimum is also attained by r-out-of-n systems. $\mathrm{JE}(T : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}) \geq 0$ quantifies how much the system with a signature $\mathbf{s}$ is more complex than an r-out-of-n system with homogeneous components. For this two reasons, JE divergence can be used as the alternative information criteria for comparing mixed systems with homogeneous components. Moreover, consider a mixed system with signature $\mathbf{s}$ having n iid components. Let T^X (T^Y) be the lifetime of the system if its components have lifetime X (Y). If $X \leq_{\mathrm{disp}} Y$, it follows from (4.2) that

$$\mathrm{JE}(T^X : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}) \leq \mathrm{JE}(T^Y : Y_{1:n}, \ldots, Y_{n:n}, \mathbf{s}),$$

where $Y_{1:n}, \ldots, Y_{n:n}$ are order statistics associated with Y.

Recall that the relative extropy between X with pdf f and Y with pdf g is defined by Lad, Sanfilippo, and Agrò [14] as

$$\mathrm{D}(X : Y) = \frac{1}{2} \int_{-\infty}^{\infty} [f(x) - g(x)]^2 \, dx.$$

Next we will show that JE divergence can be represented by relative extropy. From the following Proposition 4.1, we see that JE divergence maps the weighted information discrepancy between the lifetimes of the system and the order statistics.

PROPOSITION 4.1: *The* JE *divergence between* T *with pdf* f_T *given in* (1.1) *and* $X_{1:n}, \ldots, X_{n:n}$ *can be expressed as*

$$\mathrm{JE}(T : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}) = \sum_{i=1}^{n} s_i \mathrm{D}(T : X_{i:n}).$$

PROOF: By (1.1) and (2.3), we have

$$\begin{aligned}
\sum_{i=1}^{n} s_i \mathrm{D}(T : X_{i:n}) &= \frac{1}{2} \sum_{i=1}^{n} s_i \int_0^{\infty} \left[\sum_{i=1}^{n} s_i f_{i:n}(x) - f_{i:n}(x) \right]^2 dx \\
&= \frac{1}{2} \sum_{i=1}^{n} s_i \int_0^1 [g_V(v) - g_i(v)]^2 f(F^{-1}(v)) \, dv \\
&= \frac{1}{2} \sum_{i=1}^{n} s_i \int_0^1 \left[g_V^2(v) - 2 g_i(v) g_V(v) + g_i^2(v) \right] f(F^{-1}(v)) \, dv \\
&= \frac{1}{2} \int_0^1 \left[\sum_{i=1}^{n} s_i g_V^2(v) - 2 g_V^2(v) + \sum_{i=1}^{n} s_i g_i^2(v) \right] f(F^{-1}(v)) \, dv \\
&= -\frac{1}{2} \int_0^1 \left[g_V^2(v) - \sum_{i=1}^{n} s_i g_i^2(v) \right] f(F^{-1}(v)) \, dv \\
&= \mathrm{JE}(T : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}),
\end{aligned}$$

where the fifth equality follows from $\sum_{i=1}^{n} s_i = 1$ and the last equality comes from (4.2). ■

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

Some properties of the relative extropy have important consequences for the JE divergence of a system. To illustrate this point, we use Proposition 4.1 to prove that the JE divergence of a mixed system equals to that of its dual system.

THEOREM 4.2: *Let T be the lifetime of a mixed system with signature $\mathbf{s}$ consisting of n iid components. If the components of the mixed system have a symmetric pdf f, then*

$$\mathrm{JE}(T : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}) = \mathrm{JE}(T^D : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}^D),$$

where T^D is the lifetime of the dual system with signature $\mathbf{s}^D$.

PROOF: For all $i = 1, \ldots, n$, since $g_{n-i+1}(v) = g_i(1-v)$, we have

$$\begin{aligned}
\mathrm{D}(T^D : X_{n-i-1:n}) &= \frac{1}{2}\int_0^\infty \left[\sum_{k=1}^n s_{n-k+1} f_{k:n}(x) - f_{n-i+1:n}(x)\right]^2 dx \\
&= \frac{1}{2}\int_0^1 \left[\sum_{k=1}^n s_{n-k+1} g_k(v) - g_{n-k+1}(v)\right]^2 f(F^{-1}(v))\, dv \\
&= \frac{1}{2}\int_0^1 \left[\sum_{r=1}^n s_r g_{n-r+1}(v) - g_{n-i+1}(v)\right]^2 f(F^{-1}(v))\, dv \\
&= \frac{1}{2}\int_0^1 \left[\sum_{r=1}^n s_r g_r(1-v) - g_i(1-v)\right]^2 f(F^{-1}(1-v))\, dv \\
&= \frac{1}{2}\int_0^1 \left[\sum_{r=1}^n s_r g_r(v) - g_i(v)\right]^2 f(F^{-1}(v))\, dv \\
&= \mathrm{D}(T : X_{i:n}),
\end{aligned}$$

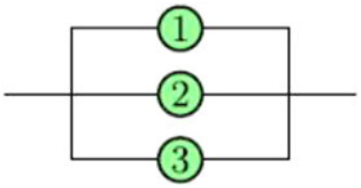
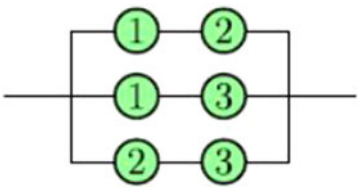
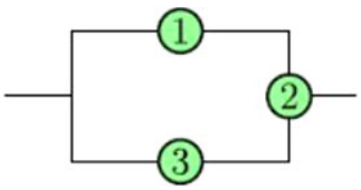
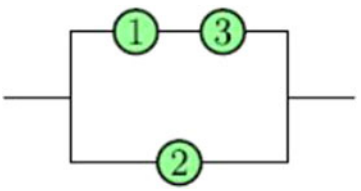
where the forth equality follows from the assumption that f is symmetric and Lemma 2.9. Thus, it follows from Proposition 4.1 that

$$\begin{aligned}
\mathrm{JE}(T^D : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}^D) &= \sum_{i=1}^n s_{n-i+1} \mathrm{D}(T^D : X_{i:n}) \\
&= \sum_{r=1}^n s_r \mathrm{D}(T^D : X_{n-r+1:n}) \\
&= \sum_{r=1}^n s_r \mathrm{D}(T : X_{r:n}) \\
&= \mathrm{JE}(T : X_{1:n}, \ldots, X_{n:n}, \mathbf{s}).
\end{aligned}$$

This completes the proof. ■

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

TABLE 1. Mixed Systems with Three Components

	System	Structure	Signature $\mathbf{s}$	JE divergence
A		$X_{1:3}$ (Series)	$(1,0,0)$	0
B		$X_{3:3}$ (Parallel)	$(0,0,1)$	0
C		$X_{2:3}$ (2-out-of-3)	$(0,1,0)$	0
D		$\min\{\max\{X_1, X_3\}, X_2\}$	$(\frac{1}{3}, \frac{2}{3}, 0)$	0.100000
E		$\max\{\min\{X_1, X_3\}, X_2\}$	$(0, \frac{2}{3}, \frac{1}{3})$	0.033333

Remark 4.3: Let $W_i, i = 1, \ldots, n$ have pdf g_i as defined in (2.2). Using the similar manner in the proof of Theorem 4.2, we can show that

$$\sum_{i=1}^{n} s_i \mathrm{D}(W_i : W_j) = \sum_{i=1}^{n} s_i \mathrm{D}(W_i : W_{n-j+1})$$

for all $j = 1, \ldots, n$, whenever the vector $\mathbf{s}$ is symmetric, that is, $s_i = s_{n-i+1}$ for all $i = 1, \ldots, n$. Furthermore, if the underlying distribution of order statistics $X_{i:n}$ has a symmetric pdf and the vector $\mathbf{s}$ is symmetric, then for all $j = 1, \ldots, n$,

$$\sum_{i=1}^{n} s_i \mathrm{D}(X_{i:n} : X_{j:n}) = \sum_{i=1}^{n} s_i \mathrm{D}(X_{i:n} : X_{n-j+1:n}).$$

Next, we will illustrate applications of JE divergence using two sets of systems with three and four components discussed by Shaked and Suarez-Llorens [24]. Recently, Asadi, Ebrahimi, Soofi, and Zohrevand [3] considered these two sets of systems to show the applications of JS divergence. The former paper ordered systems in each set by defining a

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

TABLE 2. Mixed Systems with Four Components

	System and Structure	Signature **s**	JE divergence
	Group 1		
A	$X_{1:4}$ (Series)	$(1,0,0,0)$	0
T	$X_{4:4}$ (Parallel)	$(0,0,0,1)$	0
F	$X_{3:4}$ (3-out-of-4)	$(0,0,1,0)$	0
O	$X_{2:4}$ (2-out-of-4)	$(0,1,0,0)$	0
	Group 2		
G	$\max\{X_{1:2}, \min\{X_1, X_3, X_4\}, \min\{X_2, X_3, X_4\}\}$	$(0, \frac{5}{6}, \frac{1}{6}, 0)$	0.023810
N	$\min\{X_{2:2}, \max\{X_1, X_3, X_4\}, \max\{X_2, X_3, X_4\}\}$	$(0, \frac{1}{6}, \frac{5}{6}, 0)$	0.023810
	Group 3		
H	$\max\{X_{1:2}, \min\{X_3, X_4\}\}$	$(0, \frac{2}{3}, \frac{1}{3}, 0)$	0.038095
I	$\max\{X_{1:2}, \min\{X_1, X_3\}, \min\{X_2, X_3, X_4\}\}$	$(0, \frac{2}{3}, \frac{1}{3}, 0)$	0.038095
M	$\max\{X_{2:2}, \max\{X_3, X_4\}\}$	$(0, \frac{1}{3}, \frac{2}{3}, 0)$	0.038095
L	$\max\{\min\{X_1, \max\{X_2, X_3, X_4\}\}, \min\{X_2, X_3\}\}$	$(0, \frac{1}{3}, \frac{2}{3}, 0)$	0.038095
	Group 4		
J	$\max\{X_{1:2}, \min\{X_2, X_3\}, \min\{X_3, X_4\}\}$	$(0, \frac{1}{2}, \frac{1}{2}, 0)$	0.042857
K	$\max\{\min\{X_1, \max\{X_2, X_3, X_4\}\}, \min\{X_2, X_3, X_4\}\}$	$(0, \frac{1}{2}, \frac{1}{2}, 0)$	0.042857
	Group 5		
C	$\min\{X_{1:3}, X_4\}$	$(\frac{1}{4}, \frac{3}{4}, 0, 0)$	0.107143
R	$\max\{X_{1:3}, X_4\}$	$(0, 0, \frac{1}{4}, \frac{3}{4})$	0.021429
	Group 6		
B	$\max\{X_{1:3}, \min\{X_2, X_3, X_4\}\}$	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	0.142857
S	$\max\{X_1, X_2, \min\{X_3, X_4\}\}$	$(0, 0, \frac{1}{2}, \frac{1}{2})$	0.028571
	Group 7		
D	$\min\{X_{2:2}, \max\{X_1, X_3\}, X_4\}$	$(\frac{3}{12}, \frac{7}{12}, \frac{2}{12}, 0)$	0.140476
Q	$\max\{X_1, \min\{X_2, X_4\}, \min\{X_3, X_4\}\}$	$(0, \frac{2}{12}, \frac{7}{12}, \frac{3}{12})$	0.050000
	Group 8		
E	$\min\{X_{3:3}, X_4\}$	$(\frac{1}{4}, \frac{1}{4}, \frac{2}{4}, 0)$	0.178571
Q	$\max\{X_1, \min\{X_2, X_3, X_4\}\}$	$(0, \frac{2}{4}, \frac{1}{4}, \frac{1}{4})$	0.078571

convolution order and assuming the lifetimes of the components are iid with truncated exponential pdf $f(x) = \lambda e^{-\lambda(x-\theta)}, x \geq \theta$. We assume that the lifetimes of the components are iid with standard exponential pdf $f(x) = e^{-x}, x \geq 0$.

Table 1 depicts five mixed systems (A-E) with three components, their structures for the lifetime of system T, and their signatures given in Table 1 of Shaked and Suarez-Llorens [24]. The last column of Table 1 shows the JE divergence of each system. Systems A-C are k-out-of-3, $k = 1, 2, 3$, and hence their JE divergence are zero.

Table 2 shows 20 mixed systems (A-T) with four components, their structures for the lifetime of the system T, and their signatures from Table 2 of Shaked and Suarez-Llorens [24]. As pointed in Shaked and Suarez-Llorens [24], the 20 systems in Table 2 correspond to only 17 different signatures. To make it more clear, we classify these 20 systems into

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

8 groups according to the dualities of systems. All the systems with dual signatures are comprised in the same group. We see from Group 5 that systems C and R have different JE divergences even though their signatures are dual.

Group 1 is the least complex group consisting of four k-out-of-4, $k = 1, 2, 3, 4$ systems, whose lifetimes are determined by one of the order statistics. Therefore, the JE divergence of these systems are zero. Groups 2–4 consist of systems where the second and the third order statistics determine the system's lifetime. The systems whose lifetimes are determined by either the first and second or the third and fourth order statistics are included in Groups 5–6. In the last two groups, the lifetimes of systems are determined by either the first three or the last three order statistics.

Acknowledgement

We are grateful to the referees and editors for their important and constructive comments, which improved the presentation. This work was supported by NNSF of China (Nos: 11371340, 11471303) and Key Program of Education Department of Anhui Province (No. KJ2018A0590).

References

1. Agrò, G., Lad, F., & Sanfilippo, G. (2010). Sequentially forecasting economic indices using mixture linear combinations of EP distributions. *Journal of Data Science* 8: 101–126.
2. Aliprantis, C.D. & Burkinshaw, O. (1981). *Principles of real analysis*. New York: Elsevier North-Holland Incorporated.
3. Asadi, M., Ebrahimi, N., Soofi, E.S., & Zohrevand, Y. (2016). Jensen–Shannon information of the coherent system lifetime. *Reliability Engineering and System Safety* 156: 244–255.
4. Bagai, L. & Kochar, S.C. (1986). On tail ordering and comparison of failure rates. *Communications in Statistics–Theory and Methods* 15: 1377–1388.
5. Barlow, R. & Proschan, F. (1981). *Statistical theory of reliability and life testing, probability model*. New York: Holt, Rinehart and Winnston.
6. Boland, P.J. & Samaniego, F. (2004). The signature of a coherent system and its applications in reliability. In R. Soyer, T. Mazzuchi, & N.D. Singpurwalla, (eds.), *Mathematical reliability: an expository perspective*, Boston: Kluwer Publishers, pp. 1–29.
7. Capotorti, A., Regoli, G., & Vattari, F. (2010). Correction of incoherent conditional probability assessments. *International Journal of Approximate Reasoning* 51: 718–727.
8. Castet, J.F. & Saleh, J.H. (2010). Single versus mixture Weibull distributions for non-parametric satellite reliability. *Reliability Engineering and System Safety* 95: 295–300.
9. D'andrea, A. & De Sanctis, L. (2015). The Kruskal-Katona theorem and a characterization of system signatures. *Journal of Applied Probability* 52: 508–518.
10. Fashandi, F. & Ahmadi, J. (2012). Characterizations of symmetric distributions based on Renyi entropy. *Statistics and Probability Letters* 82: 798–804.
11. Gneiting, T. & Raftery, A.E. (2007). Strictly proper scoring rules, prediction, and estimation. *Journal of the American Statistical Association* 102: 359–378.
12. Jiang, R., Zuo, M.J., & Li, H.X. (1999). Weibull and inverse Weibull mixture models allowing negative weights. *Reliability Engineering and System Safety* 66: 227–234.
13. Lad, F., Sanfilippo, G., & Agrò, G. (2012). *Completing the logarithmic scoring rule for assessing probability distributions. 11th Brazilian Meeting on Bayesian Statistics, Amparo, Brazil*. In J.M. Stern, M.D. Lauretto, & Polpo, A. (eds.), AIP Conference Proceedings, Vol. 1490, pp. 13–30.
14. Lad, F., Sanfilippo, G., & Agrò, G. (2015). Extropy: complementary dual of entropy. *Statistical Science* 30: 40–58.
15. Müller, A. & Stoyan, D. (2002). *Comparison methods for stochastic models and risks*. West Sussex: Wiley and Sons.
16. Murthy, D.N.P. & Jiang, R. (1997). Parametric study of sectional models involving two Weibull distributions. *Reliability Engineering and System Safety* 56: 151–159.
17. Qiu, G. (2017). The extropy of order statistics and record values. *Statistics and Probability Letters* 120: 52–60.
18. Qiu, G. & Jia, K. (2018). The residual extropy of order statistics. *Statistics and Probability Letters* 133: 15–22.

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press

19. Qiu, G. & Jia, K. (2018). Extropy estimators with applications in testing uniformity. *Journal of Nonparametric Statistics* 30: 182–196.
20. Rao, M., Chen, Y., Vemuri, B.C., & Wang, F. (2004). Cumulative residual entropy: a new measure of information. *IEEE Transactions on Information Theory* 50: 1220–1228.
21. Samaniego, F.J. (2007). *System signatures and their applications in engineering reliability*. New York: Springer.
22. Samaniego, F.J. & Navarro, J. (2016). On comparing coherent systems with heterogeneous components. *Advance in Applied Probability* 48: 88–111.
23. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders*. New York: Springer.
24. Shaked, M. & Suarez-Llorens, A. (2003). On the comparison of reliability experiments based on the convolution order. *Journal of the American Statistical Association* 98: 693–702.
25. Shannon, C.E. (1948). A mathematical theory of communication. *The Bell System Technical Journal* 27: 379–423.
26. Toomaj, A. (2017). Rènyi entropy properties of mixed systems. *Communications in Statistics–Theory and Methods* 46: 906–916.
27. Toomaj, A. & Doostparast, M. (2014). A note on signature-based expressions for the entropy of mixed r–out–of–n systems. *Naval Research Logistics* 61: 202–206.
28. Toomaj, A. & Doostparast, M. (2016). On the Kullback Leibler information for mixed systems. *International Journal of Systems Science* 47: 2458–2465.
29. Toomaj, A., Sunoj, S.M., & Navarro, J. (2017). Some properties of the cumulative residual entropy of coherent and mixed systems. *Journal of Applied Probability* 54: 379–393.

https://doi.org/10.1017/S0269964818000244 Published online by Cambridge University Press