Math. Proc. Camb. Phil. Soc. (1979), **85**, 69
Printed in Great Britain

On harmonic functions which take integer values on integer lattices

By D. H. ARMITAGE
Queen's University, Belfast

(*Received* 8 *March* 1978)

1. *Introduction.* Throughout this paper α denotes a number larger than 1 and C denotes the complex plane. Pólya (15), (16) (see also Hardy (10) and Landau (13)) proved the following results.

THEOREM A. (i) *If f is an entire function in C such that*

$$\max_{|z|=r} |f(z)| = O(\alpha^r) \quad (r \to \infty), \tag{1}$$

where $\alpha < 2$, and $f(z)$ is an integer whenever z is a non-negative integer, then f is a polynomial.

(ii) *If f is an entire function in C such that* (1) *holds for some $\alpha < \frac{1}{2}(3+\sqrt{5})$ and $f(z)$ is an integer whenever z is an integer, then f is a polynomial.*

A concise proof (essentially Pólya's with the improvements of Hardy and Landau) of Theorem A (i) is given in J. M. Whittaker's tract (19). The theorem has been generalized in several directions, and we cite (5, 7, 8, 9, 14, 17, 18) as a sample of the literature; Buck (7) contains a good bibliography of early work. Here we prove analogous results for harmonic functions in the Euclidean space R^n ($n \geqslant 2$). To do this, we shall use some results of the author (mainly (3, 4)) together with an m-dimensional generalization of Theorem A (i) due to A. Baker (5) and an analogous generalization of Theorem A (ii).

Before stating our results we give some notations and conventions. Throughout the paper m and n will denote integers such that $n \geqslant 2$ and $1 \leqslant m \leqslant n$. Arbitrary points of R^n and C^m are denoted by $x = (x_1, \ldots, x_n)$ and $z = (z_1, \ldots, z_m)$, respectively, and we identify a point $(x_1, \ldots, x_m)$ of R^m with the point $(x_1, \ldots, x_m, 0, \ldots, 0)$ of R^n, so that R^m is regarded as a subspace of R^n. We also regard R^m as a subset of C^m in the obvious way and put

$$\|z\| = |z_1| + \ldots + |z_m| \quad (z \in C^m)$$

and

$$|x| = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} \quad (x \in R^n).$$

The set of all integers (resp. non-negative integers) is denoted by Z (resp. N) and we put

$$Z^m = \{x \in R^n : x_i \in Z\,(1 \leqslant i \leqslant m), \quad x_i = 0\,(m < i \leqslant n)\}$$

and

$$N^m = \{x \in R^n : x_i \in N\,(1 \leqslant i \leqslant m), \quad x_i = 0\,(m < i \leqslant n)\}.$$

If G_1 and G_2 are subsets of R^n, we put

$$G_1 + G_2 = \{x + y : x \in G_1, y \in G_2\}.$$

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press

If f is a continuous function in C^m, we denote by $\mathscr{L}(f,r)$ the maximum value of $|f|$ over the set $\{z \in C^m: \|z\| = r\}$, and if g is a continuous function in R^n, we denote by $\mathscr{M}(g,r)$ the maximum value of $|g|$ over the sphere $\{x \in R^n: |x| = r\}$. The space of all entire functions f in C^m which satisfy

$$\mathscr{L}(f,r) = O(\alpha^r) \quad (r \to \infty)$$

is denoted by $\mathscr{E}_m(\alpha)$, and the space of all harmonic functions h in R^n which satisfy

$$\mathscr{M}(h,r) = O(\alpha^r) \quad (r \to \infty)$$

is denoted by $\mathscr{H}_n(\alpha)$.

The results from which our harmonic analogues of Theorem A will be derived are as follows.

THEOREM 1. *Suppose that $m < n$. If $h \in \mathscr{H}_n(\alpha)$, where $\alpha < 2$, and $h(N^m) \subset Z$, then there is a polynomial P in R^n such that $P = h$ on R^m. The result is false with $\alpha = 2$.*

THEOREM 2. *Suppose that $m < n$. If $h \in \mathscr{H}_n(\alpha)$, where $\alpha < \frac{1}{2}(3+\sqrt{5})$, and $h(Z^m) \subset Z$, then there is a polynomial P in R^n such that $P = h$ on R^m. The result is false with*

$$\alpha = \tfrac{1}{2}(3+\sqrt{5}).$$

It is easy to show that the polynomial P in these theorems can be taken to be harmonic (see, for example, (2), theorem 3). We shall show in Section 6 that Theorem A can be recovered from the case $n = 2$ of these theorems.

The positive parts of these theorems remain true in the case $m = n$, but in this case, by using the case $m = n-1$ of Theorems 1 and 2 and some well-known results, we shall be able to deduce the following stronger theorems. We denote the origin of R^n by O.

THEOREM 3. *Suppose that $\alpha < 2$ and that y is a point of R^n such that $0 < y_n < \pi/\log\alpha$. If $h \in \mathscr{H}_n(\alpha)$ and $h(N^{n-1} + \{O, y\}) \subset Z$, then h is a polynomial. The result fails if*

$$y_n = \pi/\log\alpha;$$

it is also false, for any y, with $\alpha = 2$.

THEOREM 4. *Suppose that $\alpha < \frac{1}{2}(3+\sqrt{5})$ and that y is a point of R^n such that*

$$0 < y_n < \pi/\log\alpha.$$

If $h \in \mathscr{H}_n(\alpha)$ and $h(Z^{n-1} + \{O, y\}) \subset Z$, then h is a polynomial. The result fails if

$$y_n = \pi/\log\alpha;$$

it is also false, for any y, with $\alpha = \frac{1}{2}(3+\sqrt{5})$.

Instead of putting integer-value conditions on h on two parallel lattices, as in Theorems 3 and 4, we can put integer-value conditions on h and $\partial h/\partial x_n$.

THEOREM 5. *Suppose that $\alpha < 2$ and that y is a point of R^n such that*

$$0 \leqslant y_n < \pi/(2\log\alpha).$$

If $h \in \mathscr{H}_n(\alpha)$ and $h(N^{n-1}) \subset Z$ and $(\partial h/\partial x_n)(N^{n-1} + \{y\}) \subset Z$, then h is a polynomial. The result fails if $y_n = \pi/(2\log\alpha)$; it is also false, for any y, with $\alpha = 2$.

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press

THEOREM 6. *Suppose that* $\alpha < \frac{1}{2}(3+\sqrt{5})$ *and that* y *is a point of* R^n *such that*

$$0 \leqslant y_n < \pi/(2\log\alpha).$$

If $h \in \mathscr{H}_n(\alpha)$ *and* $h(Z^{n-1}) \subset Z$ *and* $(\partial h/\partial x_n)\,(Z^{n-1}+\{y\}) \subset Z$, *then* h *is a polynomial. The result fails if* $y_n = \pi/(2\log\alpha)$; *it is also false, for any* y, *with* $\alpha = \frac{1}{2}(3+\sqrt{5})$.

2. *Preparatory results on entire functions in* C^m. I am grateful to the referee for indicating how Theorems 1 and 2 follow easily from the results presented in this section and an earlier result of the author (Lemma A, below). The proofs of Theorems 1 and 2 in the first draft of this paper were independent of the results of this section but were based on ideas similar in essence to the main ideas in the proofs of the following theorems.

THEOREM B. *If* $f \in \mathscr{E}_m(\alpha)$, *where* $\alpha < 2$, *and* $f(N^m) \subset Z$, *then* f *is a polynomial.*

This is a generalization of Pólya's original result (Theorem A (i)) and is the case $h = 1$ of Baker's theorem (i) in (5).

The corresponding result with Z^m replacing N^m, which we require for the proof of Theorem 2, seems not to have appeared in print hitherto, and we use Baker's technique (5) to prove it here.

THEOREM 7. *If* $f \in \mathscr{E}_m(\alpha)$, *where* $\alpha < \frac{1}{2}(3+\sqrt{5})$, *and* $f(Z^m) \subset Z$, *then* f *is a polynomial.*

It is very probable that a fuller exploitation of the techniques of (5) would yield a complete analogue (and not merely an analogue of the case $h = 1$, as is our Theorem 7) of Baker's theorem (i).

In the proofs of our results we write $\sigma = \frac{1}{2}(3+\sqrt{5})$.

We now prove Theorem 7. Let $J_1, J_2 = J_2(j_1), \ldots, J_m = J_m(j_1, \ldots, j_{m-1})$ be non-negative integers such that J_k is defined when $1 < k \leqslant m$ for each $(k-1)$-tuple $(j_1, \ldots, j_{k-1})$ of integers satisfying $|j_l| \leqslant J_l$ $(1 \leqslant l < k)$. Let $C_1, \ldots, C_m$ denote circles in C with centres at the origin, described in the positive sense. For each m-tuple of integers $(j_1, \ldots, j_m)$ with $|j_l| \leqslant J_l (1 \leqslant l \leqslant m)$, define $q_k = q_k(j_1, \ldots, j_m)$ $(1 \leqslant k \leqslant m)$ as 1 if $j_k = -J_k$ and as 0 otherwise. We write for each integer j and each complex number w

$$p_j(w) = w(w-1)\,(w+1)\,(w-2)\,(w+2)\ldots(w-j).$$

The identity

$$\frac{1}{(\zeta_1-z_1)\ldots(\zeta_m-z_m)} = \sum_{j_1=-J_1}^{J_1} \cdots \sum_{j_m=-J_m}^{J_m} \prod_{k=1}^{m} \left\{ \frac{1}{(z_k-j_k)} \left(\frac{p_{j_k}(z_k)}{p_{j_k}(\zeta_k)}\right) \left(\frac{\zeta_k-j_k}{\zeta_k-z_k}\right)^{q_k} \right\},$$

valid for all $z, \zeta \in C^m$ for which the denominators do not vanish, is easily verified for $m = 1$ by inducton on J_1 and in general by induction on m. If $z_1, \ldots, z_m$ and those $j_1, \ldots, j_m$ which occur in the denominators are interior to $C_1, \ldots, C_m$ respectively, then, on multiplying the identity by $f(\zeta)$ and integrating, we obtain

$$f(z) = \sum_{j_1=-J_1}^{J_1} \cdots \sum_{j_m=-J_m}^{J_m} a_{j_1,\ldots,j_m}(z) \prod_{k=1}^{m} \{(z_k-j_k)^{-1}\, p_{j_k}(z_k)\}, \tag{2}$$

where

$$a_{j_1,\ldots,j_m}(z) = \frac{1}{(2\pi i)^m} \int_{C_1} \cdots \int_{C_m} \frac{f(\zeta)}{p_{j_1}(\zeta_1)\ldots p_{j_m}(\zeta_m)} \prod_{k=1}^{m} \left(\frac{\zeta_k-j_k}{\zeta_k-z_k}\right)^{q_k} d\zeta_m \ldots d\zeta_1. \tag{3}$$

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press

In the case where $j_k > -J_k$ for all $k = 1, \ldots, m$, we have $q_k = 0$ for all $k = 1, \ldots, m$, so that $a_{j_1, \ldots, j_m}$ is independent of z, and, by repeated use of Cauchy's residue theorem, we obtain

$$a_{j_1, \ldots, j_m} = \Sigma f(l_1, \ldots, l_m)\{\Pi(l_1 - \lambda_1)\ldots(l_m - \lambda_m)\}^{-1},$$

where the sum is over all m-tuples $(l_1, \ldots, l_m)$ such that $l_k = 0, 1, -1, 2, -2, \ldots, j_k$ $(k = 1, \ldots, m)$ and the products are over all m-tuples $(\lambda_1, \ldots, \lambda_m)$ such that

$$\lambda_k = 0, 1, -1, 2, -2, \ldots, j_k \quad \text{and} \quad \lambda_k \neq l_k (k = 1, \ldots, m).$$

Now it is easy to see that each of these products divides $|2j_1|! \ldots |2j_n|! = X$, say, whence it follows by hypothesis that $Xa_{j_1, \ldots, j_n}$ is an integer.

Now let C_k $(k = 1, \ldots, m)$ have radius $(|j_k| + 1)/\sqrt{5} = r_k$, say. If $z_k \in C_k$ for each $k = 1, \ldots, m$, then

$$|f(z)| < c\alpha^{r_1 + \cdots + r_m},$$

where c is independent of $r_1, \ldots, r_m$, and

$$|p_{j_k}(z_k)| > r_k(r_k^2 - 1)(r_k^2 - 4)\ldots(r_k^2 - (|j_k| - 1)^2) \quad (|j_k| \geqslant 2),$$

so that

$$|p_{j_k}(z_k)| > \Gamma(r + |j_k|)/\Gamma(r - |j_k| + 1)$$

for all values of j_k. It now follows from (3) that

$$X|a_{j_1, \ldots, j_n}| \leqslant c\alpha^{r_1 + \cdots + r_m} \prod_{k=1}^{m} r_k |2j_k|! \, \Gamma(r_k - |j_k| + 1)/\Gamma(r_k + |j_k|).$$

By using Stirling's formula and performing some elementary simplifications, we find that

$$X|a_{j_1, \ldots, j_n}| = O((r_1 + \ldots + r_m)^{5m/2} (\alpha/\sigma)^{-(r_1 + \cdots + r_m)}) \quad (r_1 + \ldots + r_m \to \infty),$$

so that $Xa_{j_1, \ldots, j_m} \to 0$ as $|j_1| + \ldots + |j_m| \to \infty$. It now follows, since $Xa_{j_1, \ldots, j_m}$ is an integer, that when $|j_1| + \ldots + |j_m|$ is sufficiently large and $j_k > -J_k$ $(k = 1, \ldots, m)$ we have $a_{j_1, \ldots, j_m} = 0$.

Now let ν be a positive integer, let $J_1 = \nu$ and let

$$J_k(j_1, \ldots, j_{k-1}) = \nu - |j_1| - \ldots - |j_{k-1}| \quad (2 \leqslant k \leqslant m)$$

for each $(k-1)$-tuple $(j_1, \ldots, j_{k-1})$ with $|j_l| \leqslant J_l$ $(l = 1, \ldots, k-1)$. Also, let $R_\nu(z)$ denote the sum of the products

$$a_{j_1, \ldots, j_m}(z) \prod_{k=1}^{m} \{(z_k - j_k)^{-1} p_{j_k}(z_k)\}$$

over all m-tuples $(j_1, \ldots, j_m)$ for which $j_k = -J_k$ for some k. Plainly there are at most $(2\nu + 1)^m$ such m-tuples. Observe that for such m-tuples we have $|j_1| + \ldots + |j_m| = \nu$. Proceeding as before, we find that, for given z and given $K > 0$,

$$X\nu^K a_{j_1, \ldots, j_n}(z) \to 0 \quad (\nu \to \infty). \tag{4}$$

Further,

$$\begin{aligned} |(z_k - j_k)^{-1} p_{j_k}(z_k)| &\leqslant \{(|z_k| + 1)(|z_k| + 2)\ldots(|z_k| + |j_k|)\}^2 \\ &\leqslant (|j_k|!)^2 (1 + |j_k|)^{2|z_k|}, \end{aligned}$$

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press

so that $$|R_\nu(z)| \leqslant (2\nu+1)^m A_\nu(z) \prod_{k=1}^{m} (|j_k|!)^2 (1+|j_k|)^{2|z_k|},$$

where $A_\nu(z)$ denotes the maximum value of $|a_{j_1,\ldots,j_m}(z)|$ taken over all m-tuples $(j_1, \ldots, j_m)$ for which $j_k = -J_k$ for some k. Hence, by (4), $R_\nu(z) \to 0$ $(\nu \to \infty)$.

It follows from (2) and the results of the two preceding paragraphs that f is a polynomial.

3. *Proofs of Theorems* 1 *and* 2. With each n-tuple $a = (a_1, \ldots, a_n)$ of non-negative integers we associate the integers $|a| = a_1 + \ldots + a_n$ and $a! = a_1! \ldots a_n!$ and the differential operator

$$D^a = \partial^{|a|} / \partial x_1^{a_1} \ldots \partial x_n^{a_n}.$$

Also, given $z \in C^n$, we write $z^a = z_1^{a_1} \ldots z_n^{a_n}$.

Lemma 1 of (4) may be stated as follows.

LEMMA A. *If* $h \in \mathscr{H}_n(\alpha)$, *then*

$$D^a h(O) = O(|a|^{n-\frac{3}{2}} (\log \alpha)^{|a|}) \quad (|a| \to \infty).$$

From this we shall deduce the following lemma which is the key result for deriving Theorems 1 and 2 from Theorems B and 7.

LEMMA 1. *If* $h \in \mathscr{H}_n(\alpha)$ *and* $\beta > \alpha$, *then there is a function* $f \in \mathscr{E}_n(\beta)$ *such that* $h = f$ *on* R^n.

To prove this lemma, note that h is equal everywhere in R^n to the sum of its multiple Taylor series about O (see, for example, Hayman (11)), so that

$$h(x) = \sum_{a \in N^n} (a!)^{-1} D^a h(O)\, x^a \quad (x \in R^n). \tag{5}$$

We define $f(z)$ for each $z \in C^n$ to be the sum of the series (5) with z replacing x. Obviously $f = h$ on R^n and since, by Lemma A, $D^a h(O) = o((\log \beta)^{|a|})$ as $|a| \to \infty$, we find that $|f|$ is dominated in C^n by a constant multiple of

$$\sum_{a \in N^n} (\log \beta)^{|a|} |z_1|^{a_1} \ldots |z_n|^{a_n} / a!,$$

whence it follows that $f \in \mathscr{E}_n(\beta)$.

To prove Theorem 1, let f be the function of Lemma 1 and define g in C^m by writing

$$g(z_1, \ldots, z_m) = f(z_1, \ldots, z_m, 0, \ldots, 0).$$

Then plainly $g \in \mathscr{E}_m(\beta)$ and $g = h$ on R^m. By Theorem B, g is a polynomial and the positive part of Theorem 1 follows. The proof of the positive part of Theorem 2 is the same, except that we appeal to Theorem 7 instead of Theorem B.

To show that Theorems 1 and 2 fail in the cases $\alpha = 2$ and $\alpha = \sigma$ respectively, consider the functions u_1 and u_2 defined in R^n by the equations

$$u_1(x) = 2^{x_1} \cos(x_n \log 2)$$

and

$$u_2(x) = 2 \cosh(x_1 \log \sigma) \cos(x_n \log \sigma). \tag{6}$$

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press

These are the real parts of the entire functions $2^{x_1+ix_n}$ and $2\cosh\{\sigma(x_1+ix_n)\}$ respectively and are therefore harmonic in R^n. Clearly $u_1 \in \mathscr{H}_n(2)$ and $u_2 \in \mathscr{H}_n(\sigma)$. It is also clear that $u_1(N^{n-1}) \subset Z$; to show that $u_2(Z^{n-1}) \subset Z$, note that if $x \in R^{n-1}$, then

$$u_2(x) = \sigma^{x_1} + (\sigma - \sqrt{5})^{x_1} = \phi(x_1),$$

say, and use the relation $\phi(x_1+2) = 3\phi(x_1+1) - \phi(x_1)$.

4. *Proofs of Theorems* 3 *and* 4. These proofs depend on the case $m = n-1$ of Theorems 1 and 2 and the following lemma.

LEMMA B. *If* $h \in \mathscr{H}_n(\alpha)$ *and* $h = 0$ *on* $R^{n-1} + \{O, y\}$, *where* y *is a point of* R^n *such that* $0 < y_n < \pi/\log\alpha$, *then* $h \equiv 0$.

This is an easy corollary of (3), theorem 2, which includes the result that if $h \in \mathscr{H}_n(\beta)$, where $\beta < e^\pi$ and $h = 0$ on $R^{n-1} + \{O, y^*\}$, where $y^* = (0, \ldots, 0, 1)$, then $h \equiv 0$. To prove Lemma B, it is enough to apply the latter result to the function $x \rightsquigarrow h(y_n x)$.

If the hypotheses of Theorem 3 are satisfied, then, by Theorem 1, there is a polynomial P_1 in R^n such that $P_1 = h$ on R^{n-1}. Similarly, since, by the maximum principle, the class $\mathscr{H}_n(\alpha)$ is invariant under a translation of the axes, we find that there is a polynomial P_2 in R^n such that $P_2 = h$ on $R^{n-1} + \{y\}$. In the same way, using Theorem 2 instead of Theorem 1, we can show that if the hypotheses of Theorem 4 are satisfied, then there are polynomials P_1 and P_2 with the properties stated above. Consider now the polynomial P, defined in R^n by the equation

$$P(x) = (1 - x_n/y_n)\,P_1(x) + (x_n/y_n)\,P_2(x).$$

This polynomial agrees with P_1 on R^{n-1} and with P_2 on $R^{n-1} + \{y\}$. Now there is a harmonic polynomial H in R^n such that $H = P$ on $R^{n-1} + \{O, y\}$ (see, for example, (1), theorem 4). Hence we have $H = h$ on $R^{n-1} + \{O, y\}$, and the proofs of the positive parts of Theorems 3 and 4 are completed by applying Lemma B to $H - h$ to obtain $h = H$.

To show that these theorems fail with $y_n = \pi/\log\alpha$, consider the function u_3, defined in R^n by the equation

$$u_3(x) = \alpha^{x_1} \sin(x_n \log\alpha). \tag{7}$$

This function is harmonic in R^n, since it is the imaginary part of the entire function $\alpha^{x_1+ix_n}$. Further, it is clear that $u_3 \in \mathscr{H}_n(\alpha)$ and that $u_3 = 0$ on $R^{n-1} + \{O, y'\}$, where $y' = (0, \ldots, 0, \pi/\log\alpha)$.

To show that Theorems 3 and 4 fail, for any y, with $\alpha = 2$ and $\alpha = \sigma$ respectively, consider the functions u_4 and u_5, defined in R^n by the equations

$$u_4(x) = \begin{cases} 2^{x_1} \sin(x_n \log 2)/\sin(y_n \log 2) & \text{if} \quad \sin(y_n \log 2) \neq 0, \\ 2^{x_1} \sin(x_n \log 2) & \text{if} \quad \sin(y_n \log 2) = 0 \end{cases}$$

and

$$u_5(x) = \begin{cases} 2\cosh(x_1 \log\sigma) \sin(x_n \log\sigma)/\sin(y_n \log\sigma) & \text{if} \quad \sin(y_n \log\sigma) \neq 0, \\ 2\cosh(x_1 \log\sigma) \sin(x_n \log\sigma) & \text{if} \quad \sin(y_n \log\sigma) = 0. \end{cases}$$

It is easy to see that $u_4 \in \mathscr{H}_n(2)$ and that $u_5 \in \mathscr{H}_n(\sigma)$. Clearly $u_4(N^{n-1} + \{O, y\}) \subset Z$. To see that $u_5(Z^{n-1} + \{O, y\}) \subset Z$, compare u_5 with the function u_2 defined by (6).

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press

5. *Proofs of Theorems* 5 *and* 6. We shall need the following lemmas.

LEMMA 2. *If* $h \in \mathscr{H}_n(\alpha)$, *then* $\partial h/\partial x_n \in \mathscr{H}_n(\alpha)$.

It is well known (see, for example, Brelot(6), Appendix) that for each $y \in R^n$

$$\left|\frac{\partial h}{\partial x_n}(y)\right| \leqslant K_n \sup_{|x-y|=1} |h(x)|,$$

where K_n depends only on n, so that, by the maximum principle

$$\left|\frac{\partial h}{\partial x_n}(y)\right| \leqslant K_n \mathscr{M}(h, |y|+1).$$

The lemma follows.

LEMMA 3. *Let* Q_1 *and* Q_2 *be polynomials in* R^n *and let* $y \in R^n$. *Then there exists a harmonic polynomial* H *in* R^n *such that* $H = Q_1$ *on* R^{n-1} *and* $\partial H/\partial x_n = Q_2$ *on* $R^{n-1}+\{y\}$.

This is well known in the case where $y_n = 0$ (see, for example, (2), theorems 3 and 5).

In dealing with the other cases, it is enough to show that there are harmonic polynomials H_1 and H_2 in R^n such that $H_1 = Q_1$ on R^{n-1}, $\partial H_1/\partial x_n = 0$ on $R^{n-1}+\{y\}$, $H_2 = 0$ on R^{m-1} and $\partial H_2/\partial x_n = Q_2$ on $R^{n-1}+\{y\}$.

To show the existence of H_1, let Q_3 be the polynomial defined in R^n by

$$Q_3(x) = Q_1(x_1, \ldots, x_{n-1}, 0, \ldots, 0).$$

There exists a harmonic polynomial H_3 in R^n such that $H_3 = P_3$ on $R^{n-1}+\{O, 2y\}$ (see, for example, (1), theorem 4). It is easy to verify that the function H_1, defined in R^n by the equation

$$H_1(x_1, \ldots, x_n) = \tfrac{1}{2}\{H_3(x_1, \ldots, x_n) + H_3(x_1, \ldots, x_{n-1}, 2y_n - x_n)\}$$

has the required properties.

A similar argument shows that there is a harmonic polynomial H_4 in R^n such that $H_4(x) = (x_n/y_n)Q_2(x_1, \ldots, x_{n-1}, y_n)$ whenever $x \in R^{n-1}+\{-y, y\}$. Now there exists a harmonic polynomial H_5 in R^n such that $\partial H_5/\partial x_n = H_4$ (see Kuran(12), lemma 3). It is easy to see that the function H_2, defined in R^n by the equation

$$H_2(x_1, \ldots, x_n) = \tfrac{1}{2}\{H_5(x_1, \ldots, x_n) - H_5(x_1, \ldots, x_{n-1}, -x_n)\},$$

has the required properties.

LEMMA 4. *If* $h \in \mathscr{H}_n(\alpha)$, $h = 0$ *on* R^{n-1} *and* $\partial h/\partial x_n = 0$ *on* $R^{n-1}+\{y\}$, *where*

$$0 \leqslant y_n < \pi/(2\log\alpha),$$

then $h \equiv 0$.

Clearly it is enough to prove that $\partial h/\partial x_n \equiv 0$.

In the case $y_n = 0$, the hypothesis $h \in \mathscr{H}_n(\alpha)$ is redundant, and we deal with this case first. Since $h = 0$ on R^{n-1}, the reflexion principle and the uniqueness of harmonic continuations imply that h is an odd function of x_n and hence that $\partial h/\partial x_n$ is an even function of x_n. Similarly, since $\partial h/\partial x_n = 0$ on R^{n-1}, $\partial h/\partial x_n$ is an odd function of x_n. Hence $\partial h/\partial x_n \equiv 0$.

Now suppose that $y_n \neq 0$. Since $h = 0$ on R^{n-1}, we have, as before, that $\partial h/\partial x_n$ is an

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press

even function of x_n. Hence $\partial h/\partial x_n = 0$ on $R^{n-1}+\{-y, y\}$. By using Lemma 2, the invariance of the class $\mathscr{H}_n(\alpha)$ under translation and Lemma B, we find that $\partial h/\partial x_n \equiv 0$.

Suppose now that h satisfies the hypotheses of Theorem 5. By Theorem 1, there exists a polynomial Q_1 in R^n such that $h = Q_1$ on R^{n-1}. By Theorem 1, Lemma 2 and the invariance of $\mathscr{H}_n(\alpha)$ under translations, there is a polynomial Q_2 in R^n such that $\partial h/\partial x_n = Q_2$ on $R^{n-1}+\{y\}$. Hence, by Lemma 3, there is a harmonic polynomial H in R^n such that $H = h$ on R^{n-1} and $\partial H/\partial x_n = \partial h/\partial x_n$ on $R^{n-1}+\{y\}$.

The same argument, with Theorem 2 replacing Theorem 1, establishes the existence of a harmonic polynomial H with the above properties in the case where h satisfies the hypotheses of Theorem 6.

The proofs of the positive parts of Theorems 5 and 6 are completed by applying Lemma 4 to $h-H$ to obtain $h = H$.

Consideration of the function u_3, defined by (7), shows that these theorems fail with $y_n = \pi/(2\log\alpha)$.

To show that they fail, for any y, with $\alpha = 2$ and $\alpha = \sigma$ respectively, consider the functions u_6 and u_7, defined in R^n by the equations

$$u_6(x) = \begin{cases} 2^{x_1}\sin(x_n\log 2)/\{(\log 2)\cos(y_n\log 2)\} & \text{if} \quad \cos(y_n\log 2) \neq 0 \\ 2^{x_1}\sin(x_n\log 2) & \text{if} \quad \cos(y_n\log 2) = 0 \end{cases}$$

$$u_7(x) = \begin{cases} 2\cosh(x_1\log\sigma)\sin(x_n\log\sigma)/\{(\log\sigma)\cos(y_n\log\sigma)\} & \text{if} \quad \cos(y_n\log\sigma) \neq 0 \\ 2\cosh(x_1\log\sigma)\sin(x_n\log\sigma) & \text{if} \quad \cos(y_n\log\sigma) = 0. \end{cases}$$

The verifications are left to the reader.

6. *The relation of Theorem A and Theorems* 1 *and* 2 *with* $n = 2$. In proving the case $n = 2$ of Theorems 1 and 2, we used Lemma A and the case $m = 1$ of Theorems B and 7 respectively. Since the case $m = 1$ of these theorems reduces to Theorem A, we see that Theorems 1 and 2 are consequences of Lemma A and Theorem A.

Conversely, as claimed in Section 1, Theorem A can be recovered from Theorems 1 and 2. To see this, suppose that f satisfies the hypotheses of Theorem A(i) and write $f = u+iv$. Then u and v satisfy the hypotheses of Theorem 1 and hence their restrictions to R are polynomials. It follows that the restrictions of $\partial u/\partial x_1$ and $\partial v/\partial x_1$ to R are polynomials and hence, by the Cauchy–Riemann equations, the restrictions of $\partial u/\partial x_2$ and $\partial v/\partial x_2$ to R are polynomials. Since u and $\partial u/\partial x_2$, restricted to R, are polynomials, it follows from Lemmas 3 and 4 that u is a polynomial. Similarly v is a polynomial, and hence so also is f. The deduction of Theorem A(ii) from Theorem 2 is similar.

REFERENCES

(1) ARMITAGE, D. H. A linear function from a space of polynomials onto a space of harmonic polynomials. *J. London Math. Soc.* (2) **5** (1972), 529–538.

(2) ARMITAGE, D. H. On harmonic polynomials. *Proc. London Math. Soc.* (3) **34** (1977), 173–192.

(3) ARMITAGE, D. H. Uniqueness theorems for harmonic function which vanish at lattice points. *J. Approximaton Theory* (submitted).

(4) ARMITAGE, D. H. On the derivatives at the origin of entire harmonic functions. *Proc. Glasgow Math. Assoc.* (in the Press).

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press

(5) BAKER, A. A note on integral integer-valued function of several variables. *Proc. Cambridge Philos. Soc.* **63** (1967), 715–720.
(6) BRELOT, M. *Éléments de la théorie classique du potentiel* (3rd ed. Paris, 1965).
(7) BUCK, R. C. Integral valued entire functions. *Duke Math. J.* **15** (1948), 879–891.
(8) GELFOND, A. O. Sur les propriétés arithmétiques des fonctions entières. *Tôhoku Math. J.* **30** (1929), 280–285.
(9) GRAMAIN, F. Sur les fonctions entières de plusieurs variables complexes prenant des valeurs algébriques aux points entiers. *C. R. Acad. Sci. Paris* **284** (1977), 17–19.
(10) HARDY, G. H. On a theorem of Mr G. Pólya. *Proc. Cambridge Philos. Soc.* **19** (1920), 60–63.
(11) HAYMAN, W. K. Power series expansions for harmonic functions. *Bull. London Math. Soc.* **2** (1970), 152–158.
(12) KURAN, Ü. On Brelot–Choquet axial polynomials. *J. London Math. Soc.* (2) **4** (1971), 15–26.
(13) LANDAU, E. On Mr. Hardy's extension of a theorem of Mr. Pólya. *Proc. Cambridge Philos. Soc.* **20** (1920), 14–15.
(14) LANG, S. Algebraic values of meromorphic functions. *Topology* **3** (1965), 183–191.
(15) PÓLYA, G. Über ganzwertige ganze Funktionen. *Rend. Circ. Mat. Palermo* **40** (1915), 1–16.
(16) PÓLYA, G. Über ganze ganzwertige Funktionen. *Göttingen Nachrichten* (1920), 1–10.
(17) SELBERG, A. Über einen Satz von A. Gelfond. *Arch. Math. Naturvid.* **44** (1941), 159–170.
(18) STRAUS, E. G. On entire functions with algebraic derivatives at certain algebraic points. *Ann. of Math.* (2) **52** (1950), 188–198.
(19) WHITTAKER, J. M. *Interpolatory function theory* (Cambridge, 1935).

https://doi.org/10.1017/S0305004100055511 Published online by Cambridge University Press