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Attractive interaction between ions inside a quantum plasma structure

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We construct the model of a quantum spherically symmetric plasma structure based on radial oscillations of ions. We suppose that ions are involved in ion-acoustic plasma oscillations. We find the exact solution of the Schrödinger equation for an ion moving in the self-consistent oscillatory potential of an ion-acoustic wave. The system of ions is secondly quantized and its ground state is constructed. Then, we consider the interaction between ions by the exchange of an acoustic wave. It is shown that this interaction can be attractive. We describe the formation of pairs of ions inside a plasma structure and demonstrate that such a plasmoid can exist in a dense astrophysical medium corresponding to the outer core of a neutron star (NS).

1. Introduction

The Coulomb interaction is undoubtedly dominant for charged particles in plasma. For instance, various oscillatory processes, like Langmuir waves are driven by the Coulomb interaction. However, in the last decades there is a growing interest for various effective interactions which can arise between charged particles in plasmas.

First, we mention that Nambu et al. (1995) studied the wakefield interaction between particles of the same polarity moving in plasma. Under certain conditions, the wakefield interaction can be attractive. Morfill and Ivlev (2009) found that this kind of interaction can result in the formation of complex structures in dusty plasmas. Recently, Carstensen et al. (2012) observed the attractive wakefield interaction in the ions system in a laboratory experiment.

Besides the wakefield interaction which arises mainly in classical plasmas, Bonitz et al. (2008) showed that an effective interaction between charged particles in dense strongly correlated plasmas is responsible for the formation of Coulomb crystals. Note that Coulomb crystals can exist in both classical and quantum plasmas. Grimes and Adams (1979) observed Coulomb crystals in liquid helium, Birkl et al. (1992) in a system of trapped laser cooled ions, and Chu and I (1994) in dusty plasmas. Baiko (2009) suggested that magnetized Coulomb crystals play an important role in the evolution of NSs.

Another type of the effective interaction between charged particles in plasma is based on the exchange of a virtual acoustic wave. Vlasov and Yakovlev (1978)

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proposed that this effect can explain the stability of atmospheric plasmoids. Recently, Dvornikov (2013) studied analogous effective interaction to describe the pairing of ions performing spherically symmetric quantum oscillations. The weakness of the description of quantum plasmoids undertaken by Dvornikov (2013) consists in the assumption that ions participate in Langmuir oscillations which is valid only in the short waves limit.

In the present work, we continue the study of spatially localized quantum plasma structures based on radial oscillations of charged particles. To describe the motion of charged particles, we do not use quantum hydrodynamics since in that approach quantum effects are accounted for perturbatively. In Sec. 2, we find the exact solution of the Schrödinger equation describing the motion of an ion in the self-consistent potential of an ion-acoustic wave. Using the second quantization method, we construct the ground state of a plasmoid corresponding to the oscillatory motion of ions. In Sec. 3, we consider the effective interaction between oscillating ions by the exchange of an acoustic wave. In particular, we show that this interaction can be attractive. In Sec. 4, we discuss the pairing of ions due to this attractive interaction and demonstrate that a plasma structure, where bound states of ions are formed, can well exist in a dense matter of the outer core of a NS.

Finally, in Sec. 5, our results are briefly summarized. The mathematical details of the analysis of ions wave functions in the coordinate representation are given in Appendix A.

2. Quantum states of ions in a spherical plasmoid

In this section, we develop the model of a quantum plasmoid based on radial oscillations of ions in plasma. We find the exact solution of the Schrödinger equation describing a spherically symmetric motion of ions participating in ion-acoustic oscillations. Then this system is secondly quantized and its ground state is constructed. We formulate the criterion of the stability of quantum oscillations and consider the description of the plasmoid in the short waves limit.

One of the most popular approaches to account for quantum effects in plasmas is based on the quantum hydrodynamics, recently reviewed by Haas (2011). In this method, the classical hydrodynamic equations are used to study the plasma dynamics. Quantum effects are accounted for by adding the quantum pressure term. Thus, by construction, within the quantum hydrodynamics, plasmas are mainly classical and quantum effects are treated perturbatively. It is a reasonable approximation for a high temperature plasma.

On the contrary, if one studies a strongly correlated system of charged particles, e.g. a degenerate plasma (as an example, see Sec. 3 below), the dynamics of particles in such a medium should obey mainly quantum mechanical laws. Bonitz et al. (2013) showed that the methods of the quantum hydrodynamics are not applicable in this situation.

In our work, we consider the model of a quantum plasma structure based on spherically symmetric oscillations of charged particles. As a rule, plasma oscillations are dominated by the motion of an electron component in plasma. It happens since an electron is a very light particle compared to an ion. Despite the amplitude and the frequency of electrons oscillations are much higher than that of ions, in realistic situation we cannot neglect the ions motion. Typically ions have lower temperature. Thus, quantum effects can be more pronounced for them. Note that, in Sec. 4, we shall study the effective interaction of ions in a very dense matter, with the temperature

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being much less than the chemical potential. It justifies the application of the quantum approach for the description of the ions motion.

Let us suppose that ions in plasma oscillate with the frequency ω. If we consider an ion as a test particle, the stationary Schrödinger equation for the ion wave function ψ has the form,

$$E\psi = \hat{H}\psi, \quad \hat{H} = \frac{\hat{\mathbf{p}}^2}{2m_i} + \frac{m_i\omega^2\hat{\mathbf{r}}^2}{2}, \tag{1}$$

where $\hat{\mathbf{p}}$ and $\hat{\mathbf{r}}$ are the operators of the momentum and the coordinate, E is the ion energy, and m_i is its mass.

Dvornikov (2013) studied the quantization of the motion of charged particles participating in Langmuir oscillations, which is a rough model in case we deal with ions. Let us suppose that ions in plasma are involved in collective oscillations corresponding to ion-acoustic waves. Thus, the frequency ω in (1) obeys the dispersion relation (Lifshitz and Pitaevskiĭ 2010),

$$\omega^2 = \omega_i^2\frac{\lambda_e^2k^2}{1+\lambda_e^2k^2}, \tag{2}$$

where ω_i is the Langmuir frequency for ions, λ_e is the Debye length for electrons, and $k = |\mathbf{k}|$ is the absolute value of the wave vector. If we consider a classical electron-ion plasma, then $\omega_i = \sqrt{4\pi e^2 n_i^{(0)}/m_i}$ and $\lambda_e = \sqrt{T_e/4\pi e^2 n_e^{(0)}}$, where T_e is the electron temperature and $n_{i,e}^{(0)}$ are the unperturbed densities of ions and electrons.

In Sec. 4, we shall study quantum plamoids in a degenerate plasma consisting of ultrarelativistic electrons and non-relativistic ions (protons). Using the results of Braaten and Segel (1993), one gets that the dispersion relation for ion-acoustic waves in this case has the same form as in (2). However, $\lambda_e = \frac{c}{3\omega_e}$, where c is the speed of light. The Langmuir frequency for electrons ω_e as well as ω_i have the form (Braaten and Segel 1993),

$$\omega_e^2 = \frac{4\alpha_{\mathrm{em}}}{3\pi}[3\pi^2 n_e^{(0)}c^3]^{2/3}, \quad \omega_i^2 = \frac{4\alpha_{\mathrm{em}}}{3\pi}\frac{3\pi^2 n_i^{(0)}}{m_i}\hbar c, \tag{3}$$

where $\alpha_{\mathrm{em}} \approx 7.3\times10^{-3}$ is the fine structure constant.

We shall choose the momentum representation in (1), i.e. $\hat{\mathbf{p}} = \mathbf{p} = \hbar\mathbf{k}$ and $\hat{\mathbf{r}} = \mathrm{i}\hbar\nabla_{\mathbf{p}}$, where $\nabla_{\mathbf{p}} = \frac{\partial}{\partial\mathbf{p}}$. Using (2), we can rewrite (1) as

$$\left(\frac{p^2}{2m_i} - \frac{m_i\omega_i^2}{2}\frac{\lambda_e^2p^2}{\hbar^2+\lambda_e^2p^2}\hbar^2\nabla_{\mathbf{p}}^2\right)\psi = E\psi, \tag{4}$$

where $\psi = \psi(\mathbf{p})$. The coefficients in (4) depend on the absolute value of the momentum $p = |\mathbf{p}|$. Thus, we can look for the solution of (4) as $\psi(\mathbf{p}) = R(p)Y_{l\mathrm{m}}(\theta,\phi)$, where R is the radial part of the wave function, $Y_{l\mathrm{m}}(\theta,\phi)$ is the spherical harmonic corresponding to the orbital and magnetic quantum numbers: $l = 0, 1, 2, \ldots$ and $\mathrm{m} = 0, \pm1, \ldots, \pm l$, θ and ϕ are the spherical angles fixing the momentum direction. Note that (4) is analogous to the equations studied by Alhaidari (2002); Schmidt (2007).

Let us introduce the new variable $\rho = p^2/m_i\omega_i\hbar$ and the new unknown function u, as $R = e^{-\rho/2}\rho^{l'/2}u(\rho)$, where

$$l' = \frac{1}{2}\left\{\left[(2l+1)^2 - \frac{8E}{m_i\omega_i^2\lambda_e^2}\right]^{1/2} - 1\right\}, \tag{5}$$

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is the effective orbital quantum number. Note that $-1/2 \leqslant l' \leqslant l$. In these new variables (4) takes the form,

$$\rho \frac{\mathrm{d}^2 u}{\mathrm{d}\rho^2} + \left(\frac{3}{2} + l' - \rho \right) \frac{\mathrm{d}u}{\mathrm{d}\rho} + \left(\frac{E'}{2\hbar\omega_0} - \frac{3}{4} - \frac{l'}{2} \right) u = 0, \tag{6}$$

where $E' = E - \frac{\hbar^2}{2m_i\lambda_e^2}$ is the effective energy.

The solution of (6) can be expressed using (9.216) on page 1024 in Gradshteyn and Ryzhik (2007),

$$u = {}_1F_1 \left(-\frac{E'}{2\hbar\omega_0} + \frac{l'}{2} + \frac{3}{4}; l' + \frac{3}{2}; \rho \right), \tag{7}$$

where ${}_1F_1(a;b;z)$ is the Kummer's confluent hypergeometric function. The hypergeometric function in (7) should be finite at $\rho \to \infty$. Thus, the energy of an ion should satisfy the relation,

$$E_{nl} = \hbar\omega_i \left\{ 2n + 1 - \frac{\hbar}{2m_i\omega_i\lambda_e^2} + \left[\left(l + \frac{1}{2} \right)^2 - 4 \left(n + \frac{1}{2} \right) \frac{\hbar}{m_i\omega_i\lambda_e^2} \right]^{1/2} \right\}, \tag{8}$$

where $n = 0, 1, 2, \ldots$ is the radial quantum number. It should be noted that, at big n, ions oscillations become unstable since the energy in (8) acquires the imaginary part. Thus, we should impose a restriction,

$$n < n_{\mathrm{cr}} = \left(l + \frac{1}{2} \right)^2 \frac{m_i\omega_i\lambda_e^2}{4\hbar} - \frac{1}{2}, \tag{9}$$

to guarantee the stability of oscillations.

In the following, we shall study spherically symmetric plasma oscillations. Thus, we should consider wave functions independent of θ and ϕ. This case corresponds to $l = 0$. Using (7), one finds the properly normalized total wave function, which also includes the spin variables, in the following form:

$$\begin{aligned} \psi_{n\sigma}(p) = & 2\pi \left[\frac{n!}{\Gamma(l' + n + 3/2)} \right]^{1/2} \left(\frac{\hbar}{m_i\omega_i} \right)^{3/4} \left(\frac{p^2}{m_i\omega_i\hbar} \right)^{l'/2} \\ & \times \exp\left[-\frac{p^2}{2m_i\omega_i\hbar} \right] L_n^{l'+1/2} \left(\frac{p^2}{m_i\omega_i\hbar} \right) \chi_\sigma, \end{aligned} \tag{10}$$

where $L_n^\alpha(z)$ is the associated Laguerre polynomial, $\Gamma(z)$ is the Euler gamma function, χ_σ is the spin wave function, and σ is the spin variable.

Note that both (1) and (4) correspond to a spinless ion. The spin dependence was introduced as a direct product of the coordinate wave function $\psi_n(p)$ and the spin wave function χ_σ. Thus, we do not consider here effects related to the spin-orbit interaction like in the Dirac theory.

The new expression for l' has the form,

$$l' = \frac{1}{2} \left\{ \left[1 - \frac{8E}{m_i\omega_i^2\lambda_e^2} \right]^{1/2} - 1 \right\}. \tag{11}$$

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Note that now $-1/2 \leqslant l' \leqslant 0$. The energy levels corresponding to the states described by $\psi_{n\sigma}$ in (10) have the form,

$$E_n = \hbar\omega_i \left\{ 2n + 1 - \frac{\hbar}{2m_i\omega_i\lambda_e^2} + \left[\frac{1}{4} - 4\left(n + \frac{1}{2}\right) \frac{\hbar}{m_i\omega_i\lambda_e^2} \right]^{1/2} \right\}. \tag{12}$$

Note that (11) and (12) can be obtained directly from (5) and (8) and correspond to the case $l = 0$.

We shall assume that ions are singly ionized. Thus, they should be fermions. Indeed, a typical neutral atom has an integer spin. Thus, if we remove one electron, an ion becomes a fermion. For simplicity, we shall assume that ions have the lowest possible spin, i.e. they are 1/2-spin particles. In Sec. 4, we shall study a degenerate electron-proton plasma in the NS outer core. In this case, an ion (a proton) certainly has the spin equal to 1/2. It means that $\sigma = \pm 1$ in (10).

Now the ground state of the system can be constructed. Suppose that we have N ions performing ion-acoustic oscillations. On the basis of (10), we can introduce the operator valued wave function,

$$\hat{\psi} = \sum_{n\sigma} \psi_{n\sigma} \hat{a}_{n\sigma}, \tag{13}$$

and the analogous expression for $\hat{\psi}^\dagger$, which contains $\hat{a}^\dagger_{n\sigma}$. Here $\hat{a}^\dagger_{n\sigma}$ and $\hat{a}_{n\sigma}$ are the creation and annihilation operators of the states corresponding to ions oscillations. These operators obey the usual anticommutation relation, $[\hat{a}_{n\sigma}, \hat{a}^\dagger_{n'\sigma'}]_+ = \delta_{nn'}\delta_{\sigma\sigma'}$, with other anticommutators being equal to zero. In this situation, each energy state can be occupied by no more than two particles. Thus, we do not contradict the Pauli principle.

The constructed ground state corresponds to the collective motion of ions in a spherically symmetric plasma structure, since the occupied energy levels are associated with particles involved in ion-acoustic oscillations. It should be noted that in our model we consider ions as test particles oscillating with the same frequency ω corresponding to an ion-acoustic wave; cf. the effective potential in (1). This approximation is valid if we neglect temperature effects (Dawson 1959).

From the formal point of view, we can choose any complete set of basis wave functions to build a ground state. However, for the description of the motion of ions which are involved in spherically symmetric oscillations, the ground state constructed in our work is preferable since it accounts for the dynamical features of the system and its geometric symmetry. It is not the case for a ground state based on plane waves, $\sim \exp(\mathrm{i}\mathbf{kr})$, which are typically chosen to describe the evolution of test particles in quantum plasmas. In other words, the ground state constructed takes into account the electromagnetic interactions between charged particles in plasma in all orders in a perturbative expansion. Indeed, plasma oscillations in our system are driven by electromagnetic interactions and we use the basis wave functions corresponding to a 3D harmonic oscillator.

The energy of the ground state can be found as $E_0 = \langle \hat{H}_{\mathrm{ion}} \rangle$, where

$$\hat{H}_{\mathrm{ion}} = \sum_{n\sigma} E_n \hat{a}^\dagger_{n\sigma} \hat{a}_{n\sigma}, \tag{14}$$

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is the ground state Hamiltonian of ions and the values of the energy are given in (12). Note that, since we have a great but limited number of ions involved in oscillations,

$$E_0 \approx 2 \sum_{n<n_\mathrm{F}} E_n, \tag{15}$$

where n_F is the number of occupied energy states. We shall call it the *Fermi number*.

We can define the 'size' of a spherically symmetric plasmoid in the momentum space as the last maximum of the function $|\psi_{n_\mathrm{F}}(p)|^2$. Blokhintsev (1964) showed that this maximum is approximately achieved at the classical turn point, $p_{\max}^2 = 2m_i E_n$. Supposing that $n_\mathrm{F} \sim n_\mathrm{cr}$ and using (9), we get that $p_{\max} \lesssim m_i \omega_i \lambda_e$. If we study a non-degenerate electron-ion plasma and employ the quasiclassical limit, we can take that $p_{\max} \sim m_i v_i$, where v_i is the typical ion velocity. Thus, we get that the constraint in (9) is equivalent to $L_i \sim v_i/\omega_i \lesssim \lambda_e$, where L_i is the typical particle displacement in ion oscillations. It is the well-known condition for the existence of ion-acoustic waves in a non-degenerate plasma (Lifshitz and Pitaevskiĭ 2010).

Note that for the sufficiently short waves, the frequency of ion oscillations is constant $\omega = \omega_i$; cf. (2). In the quantum description, this limit is equivalent to $\lambda_e \gg \sqrt{\hbar/m_i\omega_i}$. In this case, the energy spectrum in (12) coincides with that corresponding to Langmuir oscillations of ions obtained by Dvornikov (2013). Besides the coincidence of the spectra for big λ_e, the wave function in (10) is also consistent with the result of Dvornikov (2013). Indeed, taking into account that $l' \to 0$ at $\lambda_e \gg \sqrt{\hbar/m_i\omega_i}$, we get the following expression for the Fourier transform of the wave function in (10):

$$\psi_n(\mathbf{r}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3} e^{\mathrm{i}\frac{(\mathbf{pr})}{\hbar}} \psi_n(p) = \frac{1}{\sqrt{4\pi(2n+1)!}} \frac{1}{2^n} \left(\frac{m_i^3\omega_i^3}{\pi\hbar^3}\right)^{1/4} \times \frac{r_0}{r} \exp\left(-\frac{r^2}{2r_0^2}\right) H_{2n+1}\left(\frac{r}{r_0}\right), \tag{16}$$

where $H_n(z)$ is the Hermite polynomial and $r_0 = \sqrt{\hbar/m_i\omega_i}$. One can notice that in (16), we reproduce the result of Dvornikov (2013) up to the sign factor. The details of the derivation of (16) are provided in Appendix A; cf. (A 1). Note that in (16), we consider only the coordinate part of the wave function omitting χ_σ.

3. The effective interaction of ions

In this section, we study the effective interaction of ions in a spherically symmetric plasmoid owing to the exchange of virtual acoustic waves. It is demonstrated that this interaction can be attractive.

We consider the situation when the plasma temperature is not so high. It means that a neutral component can be present. In Sec. 4, we will study plasma structures in the NS outer core, where a neutral component, consisting of neutrons, is always present. In this case, rapidly oscillating ions will collide with neutral particles and generate acoustic waves. If an acoustic wave is coherently absorbed inside the system, it will result in the effective interaction between charged particles. One can expect that this effective interaction is more efficient for ions rather than for electrons.

We shall study oscillating ions in highly excited states with $n \gg 1$. Using (A 16) and (A 22), we get that the asymptotic form of the wave functions of such ions in the coordinate representation coincides with that in (16). Note that the analogous form of the wave function for a charged particle performing radial oscillations in plasma was derived by Dvornikov and Dvornikov (2006). One can prove it using (8.955.2) on

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page 997 in Gradshteyn and Ryzhik (2007). Therefore, for $n \gg 1$, the effect of the spatial dispersion of ion-acoustic waves does not significantly contribute to the form of the ions wave functions.

If $\delta n(\mathbf{r})$ is the perturbation of the neutral particles density because of the acoustic wave propagation, then the energy of the interaction of an ion with the field of acoustic waves is

$$U_{\text{int}}(\mathbf{r}) = \int \mathrm{d}^3\mathbf{r}' K(\mathbf{r}-\mathbf{r}')\delta n(\mathbf{r}'), \tag{17}$$

where $K(\mathbf{r}-\mathbf{r}')$ is the potential of the interaction between an ion and a neutral particle which are at the points $\mathbf{r}$ and $\mathbf{r}'$ respectively. Using (17) and the results of Vlasov and Yakovlev (1978), we derive the secondly quantized Hamiltonian describing the interaction between ions and neutral particles as,

$$\hat{H}_{\text{int}} = \int \mathrm{d}^3\mathbf{r}\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})U_{\text{int}}(\mathbf{r}) = \int \mathrm{d}^3\mathbf{r}\mathrm{d}^3\mathbf{r}' K(\mathbf{r}-\mathbf{r}')\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\delta\hat{n}(\mathbf{r}'), \tag{18}$$

where the expression for $\hat{\psi}$ is given in (13). Note that in (18), we replaced $\delta n(\mathbf{r})$ with its secondly quantized analog $\delta\hat{n}(\mathbf{r})$; cf. (19) below.

In the following, we shall use the approximation of the contact interaction. It means that $K(\mathbf{r}-\mathbf{r}') = K_0\delta^3(\mathbf{r}-\mathbf{r}')$, where K_0 is the phenomenological constant characterizing the strength of the interaction. This approximation corresponds to a very short range interaction between ions and neutral particles. In Sec. 4, we shall study a particular case of the proton–neutron interaction driven by nuclear forces which are short range. Thus, the contact interaction approximation is justified.

The expression for $\delta\hat{n}(\mathbf{r})$ was obtained by Dvornikov (2013),

$$\delta\hat{n}(r) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \left(\frac{\hbar n_n^{(0)}}{2m_n\omega_k}\right)^{1/2} k f_k \left(\hat{b}_k + \hat{b}_k^\dagger\right), \tag{19}$$

where $f_k = \frac{\sin(kr)}{kr}$ is the spherically symmetric solution of the wave equation for acoustic waves, $\hat{b}_k^\dagger$ and $\hat{b}_k$ are the creation and annihilation operators for phonons, ω_k is the frequency of acoustic oscillations, $n_n^{(0)}$ is the unperturbed density of neutral particles, and m_n is the mass of a neutral particle.

Using (19), we can rewrite (18) in the form,

$$\hat{H}_{\text{int}} = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{ns\sigma} D_{ns}(k)\hat{a}_{n\sigma}^\dagger\hat{a}_{s\sigma}\left(\hat{b}_k + \hat{b}_k^\dagger\right), \tag{20}$$

where D_{ns} is the matrix element of this interaction. If we study the effective interaction between ions occupying the same energy level, we have for $D_n \equiv D_{nn}$,

$$\begin{aligned} D_n(k) =& K_0 k \left(\frac{n_n^{(0)} m_i\omega_i}{2m_n\omega_k}\right)^{1/2} \int \mathrm{d}^3\mathbf{r}\psi_n^2(r) f_k(r) \\ \approx& \frac{K_0}{4\sqrt{n}} \left(\frac{n_n^{(0)} m_i\omega_i}{2m_n\omega_k}\right)^{1/2} [1 - \operatorname{sgn}(\xi - 4\sqrt{n})], \end{aligned} \tag{21}$$

where $\xi = k\sqrt{\hbar/m\omega_i}$ and $\operatorname{sgn}(z)$ is the sign function.

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After the standard elimination of the acoustic degrees of freedom with help of the canonical transformation,

$$\hat{H}_{\rm int} \to \exp\left(-\hat{S}\right)\hat{H}_{\rm int}\exp\left(\hat{S}\right),$$
$$\hat{S} = \int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^{3/2}} \sum_{ns\sigma} D_{ns}(k)\hat{a}^\dagger_{n\sigma}\hat{a}_{s\sigma}\left(\frac{\hat{b}_k}{E_s - E_n - \hbar\omega_k} + \frac{\hat{b}^\dagger_k}{E_s - E_n + \hbar\omega_k}\right), \tag{22}$$

the total Hamiltonian, which includes $\hat{H}_{\rm ion}$ given in (14), takes the form,

$$\hat{H} = \sum_{n\sigma} E_n \hat{a}^\dagger_{n\sigma}\hat{a}_{n\sigma} - \sum_{nn'\sigma} F_{nn'}\hat{a}^\dagger_{n\sigma}\hat{a}^\dagger_{n',-\sigma}\hat{a}_{n',-\sigma}\hat{a}_{n\sigma}. \tag{23}$$

To derive (23), it is important to assume that two interacting ions are at the same energy level. In (23), we also account for the fact that these ions must have oppositely directed spins because of the Pauli principle.

The amplitude of the effective interaction $F_{nn'}$ in (23) has the form,

$$F_{nn'} = \frac{K_0^2 n_n^{(0)} m_i \omega_i}{32 m_n \hbar\sqrt{nn'}} \int \frac{{\rm d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{\omega_k^2}[1 - {\rm sgn}(\xi - 4\sqrt{n})][1 - {\rm sgn}(\xi - 4\sqrt{n'})]. \tag{24}$$

Now, we should take into account the fact that ions and neutral particles have practically the same mass. Thus, the energy transfer in their collisions occurs rather effectively. Therefore, we should take that the frequency of virtual acoustic waves in (24) is close to the frequency of ion-acoustic oscillations given in (2). After the integration, one has the following expression for $F_{nn'}$:

$$F_{nn'} = \frac{4K_0^2 n_n^{(0)} m_i}{3\pi^2\hbar\omega_i m_n}\left(\frac{m_i\omega_i}{\hbar}\right)^{3/2}\frac{\tilde{n}^{3/2}}{\sqrt{nn'}}\left(1 + \frac{3\hbar\omega_i}{16 m_i c_s^2 \tilde{n}}\right), \tag{25}$$

where $c_s = \lambda_e\omega_i$ is the sound velocity and $\tilde{n} = \min(n, n')$.

Note that $F_{nn'}$ in (25) is positive. It means that the effective interaction described by the Hamiltonian in (23) is attractive.

Now let us discuss the ions motion which corresponds to short waves. In this situation, ion-acoustic waves are transformed into Langmuir oscillations of ions. As we have seen in Sec. 2, it happens in the limit $\lambda_e \gg \sqrt{\hbar/m_i\omega_i}$, that is equivalent to $c_s \gg \sqrt{\hbar\omega_i/m_i}$. Thus, using (25), we get that the matrix element of the effective interaction for Langmuir oscillations takes the form,

$$F^{\rm (Lang)}_{nn'} = \frac{4K_0^2 n_n^{(0)} m_i}{3\pi^2\hbar\omega_i m_n}\left(\frac{m_i\omega_i}{\hbar}\right)^{3/2}\frac{\tilde{n}^{3/2}}{\sqrt{nn'}}. \tag{26}$$

Note that (26) corrects the result of Dvornikov (2013). The matrix element derived by Dvornikov (2013) was erroneous since the incorrect dispersion relation for virtual acoustic waves was used in the calculation of the integral in (24).

To complete the analysis of the effective interaction, we should define the constant K_0. We recall that we use the approximation of the contact interaction between ions and neutral particles, which implies that the potential of this interaction is $K(\mathbf{r}) = K_0\delta^3(\mathbf{r})$. Using the Born approximation (Cohen-Tannoudji et al. 1977) for the scattering of ions in this potential, we get the total cross section as $\sigma_s = K_0^2 m_i^2/\pi\hbar^4$. Therefore, we obtain that $K_0 = \hbar^2\sqrt{\pi\sigma_s}/m_i$. The value of σ_s can be extracted from experimental results in any particular case.

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4. Pairing of ions

In this section, we study the possibility for the formation of bound states of ions owing to the effective attraction described in Sec. 3. Using this effective interaction, we discuss the pairing of protons in a dense matter of NS.

As in Sec. 3, we shall consider two ions occupying the same energy level. On the basis of (23), one can see that these ions can form a bound state if the amplitude of the effective interaction exceeds the kinetic energy of an ion,

$$E_n < F_{nn}, \tag{27}$$

where E_n is given in (12) and F_{nn} in (25). Introducing the properly defined Bogoliubov transformation, one can show that, if the condition in (27) is satisfied, the ground state of the system is transformed into the new one corresponding to the lower energy (Dvornikov 2013).

Let us first examine the case of short ion-acoustic waves which is equivalent to Langmuir oscillations of ions. Using (26), one gets that the pairing of ions takes place, i.e. (27) is fulfilled, when

$$n < n_0 = \frac{4}{9\pi^2}\frac{\hbar m_i}{\omega_i}\left(\frac{n_n^{(0)}\sigma_s}{m_n}\right)^2 - \frac{3}{2}. \tag{28}$$

Thus, there is an upper limit on the number of occupied states.

In Sec. 2, we defined the Fermi number corresponding to the maximal possible number of occupied states. If one studies short ion-acoustic waves, we can neglect the spatial dispersion. Thus, we can define ions wave functions in the coordinate space rather than in the momentum space; cf. (16). Therefore, the Fermi number can be related to the effective radius of a plasmoid R_{eff} as $n_{\text{F}} = m_i\omega_i R_{\text{eff}}^2/4\hbar$ (Dvornikov 2013). If we assume that all ions inside a plasmoid formed bound states, i.e. $n_{\text{F}} < n_0$, we get that

$$R_{\text{eff}} < R_0 = \frac{4\hbar n_n^{(0)}\sigma_s}{3\pi m_n\omega_i}, \tag{29}$$

which is the upper bound on the plasmoid radius.

We can expect that the described phenomenon of the ions pairing can happen in dense matter. Let us consider a spherical plasma structure excited in the outer core of NS. It should be noted that such a background matter is mainly composed of neutrons and has $n_n^{(0)} = 10^{38}\,\text{cm}^{-3}$. Nevertheless, various equations of state of NS matter predict that a certain fraction of protons can be also present in the NS core (Haensel et al. 2007). We shall assume that the proton density $n_p^{(0)} = 10^{36}\,\text{cm}^{-3}$, i.e. it is about 1% of the neutron density.

Yakovlev et al. (1999) mentioned that neutrons and protons form strongly non-ideal Fermi liquid in the NS outer core. We can use the formalism of quasiparticles with the effective masses $m_{i,n}^*$ for the description of plasma oscillations. Onsi and Pearson (2002) found that $m_{i,n}^* \approx 0.8m_{i,n}$. Thus, using (3), we get that $\omega_i \approx 1.48 \times 10^{21}\,\text{s}^{-1}$ in the NS outer core.

We shall be interested in the pairing of protons. The Fermi momentum of charged particles in the NS core corresponding to $n_p^{(0)} = 10^{36}\,\text{cm}^{-3}$ is $\sim 60\,\text{MeV}c^{-1}$. The temperature of these particles does not exceed $10^{10}\,\text{K} \sim 1\,\text{MeV}$ (Haensel et al. 2007). Thus these particles are degenerate, with protons being non-relativistic whereas electrons being ultrarelativistic. It means that we can use our results for the description of quantum plasmoids in NS matter. Note that the application of the

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quantum theory is essential because of the degeneracy of the proton component of plasma.

The Fermi energy of protons in the NS outer core is (3–6) MeV (Haensel et al. 2007). Chadwick et al. (2006) obtained that in the MeV energy range the cross section of the proton–neutron scattering is approximately constant and equals to $\sigma_s = 2 \times 10^{-23}\,\mathrm{cm}^2$. It should be noted that the scattering off a delta-function potential also gives the energy independent cross section. Thus, the approximation of the contact interaction adopted in Sec. 3 is valid. Using (29) and accounting for the effective mass of a neutron, one obtains that $R_0 \approx 4.47 \times 10^{-10}\,\mathrm{cm}$. Note that there are $N = \frac{4}{3}\pi R_0^3 n_p^{(0)} \approx 3.77 \times 10^8$ protons inside such a plasmoid. Thus the approximation of $n \gg 1$, used in our work is valid.

Our estimate means that the protons with oppositely directed spins can form bound states. This phenomenon is similar to the formation of Cooper pairs in metals. This result is in agreement with the predictions of various theoretical models, recently confirmed by astronomical observations (Heinke and Ho 2010), that a superfluid and superconducting phases of nucleons can be present in the NS outer core. The conventional approach for the description of the proton superconductivity in the NS matter, reviewed, e.g. by Yakovlev et al. (1999), implies the formation of singlet states of protons (Cooper pairs) owing to the directs nuclear interactions between these particles. Yakovlev et al. (1999) mentioned that this superconducting phase can be important for the NS cooling. We suggest another mechanism for the protons pairing inside the NS based on the exchange of a virtual acoustic wave. Although Cooper pairing due to direct nuclear interactions is likely to be dominant, our mechanism is not ruled out by the present knowledge about the NS structure.

Now, we study the pairing of ions participating in longer ion-acoustic waves. In this situation, one should account for the effects of the spatial dispersion. In the quantum description, it is equivalent to the consideration of the exact energy spectrum in (12). For long ion-acoustic waves, we have that $\omega < \omega_i$. Thus, we can expect that the effective attraction will be stronger in this case. If we consider the case of slightly decreased frequency of ions oscillations, i.e. $\omega \lesssim \omega_i$, we can still assume that a plasmoid is localized in coordinate space and study the corrections to n_0 and R_0 defined above. We expect that the new values of n_0 and R_0 will increase if we account for the spatial dispersion of ion-acoustic waves.

If the ratio $\kappa = \hbar/m_i\omega_i\lambda_e^2$ is small but nonzero, the value of $n_0^{(\kappa)}$ is now determined by the nonlinear algebraic equation $F_{nn} = E_n$. After we find $n_0^{(\kappa)}$, we can calculate $R_0^{(\kappa)}$ taking that $n_\mathrm{F} = n_0^{(\kappa)}$. In Fig. 1, we show the parameters $\delta_n = (n_0^{(\kappa)} - n_0)/n_0$ and $\delta_R = (R_0^{(\kappa)} - R_0)/R_0$, where $n_0^{(\kappa)}$ and $R_0^{(\kappa)}$ are the critical occupation number and the plasmoid radius corresponding to nonzero κ, versus κ for $n_0 \sim 10^3$. As one can see, if we account for the spatial dispersion of ion-acoustic waves, the corrections to n_0 and R_0 (we recall that n_0 and R_0 are defined in (28) and (29) and correspond to Langmuir oscillations of ions) are small but positive.

We have found that, for $n \gg 1$, there is a very small enhancement of the plasmoid radius if we study long ion-acoustic waves. We obtained such a small effect because we used the ions wave functions in (16), which correspond to Langmuir oscillations, for the calculation of the matrix elements. Although, we have shown that these wave functions are the correct asymptotics for ψ_n at $n \gg 1$, if we consider great but limited values of n, we should use the exact $\psi_{n\sigma}(p)$ given in (10). In this situation, the effect of the nonzero spatial dispersion will be more significant. This fact also results from the

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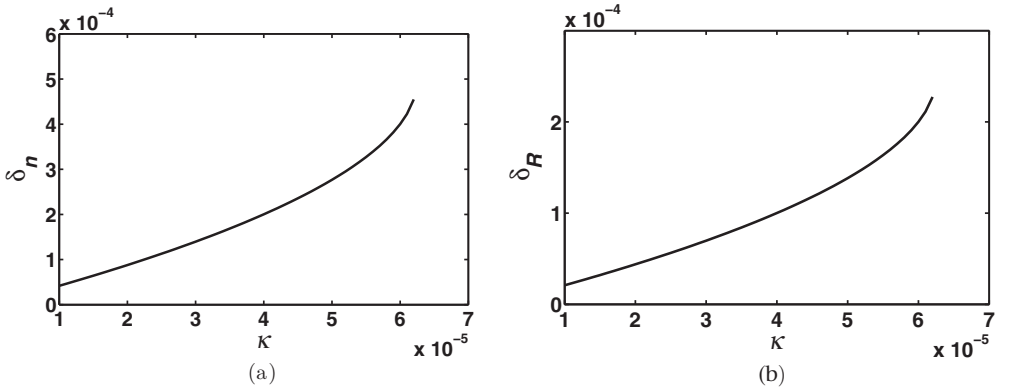


FIGURE 1. The dependence of the plasmoid parameters versus $\kappa = \hbar/m_i\omega_i\lambda_e^2$ for $n_0 \sim 10^3$. (a) The relative change of the critical occupation number $\delta_n = (n_0^{(\kappa)} - n_0)/n_0$. (b) The relative change of the critical plasmoid radius $\delta_R = (R_0^{(\kappa)} - R_0)/R_0$.

quantum mechanical uncertainty principle. Unfortunately, this case can be analyzed only numerically.

Let us estimate $n_{\rm cr}$ in (9) for a degenerate plasma in NS studied above. We shall consider the case when the plasmoid radius is comparable with λ_e. Of course, R_0 should be less than λ_e for the stochastic motion of charged particles not to destroy the plasmoid. Using the value of $R_0 \sim 10^{-10}$ cm found above we get that $n_{\rm cr} \sim 10^4$, which is much less than $N \sim 10^{10}$. This estimate shows that the plasma instability mentioned in Sec. 2 will decrease the effective plasmoid radius. Therefore, this process will counteract the enhancement of the radius due to the accounting for the dispersion relation of ion-acoustic waves.

5. Conclusion

In conclusion, we mention that in the present work we have constructed the model of a spherical quantum plasmoid based on radial oscillations of ions. In our analysis, we have suggested that the electron component of plasma is uniformly distributed in space and ions participate in ion-acoustic oscillations.

In Sec. 2, we have found the exact solution of the Schrödinger equation for an ion moving in the self-consistent field of an ion-acoustic wave. We have shown that in the limiting case of short waves, the obtained wave functions of ions transform into the previously found solution of the Schrödinger equation for a charged particle performing Langmuir oscillations. Then, we have secondly quantized our system. The creation and annihilation operators for oscillatory states of ions have been introduced and the ground state of the system has been constructed. This ground state corresponds to the collective oscillatory motion of ions. Note that the wave functions of the 3D harmonic oscillator are more appropriate for the ground state of the system since they exactly account for the dynamical features of the ions motion and the geometrical form of the plasma structure. We have also defined the Fermi number as the maximal possible occupation number. Note that the quantization of the ions motion is justified since later, in Sec. 4, we have considered plasma structures in the very dense matter of the NS outer core, where quantum effects are significant.

In Sec. 3, we have studied the effective interaction between oscillating ions by the exchange of a virtual acoustic wave. Discussing the situation of plasmoids containing

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a very great number of excited states $n \gg 1$, we have shown that two ions occupying the same energy level can attract each other.

The possibility of the formation of bound states of these ions have been analyzed in Sec. 4. Considering the short waves limit, we have derived the critical occupation number and the characteristic plasmoid radius corresponding to a plasma structure in which all ions are in bound states. As an application of our results, we have discussed the pairing of protons inside a plasmoid in the outer core of NS. We have shown that the existence of such a plasma structure is quite possible. It should be noted that, besides our approach for the description of the protons pairing in the outer core of NS, another mechanism for the formation of singlet states of protons, based on their direct nuclear interaction, is also discussed by Haensel et al. (2007).

Since the spins of ions, which formed a bound state, should be antiparallel, this bound state is analogous to a Cooper pair of electrons in a metal. It is known that the formation of Cooper pairs underlies the phenomenon of superconductivity. Our result that protons can form bound states in the NS outer core agrees with the hypothesis that the proton superconductivity should be present in this astrophysical environment (Yakovlev et al. 1999; Haensel et al. 2007; Heinke and Ho 2010).

It should be noted that in our work we have adopted rather simplified analysis of the ion's motion in plasma. Sagdeev and Galeev (1969) elaborated a more detailed description of the interaction between a charged test particle and a plasma wave in frames of the classical electrodynamics. Various instabilities, which arise in this system, as well as the dynamics of the turbulence were also studied by Sagdeev and Galeev (1969) on the classical level. Our description of quantum plasmoids is valid if we neglect temperature effects, which is the case for a degenerate plasma considered in Sec. 4.

In Sec. 4, we have studied the contribution of the spatial dispersion of ion-acoustic waves to the dynamics of the system. We have obtained that, under the assumption of great number of excited states, this contribution to the critical occupation number and the effective plasmoid radius is small. However, if one studies a plasma structure with a significant but limited n, we expect that, e.g. the effective radius can be considerably enhanced.

Although, we have examined plasma structures in a very dense medium of the outer core of NS as a possible application of our results, we may expect that the phenomenon of pairing of charged particles can happen in a terrestrial plasma. Previously Dijkhuis (1980), Zelikin (2008) and Dvornikov (2012) studied the formation of bound states of charged particles to describe some properties of stable atmospheric plasma structures. The estimates given in Sec. 4 show that our mechanism of pairing cannot be directly implemented inside plasmoids in a low density atmospheric plasma since the radius of such a structure turns out to be quite small. Nevertheless, if one discusses a plasma structure corresponding to a big but limited n, there is a possibility that the described phenomenon can take place in a terrestrial plasma as well.

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Appendix A. Wave functions in coordinate representation

In this Appendix, we present the mathematical details required to express the ion's wave function in the coordinate representation.

First, we show that at $l' = 0$ the wave function coincides up to a sign factor with that found by Dvornikov (2013). For this purpose, we rewrite the wave function in the coordinate representation as

$$\psi_n(\mathbf{r}) = \psi_n(r) = \left[\frac{n!}{\Gamma(l'+n+3/2)}\right]^{1/2} \left(\frac{m_i\omega_i}{\hbar}\right)^{3/4} \times \frac{1}{\pi\beta}\int_0^\infty \mathrm{d}x\, x^{l'+1}\sin(\beta x)\exp\left(-\frac{x^2}{2}\right) L_n^{l'+1/2}(x^2), \tag{A 1}$$

where we account for its spherical symmetry. Here $\beta = r/r_0$. Now, we can obtain (16) from (A 1) at $l' = 0$ using (8.972.3) and (7.388.2) on pages 1001 and 806 in Gradshteyn and Ryzhik (2007).

Now, let us derive the asymptotics of the wave function in (A 1) in case when $0 \neq |l'| \ll 1$. In this limit, the integral in (A 1) has the following form:

$$\int_0^\infty \mathrm{d}x\, x^{l'+1}\sin(\beta x)\exp\left(-\frac{x^2}{2}\right) L_n^{l'+1/2}(x^2) \approx \cos\left(\frac{\pi l'}{2}\right)(-1)^n\sqrt{\frac{\pi}{2}}\beta^{l'+1}\exp\left(-\frac{\beta^2}{2}\right) L_n^{l'+1/2}(\beta^2) + \sin\left(\frac{\pi l'}{2}\right) J, \tag{A 2}$$

where

$$J = \int_0^\infty \mathrm{d}x\, x^{l'+1}\exp\left(-\frac{x^2}{2}\right) L_n^{l'+1/2}(x^2)\cos\left(\beta x - \frac{\pi l'}{2}\right). \tag{A 3}$$

To derive (A 2), we use (7.421.4) on page 812 in Gradshteyn and Ryzhik (2007).

Note that for $|l'| \ll 1$, we can set $l' = 0$ in the argument of cosine and in the upper index of the associated Laguerre polynomial in (A 3) since J is already multiplied by the small factor $\sin(\pi l'/2)$ in (A 2). Thus, we rewrite J in the following way:

$$J \approx \frac{(-1)^n}{2^{2n+1}n!}J', \quad J' = \int_0^\infty \mathrm{d}x\, \exp\left(-\frac{x^2}{2}\right) H_{2n+1}(x)\cos(\beta x), \tag{A 4}$$

To study the asymptotics of J' in (A 4) at $n \gg 1$, we present $H_{2n+1}(x)$ there in the explicit form as a polynomial of $(2n+1)$th power. Then, we calculate each of the integrals in the sum using (3.952.8) on page 503 in Gradshteyn and Ryzhik (2007). Finally, with help of the following expressions:

$${}_1F_1\left(k+1;\frac{1}{2};-\frac{\beta^2}{2}\right) = \sum_{s=0}^{\infty}(-1)^s\frac{(k+1)_s}{(1/2)_s s!}\left(\frac{\beta^2}{2}\right)^s, \tag{A 5}$$

where $(z)_s = z(z+1)\cdots(z+s-1)$ is the Pochhammer symbol, and

$$\sum_{k=0}^{n}\frac{(-1)^k k!2^{3k}}{(2k+1)!(n-k)!}(k+1)_s = \frac{s!}{n!}{}_2F_1\left(-n, s+1;\frac{3}{2};2\right), \tag{A 6}$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function, we obtain J' in the form,

$$J' = (-1)^n 2\sqrt{\pi}\frac{(2n+1)!}{n!}\sum_{k=0}^{\infty}\frac{(-1)^k\beta^{2k}}{2^k\Gamma(k+1/2)}{}_2F_1\left(-n, k+1;\frac{3}{2};2\right). \tag{A 7}$$

Note that, one can study the limit $n \to \infty$ only in (A 7).

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The asymptotics at $n \gg 1$ of the Gauss hypergeometric function in (A 7) reads

$$ {}_2F_1\left(-n, k+1; \frac{3}{2}; 2\right) \sim (-1)^n \frac{2^{k-1}}{k!}\sqrt{\frac{\pi}{2}}\left(n+\frac{3}{2}\right)^{k-1/2}. \tag{A 8} $$

To derive (A 8), we take into account that

$$ \begin{aligned} {}_2F_1(a,b;c;z) = &\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-1)^a z^{-a} {}_2F_1\left(a, a+1-c; a+1-b; \frac{1}{z}\right) \\ &+ \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-1)^b z^{-b} {}_2F_1\left(b, b+1-c; b+1-a; \frac{1}{z}\right), \end{aligned} \tag{A 9} $$

and

$$ \frac{\Gamma(-n-k-1)}{\Gamma(-n)} = (-1)^{k+1}\frac{n!}{(n+k+1)!}. \tag{A 10} $$

We also use the following asymptotics:

$$ \begin{aligned} {}_2F_1\left(-n, -n-\frac{1}{2}; -n-k; \frac{1}{2}\right) &\sim 2^{-n+k-1/2}, \\ {}_2F_1\left(k+1, k+\frac{1}{2}; n+k+2; \frac{1}{2}\right) &\to 1, \end{aligned} \tag{A 11} $$

which are valid at $n \gg 1$.

It is interesting to compare (A 8) with (2.7) derived by Temme (1986), where the asymptotics of the Gauss hypergeometric function was also studied. The result of Temme (1986) reads

$$ {}_2F_1\left(-n, k+1; \frac{3}{2}; 2\right) \sim (-1)^n \frac{2^{k-1}}{k!}\sqrt{\frac{\pi}{2}}(n+1)^{k-1/2}. \tag{A 12} $$

We can see that besides the unessential difference, $n + 3/2$ versus $n + 1$, which is unimportant at big n, our result coincides with that of obtained by Temme (1986). However, Temme (1986) claimed that the asymptotics in question is valid when the argument of the hypergeometric function $z > 2$. Here, we demonstrate that the case $z = 2$ can be also described by (A 8) or (A 12) at least for the particular set of the parameters of the hypergeometric function.

The asymptotic behavior of the hypergeometric function in (A 8) is shown in Fig. A.1(a). We depict there the sequence

$$ y_n = (-1)^n \sqrt{\frac{2}{\pi}}\frac{k!\,{}_2F_1(-n, k+1; 3/2; 2)}{2^{k-1}(n+3/2)^{k-1/2}}, \tag{A 13} $$

for different values of k. One can see in Fig. A.1(a) that $y_n \to 1$. It proves the validity of the asymptotics in (A 8).

To illustrate the behavior of y_n in (A 13) at big $n \gg 1$, we can formally take that the radial quantum number is a complex number $n \to ne^{i\vartheta}$, where ϑ is the phase. In Fig. A.2(a), we plot $|y_n|$ versus n for different values of ϑ and $k = 1$. One can see that the dotted and dash-dotted lines, which correspond to positive and negative ϑ, approach to the solid line built for $\vartheta = 0$ at $|\vartheta| \to 0$. Therefore, the numerical experiment demonstrates that y_n in (A 13) in a smooth function in the complex plane. Of course, a more careful analytical analysis of this fact is required. However, this issue is beyond the scope of the present work.

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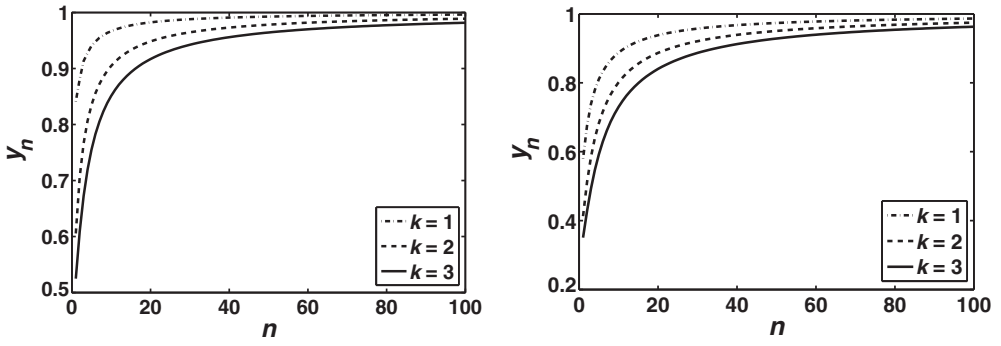


FIGURE A.1. The illustration of the asymptotic behavior of the hypergeometric functions in (A 8) and (A 20) for different values of k. (a) The sequence y_n in (A 13). (b) The sequence y_n in (A 21).

On the basis of (A 5)–(A 8), we obtain the behavior of J' as

$$J' \sim \sqrt{2} 2^{2n} n! \cos\left(2\sqrt{n+3/2}\beta\right), \tag{A 14}$$

where we use the value of the sum of the series,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+1/2)} \left(\sqrt{n+3/2}\beta\right)^{2k} = \frac{1}{\sqrt{\pi}} \cos(2\sqrt{n+3/2}\beta). \tag{A 15}$$

Using (A 2), (A 4), and (A 14), one gets the asymptotics of the wave function in (A 1), which corresponds to the case $|l'| \ll 1$ and $n \gg 1$,

$$\psi_n(r) \sim \frac{(-1)^n}{\sqrt{2}\pi\beta n^{1/4}} \left(\frac{m_i\omega_i}{\hbar}\right)^{3/4} \sin\left(2\sqrt{n}\beta\right), \tag{A 16}$$

where we keep only the leading term in l' and use (8.978.3) on page 1003 in Gradshteyn and Ryzhik (2007). One can see in (A 16) that the correction for the wave function, linear in l', vanishes in the limit $n \gg 1$.

Now, let us consider another extreme situation which corresponds to $l' = -1/2$. On the basis of (A 1), one can see that in this case it is necessary to get the asymptotics at big n of the following integral:

$$J = \int_0^{\infty} \mathrm{d}x \sqrt{x} \sin(\beta x) \exp\left(-\frac{x^2}{2}\right) L_n(x^2), \tag{A 17}$$

where $L_n(z) = L_n^0(z)$ is the Laguerre polynomial. We can analyze J in (A 17) in the same manner as J' in (A 4). We shall describe only the main steps of this analysis.

First, we represent $L_n(x^2)$ in the explicit form as a polynomial of $2n$th power. Then we use (3.952.7) on page 503 in Gradshteyn and Ryzhik (2007) and the sum of the series,

$$\sum_{k=0}^{n} \frac{(-1)^k 2^k}{k!^2(n-k)!} \Gamma\left(k+\frac{5}{4}\right) \left(k+\frac{9}{4}\right)_s = \frac{4}{5} \frac{\Gamma\left(s+\frac{9}{4}\right)}{n!} {}_3F_2\left(-n, s+\frac{9}{4}, \frac{5}{4}; 1, \frac{9}{4}; 2\right), \tag{A 18}$$

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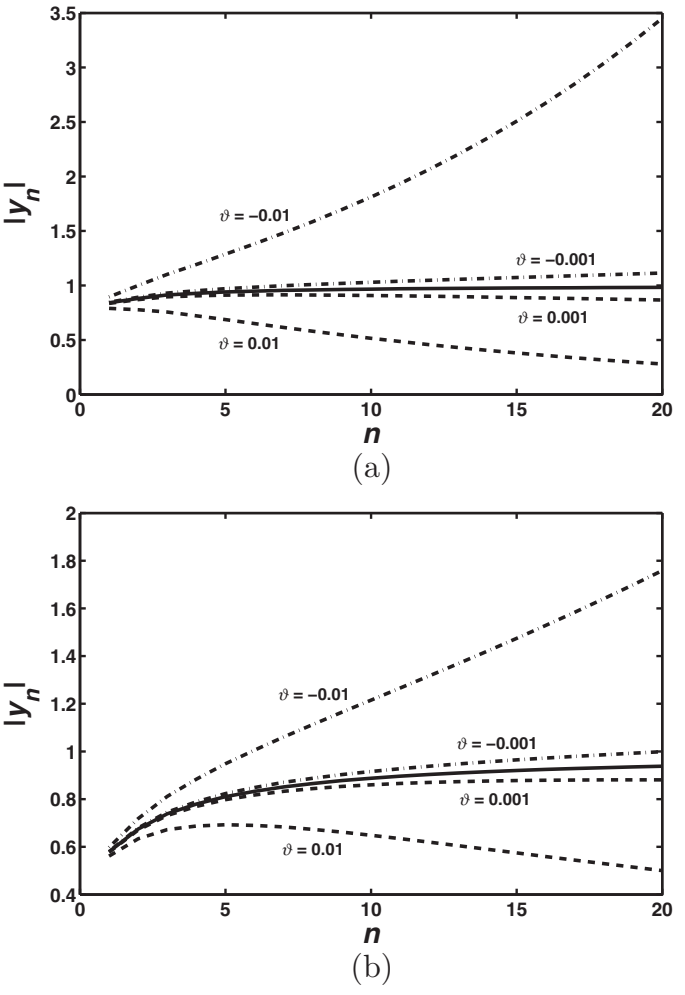


FIGURE A.2. (a) The behavior of the absolute value of y_n in (A 13) with the complex radial quantum number $n \to ne^{i\vartheta}$. Dashed lines correspond to $\vartheta > 0$ and dash-dotted to $\vartheta < 0$. The solid line is built for $\vartheta = 0$ and is also shown in Fig. A.1(a). (b) The same as in panel (a) for y_n in (A 21).

where ${}_3F_2\,(a_1, a_2, a_3; b_1, b_2; z)$ is the generalized hypergeometric function. Eventually, we get the expression for J in the form,

$$J = \frac{2^{5/4}}{5}\sqrt{\pi}\beta \sum_{k=0}^{\infty} \frac{(-1)^k \beta^{2k} \Gamma(k + 9/4)}{2^k k! \Gamma(k + 3/2)} {}_3F_2\left(-n, k + \frac{9}{4}, \frac{5}{4}; 1, \frac{9}{4}; 2\right). \tag{A 19}$$

Here, we also use the analog of (A 5).

The hypergeometric function in (A 19) has the following behavior at $n \gg 1$:

$${}_3F_2\left(-n, k + \frac{9}{4}, \frac{5}{4}; 1, \frac{9}{4}; 2\right) \sim 5\,(n+1)^{1/4} \left(n + \frac{9}{4}\right)^k \frac{(-1)^n 2^{k-7/4}}{\Gamma(k + 9/4)}. \tag{A 20}$$

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To illustrate the asymptotics of the hypergeometric function in (A 20), in Fig. A.1(b) we present the sequence

$$y_n = (-1)^n \frac{\Gamma(k+9/4)_3F_2\,(-n, k+9/4, 5/4; 1, 9/4; 2)}{2^{k-7/4}5(n+1)^{1/4}\,(n+9/4)^k}, \tag{A 21}$$

for different values of k. One can see in Fig. A.1(b) that $y_n \to 1$, as it follows from (A 20).

Analogously to Fig. A.2(a), in Fig. A.2(b) we show the behavior of $|y_n|$ in (A 21) for the complex valued radial quantum number $n \to ne^{i\vartheta}$ at $k = 1$. Again, one can see that $|y_n| \to |y_n(\vartheta = 0)|$ at $|\vartheta| \to 0$, i.e. y_n is a smooth function in the complex plane.

Finally, using (A 17)–(A 20), one obtains the expression for the wave function at $n \gg 1$,

$$\psi_n(r) \sim \frac{(-1)^n}{\sqrt{2}\pi\beta n^{1/4}} \left(\frac{m_i\omega_i}{\hbar}\right)^{3/4} \sin\left(2\sqrt{n}\beta\right). \tag{A 22}$$

To derive (A 22), we use the known value for the sum of the series,

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+3/2)} \left(\sqrt{n+9/4}\beta\right)^{2k} = \frac{\sin(2\sqrt{n+9/4}\beta)}{\sqrt{\pi}\beta\sqrt{n+9/4}}. \tag{A 23}$$

One can see in (A 22) that, in the limit $n \gg 1$, the expression for the wave function corresponding to $l' = -1/2$ again coincides with the analogous expression for $l' = 0$.

At the end of this section, we mention that as a by-product have obtained the asymptotic expression for the Hilbert transform of the Hermite function with the odd index,

$$\varphi_{2n+1}(z) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \exp\left(-\frac{z^2}{2}\right) H_{2n+1}(z). \tag{A 24}$$

The Hilbert transform $H[f](\beta)$ of the function $f(x)$ is defined as

$$H[f](\beta) = \frac{1}{\pi}\text{V.P.}\int_{-\infty}^{+\infty} \mathrm{d}x \frac{f(x)}{\beta - x}, \tag{A 25}$$

where V.P. stays for the principle value of the integral.

Using (A 14) and (A 24), we obtain the following asymptotics of the odd index Hilbert transform of the Hermite function:

$$H[\varphi_{2n+1}](\beta) \sim (-1)^{n+1} \frac{2^{3n/2+1}\sqrt{n!}}{\pi^{3/4}} \cos\left(2\sqrt{n+3/2}\beta\right). \tag{A 26}$$

It should be noted that previously, only the recurrence relation for $H[\varphi_{2n+1}]$ was known (Hahn 2000), which can be used only for small n. We have derived the explicit expression for the asymptotics of the Hilbert transform. Therefore, (A 26) can be useful, for instance, in signal processing.

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