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Invariant density functions of random β-transformations

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Abstract. We consider the random β-transformation K_β introduced by Dajani and Kraaikamp [Random β-expansions. *Ergod. Th. & Dynam. Sys.* **23** (2003), 461–479], which is defined on $\{0, 1\}^{\mathbb{N}} \times [0, [\beta]/(\beta - 1)]$. We give an explicit formula for the density function of a unique K_β-invariant probability measure absolutely continuous with respect to the product measure $m_p \otimes \lambda_\beta$, where m_p is the $(1 - p, p)$-Bernoulli measure on $\{0, 1\}^{\mathbb{N}}$ and λ_β is the normalized Lebesgue measure on $[0, [\beta]/(\beta - 1)]$. We apply the explicit formula for the density function to evaluate its upper and lower bounds and to investigate its continuity as a function of the two parameters p and β.

1. *Introduction*

Let β be a non-integer number greater than 1 and let us denote by $[\beta]$ the integer part of β. We consider an expansion of a number x in $J_\beta := [0, [\beta]/(\beta - 1)]$ of the form

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}$$

with integer digits $a_n \in \{0, 1, \ldots, [\beta]\}$. We call every expansion of x given by the above form a β-expansion of x in this paper. In contrast to the case of an integer base, a number x in J_β can have uncountably many different β-expansions in this case (see [**7**, **18**]). The largest β-expansion of x in the lexicographical order between the digit sequences of all possible β-expansions of x is called the greedy expansion of x and the smallest is called the lazy expansion of x. From dynamical viewpoints, the greedy expansion of every number x in J_β is generated by iterating the greedy map $T_{\beta,1} : J_\beta \to J_\beta$, which is defined by

$$T_{\beta,1}(x) = \begin{cases} \beta x - [\beta x] & \text{for } x \in [0, 1), \\ \beta x - [\beta] & \text{for } x \in [1, [\beta]/(\beta - 1)], \end{cases}$$

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and the lazy expansion of every number x in J_β is generated by iterating the lazy map $T_{\beta,0} : J_\beta \to J_\beta$, which is defined by

$$T_{\beta,0}(x) = l_\beta \circ T_{\beta,1} \circ l_\beta^{-1}(x)$$

for $x \in J_\beta$, where $l_\beta : J_\beta \to J_\beta$ is the map given by

$$l_\beta(x) = [\beta]/(\beta - 1) - x$$

for $x \in J_\beta$ (see [**3**]). Note that the restriction $T_{\beta,1}|_{[0,1)}$ is the classical β-transformation introduced by Rényi in [**17**]. Since the greedy map $T_{\beta,1}$ has the interval $[0, 1)$ as its invariant set, the ergodic properties of the greedy map $T_{\beta,1}$ are essentially the same as those of the classical β-transformation. In particular, the greedy map $T_{\beta,1}$ has a unique invariant probability density $h_{\beta,1}$ given by

$$h_{\beta,1}(x) = \begin{cases} \dfrac{1}{F(\beta, 1)} \displaystyle\sum_{n=0}^{\infty} \frac{1}{\beta^n} \chi_{[0, T_{\beta,1}^n(1))}(x), & x \in [0, 1), \\ 0, & x \in [1, [\beta]/(\beta - 1)], \end{cases}$$

where $F(\beta, 1)$ is the normalizing constant and χ_A denotes the indicator function of A (see [**3**, **8**, **15**]). Consequently, the lazy map $T_{\beta,0}$ also has a unique invariant probability density $h_{\beta,0}$ given by

$$h_{\beta,0}(x) = h_{\beta,1} \circ l_\beta(x)$$

for $x \in J_\beta$ by definition. In [**4**], Dajani and Kraaikamp introduced a random β-transformation K_β on $\{0, 1\}^{\mathbb{N}} \times J_\beta$ associated with the maps $T_{\beta,1}$ and $T_{\beta,0}$. By using the map K_β, they defined the so-called random β-expansion of a number x in J_β, which is given by iterating the map K_β. In fact, the random β-transformation K_β produces all β-expansions of each number x in J_β as its random β-expansions (see [**5**, Theorem 2]). In [**6**], Dajani and de Vries studied the ergodic properties of the map K_β via the skew-product map R_β on $\{0, 1\}^{\mathbb{N}} \times J_\beta$ with the left shift on $\{0, 1\}^{\mathbb{N}}$ as the base map and with the maps $T_{\beta,1}$ and $T_{\beta,0}$ as the fiber maps. They showed that the skew-product map R_β also preserves a K_β-invariant measure of the form $m_p \otimes \mu$, where m_p is the $(1 - p, p)$-Bernoulli measure on $\{0, 1\}^{\mathbb{N}}$ with a parameter $p \in (0, 1)$ and μ is a Borel probability measure on J_β. From this fact, they proved that there exists a K_β-invariant measure of the form $m_p \otimes \mu_{\beta,p}$, where $\mu_{\beta,p}$ is a probability measure absolutely continuous with respect to the normalized Lebesgue measure λ_β on J_β. Furthermore, they also proved that the product measure $m_p \otimes \mu_{\beta,p}$ is a unique K_β-invariant measure absolutely continuous with respect to $m_p \otimes \lambda_\beta$ and the dynamical system $(K_\beta, m_p \otimes \mu_{\beta,p})$ is ergodic.

In the present paper, we investigate the ergodic properties of the dynamical system $(K_\beta, m_p \otimes \mu_{\beta,p})$ via the skew-product system $(R_\beta, m_p \otimes \mu_{\beta,p})$. First, we show that the dynamical system $(K_\beta, m_p \otimes \mu_{\beta,p})$ is exact for any non-integer $\beta > 1$ and any $p \in (0, 1)$ (Theorem 3.2). Note that in [**6**] its exactness was shown in the case where β has some special algebraic property. In fact, our result is due to the exactness of the skew-product system $(R_\beta, m_p \otimes \mu_{\beta,p})$. We prove that the system $(R_\beta, m_p \otimes \mu_{\beta,p})$ is exact for any non-integer $\beta > 1$ and any $p \in (0, 1)$ by means of the specific properties of the maps $T_{\beta,1}$ and $T_{\beta,0}$ (Proposition 3.1). Second, as a further study of the system $(K_\beta, m_p \otimes \mu_{\beta,p})$, we find an explicit formula for the density function of $\mu_{\beta,p}$ through the Perron–Frobenius

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operator of the skew-product map R_β. Let us denote by $h_{\beta,p}$ the density function of $\mu_{\beta,p}$. In general, for the skew-product map associated to random expanding Lasota–Yorke-type maps, each invariant density is given implicitly as some fixed point of its Perron–Frobenius operator (see [**2**, **13**, **16**]). In the case of the skew-product map R_β, however, it is much more interesting that the density function $h_{\beta,p}$ is given explicitly since it enables us to evaluate the statistical properties of random β-expansions, such as the relative frequency of the digits $0, 1, \ldots, [\beta]$ in random β-expansions, together with the ergodicity of the system $(K_\beta, m_p \otimes \mu_{\beta,p})$. For this reason, we give an explicit formula for the density function $h_{\beta,p}$ in all cases (Theorem 4.4), which was given in the case where the greedy expansion of 1 in base β satisfies $1 = a_1/\beta + a_2/\beta^2 + \cdots + a_n/\beta^n$ with $a_i \geqq 1$ for $i = 1, \ldots, n$ and $n \geqq 2$ (see [**6**]) or in the case $p = 1/2$ (see [**9**]). From this formula, we also give lower and upper bounds of the density function $h_{\beta,p}$ explicitly and show that for each p the density function $h_{\beta,p}$ converges to $\chi_{[0,1]}$ in $L^1(\lambda)$ as $\beta \nearrow N+1$ or $\beta \nearrow +\infty$, where N is a positive integer and λ denotes the Lebesgue measure on $\mathbb{R}$ (Theorem 5.1). Finally, as an application of our explicit formula, we study the continuity of the density function $h_{\beta,p}$ as a function of two parameters p and β. We prove that for each β the function $p \mapsto h_{\beta,p}$ from $[0, 1]$ into $L^1(\lambda)$ can be extended to an analytic function on some region in $\mathbb{C}$ (Theorem 5.3). In addition, we show that for each p the map $\beta \mapsto h_{\beta,p}$ from $(1, +\infty) \setminus \mathbb{N}$ into $L^1(\lambda)$ is continuous everywhere except on $\mathcal{F}$, where the set $\mathcal{F}$ is an analog of the set of all Parry numbers for the random β-transformation (Theorem 5.7).

This paper is organized as follows. In §2, we recall the definitions of the random β-transformation K_β and the skew-product map R_β. In §3, we introduce the Perron–Frobenius operator of a skew-product map with Lasota–Yorke-type maps as the fiber maps and prove that the dynamical system $(K_\beta, m_p \otimes \mu_{\beta,p})$ is exact. Section 4 is devoted to giving an explicit formula for the density function $h_{\beta,p}$. As its application, we evaluate upper and lower bounds of the density function $h_{\beta,p}$ and study the separate continuity of the density function $h_{\beta,p}$ as a function of two variables p and β in §5.

2. *Random β-transformations*

In this section, we recall the definitions of the random β-transformation K_β and the skew-product transformation R_β following the references [**4**–**6**].

Let $\mathbb{N}$ be the set of all positive integers and β a non-integer number greater than 1 throughout this paper. We set $J_\beta = [0, [\beta]/(\beta-1)]$. Let us denote by $\mathcal{B}(J_\beta)$ the Borel field on J_β. We define the digit function $d_\beta^1 : J_\beta \to \{0, 1, \ldots, [\beta]\}$ by

$$d_\beta^1(x) = \begin{cases} [\beta x], & x \in [0, [\beta]/\beta), \\ [\beta], & x \in [[\beta]/\beta, [\beta]/(\beta-1)]. \end{cases}$$

Then the greedy map $T_{\beta,1} : J_\beta \to J_\beta$ is defined by

$$T_{\beta,1}(x) = \beta x - d_\beta^1(x)$$

for $x \in J_\beta$.

Let $l_\beta : J_\beta \to J_\beta$ be defined by

$$l_\beta(x) = \frac{[\beta]}{(\beta-1)} - x$$

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for $x \in J_\beta$. It is clear that $l_\beta^{-1} = l_\beta$. Then the lazy map $T_{\beta,0} : J_\beta \to J_\beta$ is defined by

$$T_{\beta,0}(x) = l_\beta \circ T_{\beta,1} \circ l_\beta^{-1}(x)$$

for $x \in J_\beta$. We put the function $d_\beta^0 : J_\beta \to \{0, 1, \ldots, [\beta]\}$ as

$$d_\beta^0(x) = [\beta] - d_\beta^1(l_\beta(x))$$

for $x \in J_\beta$. Then it is easy to see that

$$T_{\beta,0}(x) = \beta x - d_\beta^0(x)$$

for $x \in J_\beta$.

We define the partition $\{E_k\}_{k=0}^{[\beta]} \cup \{S_k\}_{k=1}^{[\beta]}$ of the interval J_β as follows:

$$E_k = \left(\frac{[\beta]}{\beta(\beta-1)} + \frac{k-1}{\beta}, \frac{k+1}{\beta} \right) \quad \text{for } k = 1, \ldots, [\beta] - 1,$$
$$E_0 = \left[0, \frac{1}{\beta} \right), \quad E_{[\beta]} = \left(\frac{[\beta]}{\beta(\beta-1)} + \frac{[\beta]-1}{\beta}, \frac{[\beta]}{\beta - 1} \right]$$

and

$$S_k = \left[\frac{k}{\beta}, \frac{[\beta]}{\beta(\beta-1)} + \frac{k-1}{\beta} \right] \quad \text{for } k = 1, \ldots, [\beta].$$

We set

$$E = \bigcup_{k=0}^{[\beta]} E_k, \quad S = \bigcup_{k=1}^{[\beta]} S_k.$$

Then by definition we have

$$T_{\beta,1}(x) = T_{\beta,0}(x), \quad d_\beta^1(x) = d_\beta^0(x) = k$$

if $x \in E_k$ for $k = 0, 1, \ldots, [\beta]$ and

$$T_{\beta,1}(x) = T_{\beta,0}(x) - 1, \quad d_\beta^1(x) = k, \quad d_\beta^0(x) = k - 1$$

if $x \in S_k$ for $k = 1, \ldots, [\beta]$. That is, these two maps are identical and assign the same digit k on E_k although the greedy map assigns the digit k and the lazy map assigns the digit $k-1$ on S_k for $k = 1, \ldots, [\beta]$. We call each E_k an equality region and each S_k a switch region.

Let $\{0, 1\}$ be endowed with the discrete topology and $\Omega = \{0, 1\}^{\mathbb{N}}$ the product topology. We denote by $\mathcal{B}(\Omega)$ the Borel field on Ω. Let $\sigma : \Omega \to \Omega$ be the left shift. Then the random β-transformation $K_\beta : \Omega \times J_\beta \to \Omega \times J_\beta$ is defined by

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - k) & \text{if } x \in E_k, k = 0, 1, \ldots, [\beta], \\ (\sigma\omega, \beta x - k) & \text{if } x \in S_k \text{ and } \omega_1 = 1, k = 1, \ldots, [\beta], \\ (\sigma\omega, \beta x - (k-1)) & \text{if } x \in S_k \text{ and } \omega_1 = 0, k = 1, \ldots, [\beta] \end{cases}$$

for $(\omega, x) \in \Omega \times J_\beta$, where $\omega = (\omega_n)_{n=1}^\infty$. Note that K_β is a Borel map on $(\Omega \times J_\beta, \mathcal{B}(\Omega) \otimes \mathcal{B}(J_\beta))$. By definition, it is easy to see that

$$K_\beta(\omega, x) = \begin{cases} (\omega, T_{\beta,\omega_1}(x)) & \text{if } x \in E, \\ (\sigma\omega, T_{\beta,\omega_1}(x)) & \text{if } x \in S \end{cases}$$

for $(\omega, x) \in \Omega \times J_\beta$, where $\omega = (\omega_n)_{n=1}^\infty$.

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We can obtain a β-expansion of $x \in J_\beta$ for each $\omega \in \Omega$ via the random β-transformation K_β as follows. Let us denote by $pr_1 : \Omega \times J_\beta \to \Omega$ the canonical projection onto the first coordinate and by $pr_2 : \Omega \times J_\beta \to J_\beta$ the canonical projection onto the second coordinate. Let us also denote by $\pi_1 : \Omega \to \{0, 1\}$ the canonical projection onto the first coordinate space $\{0, 1\}$.

Then by definition for $(\omega, x) \in \Omega \times J_\beta$ we have

$$pr_2(K_\beta^{n-1}(\omega, x)) = \frac{d_\beta^{\pi_1 \circ pr_1(K_\beta^{n-1}(\omega,x))}(pr_2(K_\beta^{n-1}(\omega, x)))}{\beta} + \frac{pr_2(K_\beta^n(\omega, x))}{\beta}$$

for $n \in \mathbb{N}$ and $(\omega, x) \in \Omega \times J_\beta$. This yields that

$$\begin{aligned} x &= pr_2(\omega, x) \\ &= \frac{d_\beta^{\pi_1 \circ pr_1(\omega,x)}(pr_2(\omega, x))}{\beta} + \frac{pr_2(K_\beta(\omega, x))}{\beta} \\ &= \frac{d_\beta^{\pi_1 \circ pr_1(\omega,x)}(pr_2(\omega, x))}{\beta} + \frac{d_\beta^{\pi_1 \circ pr_1(K_\beta(\omega,x))}(pr_2(K_\beta(\omega, x)))}{\beta^2} + \frac{pr_2(K_\beta^2(\omega, x))}{\beta^2} \\ &= \cdots \\ &= \sum_{n=1}^{N} \frac{d_\beta^{\pi_1 \circ pr_1(K_\beta^{n-1}(\omega,x))}(pr_2(K_\beta^{n-1}(\omega, x)))}{\beta^n} + \frac{pr_2(K_\beta^N(\omega, x))}{\beta^N} \end{aligned}$$

for $N \in \mathbb{N}$. Since $0 \leqq pr_2(K_\beta^N(\omega, x) \leqq [\beta]/(\beta - 1)$, we get a random β-expansion of x:

$$x = \sum_{n=1}^{\infty} \frac{d_\beta^{\pi_1 \circ pr_1(K_\beta^{n-1}(\omega,x))}(pr_2(K_\beta^{n-1}(\omega, x)))}{\beta^n}$$

for each $\omega \in \Omega$.

We define the skew-product map $R_\beta : \Omega \times J_\beta \to \Omega \times J_\beta$ by

$$R_\beta(\omega, x) = (\sigma\omega, T_{\beta,\omega_1}(x))$$

for $(\omega, x) \in \Omega \times J_\beta$, where $\omega = (\omega_n)_{n=1}^\infty$. Then the skew-product map R_β is also a Borel map on $(\Omega \times J_\beta, \mathcal{B}(\Omega) \otimes \mathcal{B}(J_\beta))$.

Similarly to the random β-transformation K_β, we can obtain a β-expansion of $x \in J_\beta$ for each $\omega \in \Omega$ via the skew-product map R_β. Since by definition

$$pr_2(R_\beta^{n-1}(\omega, x)) = \frac{d_\beta^{\omega_n}(pr_2(R_\beta^{n-1}(\omega, x)))}{\beta} + \frac{\pi_2(R_\beta^n(\omega, x))}{\beta}$$

for $n \in \mathbb{N}$ and $(\omega, x) \in \Omega \times J_\beta$, we get

$$x = \sum_{n=1}^{\infty} \frac{d_\beta^{\omega_n}(pr_2(R_\beta^{n-1}(\omega, x)))}{\beta^n}$$

for each $\omega = (\omega_n)_{n=1}^\infty \in \Omega$ in the same way as the case of the map K_β.

We note that for $n \in \mathbb{N}$, $\omega = (\omega_i)_{i=1}^\infty \in \Omega$ and $x \in J_\beta$,

$$pr_2(R_\beta^n(\omega, x)) = T_{\beta,\omega_n} \circ \cdots \circ T_{\beta,\omega_1}(x).$$

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In what follows, for $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$, we set

$$T_{\beta,\omega_1\cdots\omega_n} = T_{\beta,\omega_n} \circ \cdots \circ T_{\beta,\omega_1}$$

and regard $T_{\beta,\omega_1\cdots\omega_n}$ as the identity map on J_β if $n = 0$. Furthermore, let us denote by $[j_1, \ldots, j_n]_1^n$ a cylinder set $\{\omega \in \Omega;\ \omega_1 = j_1, \ldots, \omega_n = j_n\}$ and we regard $[j_1, \ldots, j_n]_1^n$ as Ω if $n = 0$.

Let m_p be the $(1-p, p)$-Bernoulli measure on $\Omega = \{0, 1\}^{\mathbb{N}}$ with a parameter $0 < p < 1$:

$$m_p([j_1, \ldots, j_n]_1^n) = p^{\sum_{i=1}^n j_i}(1-p)^{n-\sum_{i=1}^n j_i}.$$

Let us recall the following lemma.

LEMMA 2.1. [**6**, Lemma 1] *Let μ be a Borel probability measure on J_β. Then*

$$m_p \otimes \mu \circ K_\beta^{-1} = m_p \otimes \mu \circ R_\beta^{-1}.$$

This enables us to study K_β-invariant measures of the form $m_p \otimes \mu$, where μ is a probability measure absolutely continuous with respect to λ_β, through the Perron–Frobenius operator of the skew-product map R_β.

3. *Perron–Frobenius operators*

Let λ_β be the normalized Lebesgue measure on $J_\beta = [0, [\beta]/(\beta-1)]$. In this section, we study R_β-invariant probability measures of the form $m_p \otimes \mu$, where μ is a probability measure absolutely continuous with respect to λ_β, via the Perron–Frobenius operator of the skew-product map R_β. We summarize some results for the Perron–Frobenius operator of a skew-product map with Lasota–Yorke-type maps as the fiber maps as follows (for more details, see [**1**, **2**, **10**, **11**, **13**, **14**, **16**]).

Let μ be a Borel probability measure on J_β. A measurable map $T : J_\beta \to J_\beta$ is called non-singular with respect to μ if $\mu(T^{-1}A) = 0$ whenever $\mu(A) = 0$ for $A \in \mathcal{B}(J_\beta)$. If T is non-singular, the action T^* on measures defined by $T^*\mu' = \mu' \circ T^{-1}$ preserves the class of measures absolutely continuous with respect to μ. Let T be a non-singular transformation on J_β. Then the Perron–Frobenius operator $\mathcal{L}_{T,\mu} : L^1(\mu) \to L^1(\mu)$ is defined by

$$\mathcal{L}_{T,\mu} f = \frac{d}{d\mu} \int_{T^{-1}(\cdot)} f \, d\mu,$$

where $d/d\mu$ denotes the operation taking the Radon–Nikodym derivative. The Perron–Frobenius operator $\mathcal{L}_{T,\mu}$ is a bounded linear operator on $L^1(\mu)$ and $\|\mathcal{L}_{T,\mu} f\|_{1,\mu} \leqq \|f\|_{1,\mu}$ for $f \in L^1(\mu)$, where $\|\cdot\|_{1,\mu}$ denotes the L^1 norm with respect to μ.

A transformation $T : J_\beta \to J_\beta$ is called a Lasota–Yorke-type map (L–Y-type map) if there exists a partition $0 = a_0 < a_1 < \cdots < a_m = [\beta]/(\beta-1)$ such that the restriction $T|_{(a_{i-1},a_i)}$ is a strictly monotone C^2 map and can be extended to a C^2 map on $[a_{i-1}, a_i]$ for $i = 1, \ldots, m$. If T is a L–Y-type map, T is non-singular with respect to λ_β and the Perron–Frobenius operator $\mathcal{L}_{T,\lambda_\beta}$ is represented by

$$\mathcal{L}_{T,\lambda_\beta} f(x) = \sum_{x=T(y)} \frac{f(y)}{|T'(y)|} \tag{3.1}$$

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for λ_β-almost every (a.e.) $x \in J_\beta$. A L–Y-type map T is called expanding if there exists $\alpha > 1$ such that $\min_{1 \leqq i \leqq m} \inf_{x \in (a_{i-1}, a_i)} |T'(x)| \geqq \alpha$.

Let $n \in \mathbb{N}$ and $S = \{0, \ldots, n-1\}$ be endowed with the discrete topology. Let $(p_0, \ldots, p_{n-1})$ be a probability vector and π the probability measure on S defined by $\pi(\{i\}) = p_i$ for $0 \leqq i \leqq n-1$. Let us denote by $\mathcal{B}(S^{\mathbb{N}})$ the Borel field on $S^{\mathbb{N}}$ with the product topology and by $\sigma : S^{\mathbb{N}} \to S^{\mathbb{N}}$ the left shift. Let $T_0, \ldots, T_{n-1}$ be L–Y-type maps. We define the skew-product map $R : S^{\mathbb{N}} \times J_\beta \to S^{\mathbb{N}} \times J_\beta$ by

$$R(s, x) = (\sigma s, T_{s_1} x)$$

for $(s, x) \in S^{\mathbb{N}} \times J_\beta$, where $s = (s_i)_{i=1}^{\infty}$.

We define $\mathcal{L}_{R,\mu} : L^1(\mu) \to L^1(\mu)$, the Perron–Frobenius operator of R with respect to μ, by

$$\mathcal{L}_{R,\mu} f = \sum_{i=0}^{n-1} p_i \mathcal{L}_{T_i,\mu} f$$

for $f \in L^1(\mu)$, where μ is a probability measure absolutely continuous with respect to λ_β. Let us denote by μ_f the finite signed measure of the form $\int_{(\cdot)} f \, d\lambda_\beta$ for $f \in L^1(\lambda_\beta)$. By the definition of the Perron–Frobenius operator, we have that the product measure of the form $\pi^{\mathbb{N}} \otimes \mu_f$ is R-invariant if and only if $\mathcal{L}_{R,\lambda_\beta} f = f$. Therefore, we can investigate R-invariant probability measures of the form $\pi^{\mathbb{N}} \otimes \mu$, where μ is a probability measure absolutely continuous with respect to λ_β, via the Perron–Frobenius operator $\mathcal{L}_{R,\lambda_\beta}$.

By applying [**16**, Theorem 1] to the case of the skew-product map $R_{\beta,}$, Dajani and de Vries in [**6**] saw that there exists a unique function $h_{\beta,p} \in L^1(\lambda_\beta)$ such that $h_{\beta,p} \geqq 0$, $\int_{J_\beta} h_{\beta,p} \, d\lambda_\beta = 1$ and

$$\begin{aligned} \mathcal{L}_{R_\beta,\lambda_\beta} h_{\beta,p} &= p\mathcal{L}_{T_{\beta,1},\lambda_\beta} h_{\beta,p} + (1-p)\mathcal{L}_{T_{\beta,0}\lambda_\beta} h_{\beta,p} \\ &= h_{\beta,p}. \end{aligned}$$

Let us denote by $\mu_{\beta,p}$ the probability measure given by $\mu_{\beta,p}(\cdot) = \int_{(\cdot)} h_{\beta,p} \, d\lambda_\beta$. By the existence and uniqueness of the function $h_{\beta,p}$ satisfying the above conditions, it follows that the probability measure $\mu_{\beta,p}$ is the unique probability measure where $\mu_{\beta,p}$ is absolutely continuous with respect to λ_β and the product measure $m_p \otimes \mu_{\beta,p}$ is R_β-invariant. In fact, $m_p \otimes \mu_{\beta,p}$ is a unique R_β-invariant probability measure absolutely continuous with respect to $m_p \otimes \lambda_\beta$ (see [**14**, Corollary 3.1]).

In what follows, we show that the dynamical system $(K_\beta, m_p \otimes \mu_{\beta,p})$ is exact.

PROPOSITION 3.1. *The dynamical system $(R_\beta, m_p \otimes \mu_{\beta,p})$ is exact.*

Proof. Let $E \in \bigcap_{n=0}^{\infty} R_\beta^{-n}(\mathcal{B}(\Omega) \otimes \mathcal{B}(J_\beta))$. Then there exists a collection of $\mathcal{B}(\Omega) \otimes \mathcal{B}(J_\beta)$-measurable sets $\{E_n\}_{n=1}^{\infty}$ such that $E = R_\beta^{-n} E_n$ for $n \geqq 1$. Assume that $(m_p \otimes \mu_{\beta,p})(E) > 0$. Since the product Borel field $\mathcal{B}(\Omega) \otimes \mathcal{B}(J_\beta)$ is generated by the collection $\{A \times B;\, A \in \mathcal{B}(\Omega),\, B \in \mathcal{B}(J_\beta)\}$, there exist a cylinder set $[i_1 \cdots i_{n_0}]_1^{n_0}$ and an interval $I \subset J_\beta$ with $\mu_{\beta,p}(I) > 0$ such that $[i_1 \cdots i_{n_0}]_1^{n_0} \times I \subset E \bmod m_p \otimes \mu_{\beta,p}$. By the definition of the skew-product map R_β, we have

$$R_\beta^{n_0}([i_1 \cdots i_{n_0}]_1^{n_0} \times I) = \Omega \times T_{\beta,i_1 \cdots i_{n_0}}(I)$$

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and

$$R_\beta^n(\Omega \times I_{n_0}) = \Omega \times \bigcup_{\omega_1 \cdots \omega_n} T_{\beta,\omega_1 \cdots \omega_n}(I_{n_0})$$

for $n \geqq 1$, where $I_{n_0} = T_{\beta,i_1 \cdots i_{n_0}}(I)$. Let J_{n_0} be a connected piece of I_{n_0}. Since the map $T_{\beta,1}$ is expanding, the length of the images $T_{\beta,1}^n J_{n_0}$ increases until it contains a discontinuous point of $T_{\beta,1}$. By the definition of the map $T_{\beta,1}$, we see that $\lim_{x \searrow x_0} T_{\beta,1}(x) = 0$, where x_0 is a discontinuous point of $T_{\beta,1}$. As $0 \in J_\beta$ is a fixed point of $T_{\beta,0}$ and the lap of $T_{\beta,0}$ which includes 0 is full, there exist non-negative integers N, $M \geqq 0$ such that $T_{\beta,0}^M \circ T_{\beta,1}^N(J_{n_0}) = J_\beta$. By setting $L = N + M$, we obtain

$$R_\beta^L(\Omega \times I_{n_0}) = \Omega \times J_\beta \mod m_p \otimes \mu_{\beta,p}.$$

Since $m_p \otimes \mu_{\beta,p}$ is R_β-invariant and $R_\beta^L(\Omega \times I_{n_0}) \subset E_{L+n_0}$, we have that $(m_p \otimes \mu_{\beta,p})(E) = (m_p \otimes \mu_{\beta,p})(E_{L+n_0}) = 1$. This shows that the system $(R_\beta, m_p \otimes \mu_{\beta,p})$ is exact. □

We recall that the system $(K_\beta, m_p \otimes \mu_{\beta,p})$ is a factor of the system $(R_\beta, m_p \otimes \mu_{\beta,p})$ (see [**6**, §3]) and if a measure-preserving dynamical system is exact then so is its factor since the tail σ-algebra of the factor is regarded as being included in that of the exact dynamical system via the factor map. Then we have the following result.

THEOREM 3.2. *The dynamical system $(K_\beta, m_p \otimes \mu_{\beta,p})$ is exact.*

4. *An explicit formula for the density function $h_{\beta,p}$*

In this section, we give an explicit formula for the density function $h_{\beta,p}$ via the random orbits of 1 and $l_\beta 1 := l_\beta(1) = [\beta]/(\beta - 1) - 1$ by the maps $T_{\beta,1}$ and $T_{\beta,0}$.

Let us denote by χ_A the indicator function of $A \subset J_\beta$, that is,

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \notin A. \end{cases}$$

We note that for $A \in \mathcal{B}(J_\beta)$ and $N \in \mathcal{B}(J_\beta)$ with $\lambda_\beta(N) = 0$, $\chi_A(x) = \chi_{A \setminus N}(x) = \chi_{A \cup N}(x)$ for λ_β-a.e. $x \in J_\beta$. As usual, we identify measurable functions f and g on J_β if $f(x) = g(x)$ for λ_β-a.e. $x \in J_\beta$.

The next two lemmas play important roles in the proof of Theorem 4.4. In the rest of the paper, we denote the Perron–Frobenius operator of a L–Y-type map T with respect to λ_β by $\mathcal{L}_{T,\lambda_\beta} = \mathcal{L}_T$ for short.

LEMMA 4.1. *Let $\mathcal{L}_{T_{\beta,1}}$ be the Perron–Frobenius operator of the map $T_{\beta,1}$. Then we have:*

(1)
$$\mathcal{L}_{T_{\beta,1}} \chi_{[0,a]} = \frac{1}{\beta}(\chi_{[0,T_{\beta,1}(a)]} + d_\beta^1(a)\chi_{[0,1]})$$
for $a \in J_\beta$;

(2)
$$\mathcal{L}_{T_{\beta,1}} \chi_{J_\beta} = \frac{1}{\beta}(\chi_{J_\beta} + [\beta]\chi_{[0,1]});$$

(3) $\mathcal{L}_{l_\beta} f = f \circ l_\beta$ *for a function $f : J_\beta \to \mathbb{R}$.*

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Proof. (1) By an explicit formula for the Perron–Frobenius operator of a L–Y-type map (3.1), we have

$$\mathcal{L}_{T_{\beta,1}}\chi_{[0,a]}(x)=\frac{1}{\beta}\sharp\{0\leqq y\leqq a;\ x=T_{\beta,1}(y)\}$$

for $x\in J_\beta$, where $\sharp A$ denotes the cardinality of A.

By the definition of the map $T_{\beta,1}$ and the digit function d^1_β, we get

$$\sharp\{0\leqq y\leqq a;\ x=T_{\beta,1}(y)\}=\begin{cases}d^1_\beta(a)+1 & (0\leqq x\leqq T_{\beta,1}(a)),\\ d^1_\beta(a) & (T_{\beta,1}(a)<x<1),\\ 0 & (1\leqq x\leqq [\beta]/(\beta-1))\end{cases}$$

if $T_{\beta,1}(a)<1$ and

$$\sharp\{0\leqq y\leqq a;\ x=T_{\beta,1}(y)\}=\begin{cases}d^1_\beta(a)+1 & (0\leqq x<1),\\ 1 & (1\leqq x\leqq T_{\beta,1}(a)),\\ 0 & (T_{\beta,1}(a)<x\leqq [\beta]/(\beta-1))\end{cases}$$

if $T_{\beta,1}(a)\geqq 1$. Therefore, we obtain

$$\sharp\{0\leqq y\leqq a;\ x=T_{\beta,1}(y)\}=\chi_{[0,T_{\beta,1}(a)]}(x)+d^1_\beta(a)\chi_{[0,1]}(x)$$

for λ_β-a.e. $x\in J_\beta$.

(2) The statement follows from taking $a=[\beta]/(\beta-1)$ in the statement (1).

(3) Let f be a function from J_β to $\mathbb{R}$. Since $l_\beta^{-1}(x)=l_\beta(x)$ for $x\in J_\beta$, we have

$$\begin{aligned}\mathcal{L}_{l_\beta}f(x)&=f(l_\beta^{-1}(x))\\&=f(l_\beta(x))\end{aligned}$$

for λ_β-a.e. $x\in J_\beta$. □

We set $\bar{i}=1-i$ for $i\in\{0,1\}$ and $\bar{\omega}=(\bar{\omega_i})_{i=1}^\infty$ for $\omega\in\Omega$ in what follows.

LEMMA 4.2. *Let $\mathcal{L}_{T_{\beta,1}}$ be the Perron–Frobenius operator of the map $T_{\beta,1}$. Then we have*

$$\mathcal{L}_{T_{\beta,1}}\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]}=\frac{1}{\beta}\left(d^1_\beta(T_{\beta,\omega_1\cdots\omega_n}(1))\chi_{[0,1]}+\chi_{[0,T_{\beta,\omega_1\cdots\omega_n1}(1)]}\right)\tag{4.1}$$

and

$$\begin{aligned}&\mathcal{L}_{T_{\beta,1}}\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1),[\beta]/(\beta-1)]}\\&\quad=\frac{1}{\beta}\left(d^0_\beta(T_{\beta,\bar{\omega}_1\cdots\bar{\omega}_n}(1))\chi_{[0,1]}+\chi_{[T_{\beta,\omega_1\cdots\omega_n1}(l_\beta(1)),[\beta]/(\beta-1)]}\right)\end{aligned}\tag{4.2}$$

for $n\in\mathbb{N}$ and $\omega_1\cdots\omega_n\in\{0,1\}^n$ or $n=0$.

Proof. The equation (4.1) follows from taking $a=T_{\beta,\omega_n\cdots\omega_1}(1)$ in Lemma 4.1(1).

By Lemma 4.1(1) and (2), we get

$$\begin{aligned}&\mathcal{L}_{T_{\beta,1}}\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta(1)),[\beta]/(\beta-1)]}\\&\quad=\mathcal{L}_{T_{\beta,1}}\left(\chi_{J_\beta}-\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1)]}\right)\end{aligned}$$

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$$= \frac{1}{\beta}\big(\chi_{J_\beta} + [\beta]\chi_{[0,1]} - \chi_{[0,T_{\beta,\omega_1\cdots\omega_n 1}(l_\beta 1)]} - d_\beta^1(T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1))\chi_{[0,1]}\big)$$
$$= \frac{1}{\beta}\big\{\big([\beta] - d_\beta^1(l_\beta \circ T_{\beta,\bar{\omega}_1\cdots\bar{\omega}_n}(1))\big)\chi_{[0,1]} + \chi_{[T_{\beta,\omega_1\cdots\omega_n 1}(l_\beta 1),[\beta]/(\beta-1)]}\big\}$$
$$= \frac{1}{\beta}\big(d_\beta^0(T_{\beta,\bar{\omega}_1\cdots\bar{\omega}_n}(1))\chi_{[0,1]} + \chi_{[T_{\beta,\omega_1\cdots\omega_n 1}(l_\beta 1),[\beta]/(\beta-1)]}\big)$$

for $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$. The case $n = 0$ is similar. □

For $i \in \{0, 1\}$, we define the function $B^i(\beta, p) : ((1, +\infty) \setminus \mathbb{N}) \times [0, 1] \to \mathbb{R}$ associated with β-expansions of 1 by

$$B^i(\beta, p) = \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n = i}} d_\beta^i(T_{\beta,\omega_1\cdots\omega_{n-1}}(1))m_{1-p}([\omega_1 \cdots \omega_n]_1^n).$$

LEMMA 4.3. *The functions $B^0(\beta, p)$ and $B^1(\beta, p)$ have the following properties.*
(1) $B^0(\beta, p), B^1(\beta, p) \geqq 0$ *and* $B^0(\beta, 0) = B^1(\beta, 1) = 0$.
(2) $B^0(\beta, p) + B^1(\beta, p) = 1$.
(3) *The function $B^0(\beta, p)$ can be represented by*

$$B^0(\beta, p) = p \cdot \left\{1 - \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n = 1}} \chi_S(T_{\beta,\omega_1\cdots\omega_{n-1}}(1))m_{1-p}([\omega_1 \cdots \omega_n]_1^n)\right\}.$$

Proof. (1) The statement easily follows from the definition of the functions.
(2) Since

$$1 = \sum_{n=1}^{\infty} \frac{d_\beta^{\omega_n}(T_{\beta,\omega_1\cdots\omega_{n-1}}(1))}{\beta^n}$$

for $\omega = (\omega_n)_{n=1}^{\infty} \in \Omega$, we have

$$\begin{aligned} B^0(\beta, p) + B^1(\beta, p) &= \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n = 0}} d_\beta^0(T_{\beta,\omega_1\cdots\omega_{n-1}}(1))m_{1-p}([\omega_1 \cdots \omega_n]_1^n) \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n = 1}} d_\beta^1(T_{\beta,\omega_1\cdots\omega_{n-1}}(1))m_{1-p}([\omega_1 \cdots \omega_n]_1^n) \\ &= \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\omega_1\cdots\omega_n} d_\beta^{\omega_n}(T_{\beta,\omega_1\cdots\omega_{n-1}}(1))m_{1-p}([\omega_1 \cdots \omega_n]_1^n) \\ &= \int_\Omega \sum_{n=1}^{\infty} \frac{d_\beta^{\omega_n}(T_{\beta,\omega_1\cdots\omega_{n-1}}(1))}{\beta^n}\, dm_{1-p}(\omega) \\ &= \int_\Omega 1\, dm_{1-p}(\omega) \\ &= 1. \end{aligned}$$

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(3) By the statement (2), we get

$$\begin{aligned}
&B^0(\beta,\, p)\\
&\quad = 1 - B^1(\beta,\, p)\\
&\quad = 1 - \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=1}} d_\beta^1(T_{\beta,\omega_1\cdots\omega_{n-1}}(1)) m_{1-p}([\omega_1\cdots\omega_n]_1^n)\\
&\quad = 1 - \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=1}} \left(d_\beta^0(T_{\beta,\omega_1\cdots\omega_{n-1}}(1)) + \chi_S(T_{\beta,\omega_1\cdots\omega_{n-1}}(1))\right)\\
&\qquad \times m_{1-p}([\omega_1\cdots\omega_n]_1^n)\\
&\quad = 1 - \frac{1-p}{p} B^0(\beta,\, p) - \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=1}} \chi_S(T_{\beta,\omega_1\cdots\omega_{n-1}}(1)) m_{1-p}([\omega_1\cdots\omega_n]_1^n).
\end{aligned}$$

Therefore, we have

$$B^0(\beta,\, p) = p \cdot \left\{1 - \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=1}} \chi_S(T_{\beta,\omega_1\cdots\omega_{n-1}}(1)) m_{1-p}([\omega_1\cdots\omega_n]_1^n)\right\}. \quad \square$$

The main theorem in this section is the following.

THEOREM 4.4. *Let $\mu_{\beta,p}$ be the probability measure given in §3 and $h_{\beta,p}$ the density function of $\mu_{\beta,p}$. We set*

$$\begin{aligned}
\rho_{\beta,p} = {} & B^0(\beta,\, p)\left(\chi_{[0,1]} + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\omega_1\cdots\omega_n} m_p([\omega_1\cdots\omega_n]_1^n)\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]}\right)\\
& + B^0(\beta,\, 1-p)\left(\chi_{[l_\beta 1, c(\beta)]} + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\omega_1\cdots\omega_n} m_p([\omega_1\cdots\omega_n]_1^n)\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1), c(\beta)]}\right),
\end{aligned}$$

where $c(\beta) = [\beta]/(\beta-1)$. Then the function $\rho_{\beta,p}$ is of bounded variation on J_β and satisfies the functional equation $\rho_{\beta,1-p} = \rho_{\beta,p} \circ l_\beta$. Furthermore, we have

$$\rho_{\beta,p} = p\mathcal{L}_{T_{\beta,1}}\rho_{\beta,p} + (1-p)\mathcal{L}_{T_{\beta,0}}\rho_{\beta,p}. \tag{4.3}$$

Therefore, the density function $h_{\beta,p}$ is given by

$$h_{\beta,p} = \frac{1}{F(\beta,\, p)}\rho_{\beta,p},$$

where $F(\beta,\, p)$ is the normalizing constant.

Proof. It is clear that the function $\rho_{\beta,p}$ is of bounded variation on J_β by definition. Since $T_{\beta,1} = l_\beta \circ T_{\beta,0} \circ l_\beta$ and $m_p([\omega_1\cdots\omega_n]_1^n) = m_{1-p}([\bar{\omega}_1\cdots\bar{\omega}_n]_1^n)$ for $n \in \mathbb{N}$

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and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$, we have

$$\begin{aligned}
&\rho_{\beta,p} \circ l_\beta \\
&\quad = B^0(\beta, p)\Bigg(\chi_{[0,1]} \circ l_\beta + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\omega_1 \cdots \omega_n} m_p([\omega_1 \cdots \omega_n]_1^n) \chi_{[0, T_{\beta,\omega_1 \cdots \omega_n}(1)]} \circ l_\beta\Bigg) \\
&\qquad + B^0(\beta, 1-p)\Bigg(\chi_{[l_\beta 1, c(\beta)]} \circ l_\beta + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \\
&\qquad \times \sum_{\omega_1 \cdots \omega_n} m_p([\omega_1 \cdots \omega_n]_1^n) \chi_{[T_{\beta,\omega_1 \cdots \omega_n}(l_\beta 1), c(\beta)]} \circ l_\beta\Bigg) \\
&\quad = B^0(\beta, p)\Bigg(\chi_{[l_\beta(1), c(\beta)]} + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\bar\omega_1 \cdots \bar\omega_n} m_{1-p}([\bar\omega_1 \cdots \bar\omega_n]_1^n) \chi_{[T_{\beta,\bar\omega_1 \cdots \bar\omega_n}(1), c(\beta)]}\Bigg) \\
&\qquad + B^0(\beta, 1-p)\Bigg(\chi_{[0,1]} + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\bar\omega_1 \cdots \bar\omega_n} m_{1-p}([\bar\omega_1 \cdots \bar\omega_n]_1^n) \chi_{[0, T_{\beta,\bar\omega_1 \cdots \bar\omega_n}(1)]}\Bigg) \\
&\quad = \rho_{\beta,1-p}.
\end{aligned}$$

Let us give the proof of the equation (4.3) in the following. Since $\mathcal{L}_{T_{\beta,1}}$ is linear and continuous on $L^1(\lambda_\beta)$, by Lemma 4.2, we have

$$\begin{aligned}
&p\mathcal{L}_{T_{\beta,1}} \rho_{\beta,p} \\
&\quad = B^0(\beta, p)\Bigg\{\frac{p}{\beta}\big(d_\beta^1(1)\chi_{[0,1]} + \chi_{[0, T_{\beta,1}(1)]}\big) \\
&\qquad + \sum_{n=1}^{\infty} \frac{p}{\beta^{n+1}} \sum_{\omega_1 \cdots \omega_n} m_p([\omega_1 \cdots \omega_n]_1^n) \cdot \big(d_\beta^1(T_{\beta,\omega_1 \cdots \omega_n}(1))\chi_{[0,1]} + \chi_{[0, T_{\beta,\omega_1 \cdots \omega_n 1}(1)]}\big)\Bigg\} \\
&\qquad + B^0(\beta, 1-p)\Bigg\{\frac{p}{\beta}\big(d_\beta^0(1)\chi_{[0,1]} + \chi_{[T_{\beta,1}(l_\beta 1), c(\beta)]}\big) \\
&\qquad + \sum_{n=1}^{\infty} \frac{p}{\beta^{n+1}} \sum_{\omega_1 \cdots \omega_n} m_p([\omega_1 \cdots \omega_n]_1^n) \\
&\qquad \times \big(d_\beta^0(T_{\beta,\bar\omega_1 \cdots \bar\omega_n}(1))\chi_{[0,1]} + \chi_{[T_{\beta,\omega_1 \cdots \omega_n 1}(l_\beta 1), c(\beta)]}\big)\Bigg\} \\
&\quad = \Bigg\{B^0(\beta, p)\Bigg(\sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1 \cdots \omega_n; \\ \omega_n = 1}} d_\beta^1(T_{\beta,\omega_1 \cdots \omega_{n-1}}(1)) m_p([\omega_1 \cdots \omega_n]_1^n)\Bigg) \\
&\qquad + B^0(\beta, 1-p)\Bigg(\sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1 \cdots \omega_n; \\ \omega_n = 0}} d_\beta^0(T_{\beta,\omega_1 \cdots \omega_{n-1}}(1)) m_{1-p}([\omega_1 \cdots \omega_n]_1^n)\Bigg)\Bigg\}\chi_{[0,1]} \\
&\qquad + B^0(\beta, p)\Bigg(\sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\substack{\omega_1 \cdots \omega_n; \\ \omega_n = 1}} m_p([\omega_1 \cdots \omega_n]_1^n) \chi_{[0, T_{\beta,\omega_1 \cdots \omega_n}(1)]}\Bigg)
\end{aligned}$$

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$$+ B^0(\beta, 1-p)\Bigg(\sum_{n=1}^{\infty}\frac{1}{\beta^n}\sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=1}} m_p([\omega_1\cdots\omega_n]_1^n)\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1),c(\beta)]}\Bigg)$$

$$= \{B^0(\beta, p)B^1(\beta, 1-p) + B^0(\beta, 1-p)B^0(\beta, p)\}\chi_{[0,1]}$$

$$+ \sum_{n=1}^{\infty}\frac{1}{\beta^n}\sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=1}} m_p([\omega_1\cdots\omega_n]_1^n)$$

$$\times \big(B^0(\beta, p)\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]} + B^0(\beta, 1-p)\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1),c(\beta)]}\big).$$

Therefore, by Lemma 4.3(2), we get

$$p\mathcal{L}_{T_{\beta,1}}\rho_{\beta,p} = B^0(\beta, p)\chi_{[0,1]} + \sum_{n=1}^{\infty}\frac{1}{\beta^n}\sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=1}} m_p([\omega_1\cdots\omega_n]_1^n)$$

$$\times \big(B^0(\beta, p)\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]} + B^0(\beta, 1-p)\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1),c(\beta)]}\big). \quad (4.4)$$

We note that $\mathcal{L}_{T_{\beta,0}} = \mathcal{L}_{l_\beta}\circ\mathcal{L}_{T_{\beta,1}}\circ\mathcal{L}_{l_\beta}$ by the definition of the maps and the basic property of the Perron–Frobenius operators. By the equation $\rho_{\beta,1-p} = \rho_{\beta,p}\circ l_\beta$ and Lemma 4.1(3), we have

$$\begin{aligned} p\mathcal{L}_{T_{\beta,0}}\rho_{\beta,1-p} &= p\mathcal{L}_{l_\beta}\circ\mathcal{L}_{T_{\beta,1}}\circ\mathcal{L}_{l_\beta}(\rho_{\beta,p}\circ l_\beta)\\ &= \mathcal{L}_{l_\beta}(p\mathcal{L}_{T_{\beta,1}}\rho_{\beta,p}).\end{aligned}$$

By using the equation (4.4), we get

$$\mathcal{L}_{l_\beta}(p\mathcal{L}_{T_{\beta,1}}\rho_{\beta,p})$$

$$= B^0(\beta, p)\chi_{[0,1]}\circ l_\beta + \sum_{n=1}^{\infty}\frac{1}{\beta^n}\sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=1}} m_p([\omega_1\cdots\omega_n]_1^n)$$

$$\times \big(B^0(\beta, p)\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]}\circ l_\beta + B^0(\beta, 1-p)\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1),c(\beta)]}\circ l_\beta\big)$$

$$= B^0(\beta, p)\chi_{[l_\beta 1,c(\beta)]} + \sum_{n=1}^{\infty}\frac{1}{\beta^n}\sum_{\substack{\bar\omega_1\cdots\bar\omega_n;\\ \bar\omega_n=0}} m_{1-p}([\bar\omega_1\cdots\bar\omega_n]_1^n)$$

$$\times \big(B^0(\beta, 1-p)\chi_{[0,T_{\beta,\bar\omega_1\cdots\bar\omega_n}(1)]} + B^0(\beta, p)\chi_{[T_{\beta,\bar\omega_1\cdots\bar\omega_n}(l_\beta 1),c(\beta)]}\big).$$

Therefore, we have

$$(1-p)\mathcal{L}_{T_{\beta,0}}\rho_{\beta,p} = B^0(\beta, 1-p)\chi_{[l_\beta 1,c(\beta)]} + \sum_{n=1}^{\infty}\frac{1}{\beta^n}\sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=0}} m_p([\omega_1\cdots\omega_n]_1^n)$$

$$\times \big(B^0(\beta, p)\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]} + B^0(\beta, 1-p)\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1),c(\beta)]}\big). \quad (4.5)$$

By combining the equations (4.4) and (4.5), we obtain

$$p\mathcal{L}_{T_{\beta,1}}\rho_{\beta,p} + (1-p)\mathcal{L}_{T_{\beta,0}}\rho_{\beta,p} = \rho_{\beta,p}. \qquad \square$$

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Remark 4.5.

(1) In the case $p = 1/2$, we do not need to consider the contribution of the coefficient function B^0 since the coefficient function $B^0(\beta, 1/2)$ in $\rho_{\beta,1/2}$ cancels out that in the normalizing constant $F(\beta, 1/2)$. Therefore, Theorem 4.4 is an extension of [**9**, Theorem 2.1], which gives $h_{\beta,p}$ explicitly in the case $p = 1/2$.

(2) The function $\rho_{\beta,p}$ itself can be defined at $p = 1$ and $p = 0$. If $p = 1$, then

$$\rho_{\beta,1} = \sum_{n=0}^{\infty} \frac{1}{\beta^n} \chi_{[0,T_{\beta,1}^n(1)]},$$

since $B^0(\beta, 1) = 1$ and $B^0(\beta, 0) = 0$ by Lemma 4.3(1) and (2). This shows that $h_{\beta,1} = F(\beta, 1)^{-1}\rho_{\beta,1}$ coincides with the density function of the greedy map $T_{\beta,1}$ (see [**4**, **8**, **15**]). We note that $h_{\beta,0} = h_{\beta,1} \circ l_\beta$, so the function $h_{\beta,0}$ coincides with the unique invariant density of the map $T_{\beta,0}$.

If β is a univoque number, i.e. $1 \in J_\beta$ has a unique β-expansion, then $1 \in E$ and $T_{\beta,\omega_1\cdots\omega_n}(1) \in E$ for every $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$ (see [**5**, §2]). In this case, since

$$\chi_S(T_{\beta,\omega_1\cdots\omega_n}(1)) = 0$$

for every $n \in \mathbb{N}$, $\omega_1 \cdots \omega_n \in \{0, 1\}^n$ and $n = 0$, by Lemma 4.3(3), we get

$$B^0(\beta, p) = p.$$

This yields the following corollary.

COROLLARY 4.6. *Let β be a univoque number. Then the density function $h_{\beta,p}$ is given by*

$$h_{\beta,p} = \frac{1}{F(\beta, p)} \rho_{\beta,p},$$

where

$$\rho_{\beta,p} = p\left(\chi_{[0,1]} + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\omega_1\cdots\omega_n} m_p([\omega_1 \cdots \omega_n]_1^n)\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]}\right)$$
$$+ (1 - p)\left(\chi_{[l_\beta 1,c(\beta)]} + \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\omega_1\cdots\omega_n} m_p([\omega_1 \cdots \omega_n]_1^n)\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1),c(\beta)]}\right)$$

and $F(\beta, p)$ is the normalizing constant.

5. *Applications of Theorem 4.4*

In the rest of this paper, we investigate some properties of the density function $h_{\beta,p}$ as an application of Theorem 4.4.

First, we evaluate upper and lower bounds of $h_{\beta,p}$ and study the asymptotic behavior of the value $\mu_{\beta,p}(S)$ as $\beta \nearrow N + 1$, where $N \in \mathbb{N}$ or $\beta \nearrow +\infty$.

We denote by λ the Lebesgue measure on $\mathbb{R}$ throughout this section.

THEOREM 5.1. *The density function $h_{\beta,p}$ has the following properties.*

(1) *[**6**, Theorem 4] There exists a positive constant $C_{\beta,p} > 0$ such that $h_{\beta,p}(x) > C_{\beta,p}$ for $x \in J_\beta$.*

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(2) *Suppose that* $\beta > 3/2$. *Then*

$$\left(1-\frac{1}{\beta}\right)\chi_{[l_\beta 1,1]}(x) \leqq h_{\beta,p}(x) \leqq \left(1-\frac{1}{\beta}\right)^{-1}\left(2\cdot\frac{\beta-1}{[\beta]}-1\right)^{-1}\chi_{J_\beta}(x)$$

for $x \in J_\beta$.

(3) *Let* $N \in \mathbb{N}$ *and* $p \in [0, 1]$. *Then*

$$h_{\beta,p} \to \chi_{[0,1]}$$

in $L^1(\lambda)$ *as* $\beta \nearrow N+1$. *Furthermore,*

$$h_{\beta,p} \to \chi_{[0,1]}$$

in $L^1(\lambda)$ *as* $\beta \nearrow +\infty$.

(4) *Let* $N \in \mathbb{N}$. *Then*

$$\mu_{\beta,p}(S) \to 0 \quad \text{as } \beta \nearrow N+1.$$

Furthermore,

$$\mu_{\beta,p}(S) \to 0 \quad \text{as } \beta \nearrow +\infty.$$

Proof. (1) We will give an another proof of Theorem 4 in [**6**] by applying Theorem 4.4. Let $\rho_{\beta,p}$ be the function defined in Theorem 4.4. It is sufficient to show that there exists a positive constant $D_{\beta,p} > 0$ such that $\rho_{\beta,p}(x) \geqq D_{\beta,p}$ for $x \in J_\beta$.

Suppose that $\beta \geqq 3/2$. In this case, since $l_\beta 1 = [\beta]/(\beta-1) \leqq 1$, by the definition of the function $\rho_{\beta,p}$, we have

$$\begin{aligned}\rho_{\beta,p}(x) &\geqq B^0(\beta, p)\chi_{[0,1]} + B^0(\beta, 1-p)\chi_{[l_\beta 1,[\beta]/(\beta-1)]}(x)\\ &\geqq \min\{B^0(\beta, p), B^0(\beta, 1-p)\}\end{aligned}$$

for $x \in J_\beta$. We note that by definition $B^0(\beta, p), B^0(\beta, 1-p) > 0$ for $p \in (0, 1)$. Therefore, by taking $D_{\beta,p} = \min\{B^0(\beta, p), B^0(\beta, 1-p)\}$, we get the assertion.

Suppose that $1 < \beta < 3/2$. Let us take a positive integer $n \in \mathbb{N}$ such that $\beta^n > [\beta]/(\beta-1) - 1$. Then

$$\begin{aligned}\rho_{\beta,p}(x) &\geqq B^0(\beta, p)(1-p)^n\chi_{[0,T^n_{\beta,0}(1)]}(x) + B^0(\beta, 1-p)\chi_{[l_\beta 1,[\beta]/(\beta-1)]}(x)\\ &\geqq \min\{B^0(\beta, p)(1-p)^n, B^0(\beta, 1-p)\}\end{aligned}$$

for $x \in J_\beta$. Therefore, by taking $D_{\beta,p} = \min\{B^0(\beta, p)(1-p)^n, B^0(\beta, 1-p)\}$, we get the assertion.

(2) Since $\beta > 3/2$, by the definition of the function $\rho_{\beta,p}$, we have

$$\begin{aligned}\rho_{\beta,p}(x) &\geqq B^0(\beta, p)\chi_{[0,1]} + B^0(\beta, 1-p)\chi_{[l_\beta 1,[\beta]/(\beta-1)]}(x)\\ &\geqq (B^0(\beta, p) + B^0(\beta, 1-p))\chi_{[l_\beta 1,1]}(x)\end{aligned}$$

and

$$\begin{aligned}\rho_{\beta,p}(x) &\leqq \sum_{n=0}^{\infty}\frac{1}{\beta^n}(B^0(\beta, p) + B^0(\beta, 1-p))\chi_{J_\beta}(x)\\ &\leqq \left(1-\frac{1}{\beta}\right)^{-1}(B^0(\beta, p) + B^0(\beta, 1-p))\chi_{J_\beta}(x)\end{aligned}$$

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for $x \in J_\beta$. Therefore, we get

$$\begin{aligned} F(\beta, p) &\geqq (B^0(\beta, p) + B^0(\beta, 1-p))\lambda_\beta([l_\beta 1, 1]) \\ &\geqq (B^0(\beta, p) + B^0(\beta, 1-p))\left(2 \cdot \frac{\beta - 1}{[\beta]} - 1\right) \end{aligned}$$

and

$$\begin{aligned} F(\beta, p) &\leqq \left(1 - \frac{1}{\beta}\right)^{-1} (B^0(\beta, p) + B^0(\beta, 1-p))\lambda_\beta(J_\beta) \\ &\leqq \left(1 - \frac{1}{\beta}\right)^{-1} (B^0(\beta, p) + B^0(\beta, 1-p)). \end{aligned}$$

By combining these evaluations, we obtain the desired result.

(3) Let $\beta \in (N, N+1)$ with $\langle \beta \rangle > 1/\beta$. By the definition of the interval $E_{[\beta]}$, we have $1 \in \mathrm{int}(E_{[\beta]})$, where $\mathrm{int}(A)$ denotes the interior of A. Since the function $\beta \mapsto T_{\beta,1}(1)$ is continuous on $(N, N+1)$, we have

$$T_{\beta,1}(1) = \langle \beta \rangle \to 1$$

as $\beta \nearrow N+1$. We note that $T_{\beta,1}(x) = T_{\beta,0}(x)$ for $x \in E_{[\beta]}$. Therefore, for $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$, we have

$$T_{\beta,\omega_1 \cdots \omega_n}(1) \to 1$$

as $\beta \nearrow N+1$. In addition, since $l_\beta 1 = [\beta]/(\beta - 1) - 1 \searrow 0$ as $\beta \nearrow N+1$ and $T_{\beta,\omega_n \cdots \omega_1}(l_\beta 1) = l_\beta \circ T_{\beta,\bar{\omega}_n \cdots \bar{\omega}_1}(1)$, we get

$$T_{\beta,\omega_1 \cdots \omega_n}(l_\beta 1) \to 0$$

as $\beta \nearrow N+1$. Then it is easy to see that

$$\begin{aligned} \chi_{[0, T_{\beta,\omega_1 \cdots \omega_n}(1)]}(x) &\to \chi_{[0,1]}(x), \\ \chi_{[T_{\beta,\omega_1 \cdots \omega_n}(l_\beta 1), [\beta]/(\beta-1)]}(x) &\to \chi_{[0,1]}(x) \end{aligned}$$

and

$$\chi_{[l_\beta 1, [\beta]/(\beta-1)]}(x) \to \chi_{[0,1]}(x)$$

as $\beta \nearrow N+1$ for λ-a.e. $x \in [0, +\infty)$, where $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$. It is also easy to see that $B^0(\beta, p) \to p$ as $\beta \nearrow N+1$ by Lemma 4.3(3) and $T_{\beta,\omega_n \cdots \omega_1}(1) \in \mathrm{int}(E_{[\beta]})$ for $0 \leqq n \leqq m$, $\omega_1 \cdots \omega_n \in \{0, 1\}^n$ if β is sufficiently close to $N+1$.

Let

$$\begin{aligned} \rho^m_{\beta,p} = B^0(\beta, p)&\left(\chi_{[0,1]} + \sum_{n=1}^{m} \frac{1}{\beta^n} \sum_{\omega_1 \cdots \omega_n} m_p([\omega_1 \cdots \omega_n]^n_1)\chi_{[0, T_{\beta,\omega_1 \cdots \omega_n}(1)]}\right) \\ &+ B^0(\beta, 1-p)\left(\chi_{[l_\beta 1, c(\beta)]} + \sum_{n=1}^{m} \frac{1}{\beta^n}\right. \\ &\left. \times \sum_{\omega_1 \cdots \omega_n} m_p([\omega_1 \cdots \omega_n]^n_1)\chi_{[T_{\beta,\omega_1 \cdots \omega_n}(l_\beta 1), c(\beta)]}\right) \end{aligned}$$

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for $m \in \mathbb{N}$, where $c(\beta) = [\beta]/(\beta - 1)$. If $\beta \in (N + 1/2, N + 1)$, then we get

$$\begin{aligned} |\rho_{\beta,p}^m(x)| &\leqq 2 \cdot \left(1 - \frac{1}{\beta}\right)^{-1} \chi_{[0,[\beta]/(\beta-1)]}(x) \\ &\leqq 2 \cdot \left(1 - \frac{1}{N+1/2}\right)^{-1} \chi_{[0,(N+1)/(N+1/2)]}(x) \end{aligned}$$

for every $m \in \mathbb{N}$. Hence, by the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned} \rho_{\beta,p} &\to \sum_{n=0}^{\infty} \frac{1}{(N+1)^n} \big(p\chi_{[0,1]} + (1-p)\chi_{[0,1]}\big) \\ &= \left(1 - \frac{1}{N+1}\right)^{-1} \chi_{[0,1]} \end{aligned}$$

as $\beta \to N + 1$ in $L^1(\lambda)$. This yields that

$$\begin{aligned} h_{\beta,p} &= F(\beta, p)^{-1} \rho_{\beta,p} \\ &\to \chi_{[0,1]} \end{aligned}$$

as $\beta \to N + 1$ in $L^1(\lambda)$.

We note that

$$1 - \frac{1}{\beta}, \quad 2 \cdot \frac{\beta - 1}{[\beta]} - 1 \to 1 \quad \text{as } \beta \nearrow +\infty$$

and

$$\chi_{[l_\beta 1,1]}(x), \quad \chi_{J_\beta}(x) \to \chi_{[0,1]}(x) \quad \text{as } \beta \nearrow +\infty$$

for λ-a.e. $x \in [0, +\infty)$. Therefore, by the statement (2) and the Lebesgue bounded convergence theorem, we have $h_{\beta,p} \to \chi_{[0,1]}$ as $\beta \nearrow +\infty$ in $L^1(\lambda)$.

(4) By the definition of the set S, we have

$$\begin{aligned} \lambda_\beta(S) &= \left(\frac{[\beta]}{\beta - 1}\right)^{-1} \frac{[\beta]}{\beta(\beta - 1)} (1 - \langle\beta\rangle) \\ &= \frac{1 - \langle\beta\rangle}{\beta}, \end{aligned}$$

where $\langle\beta\rangle$ denotes the fractional part of β. Since $(1 - \langle\beta\rangle)/\beta \to 0$ as $\beta \nearrow N + 1$ or $\beta \nearrow +\infty$, we get the assertion by the statement (3). □

Next, we study the continuity of the density function $h_{\beta,p}$ as a function of p.

LEMMA 5.2. *Let $\beta \in (1, +\infty) \setminus \mathbb{N}$ and $A \in \mathcal{B}(J_\beta)$. Then each of the following functions can be extended to an analytic function on some region in $\mathbb{C}$.*

(1) *The function $p \mapsto B^0(\beta, p)$ from $[0, 1]$ into $\mathbb{R}$.*
(2) *The function $p \mapsto \rho_{\beta,p}$ from $[0, 1]$ into $L^1(\lambda)$.*
(3) *The function $p \mapsto \int_A \rho_{\beta,p}\, d\lambda_\beta$ from $[0, 1]$ into $\mathbb{R}$.*

Proof. Let X be a Banach space with the norm $\|\cdot\|$. We shall show that a series of the form

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\omega_1 \cdots \omega_n} z^{\sum_{i=1}^n \omega_i} \cdot (1 - z)^{n - \sum_{i=1}^n \omega_i} \cdot c_{\omega_1 \cdots \omega_n},$$

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where $c_{\omega_1\cdots\omega_n} \in X$ is such that $\{c_{\omega_1\cdots\omega_n}\}$ is bounded, that is, there exists $C > 0$ such that $\|c_{\omega_1\cdots\omega_n}\| \leqq C$ for $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$, converges absolutely and locally uniformly on some region in $\mathbb{C}$ including the interval $[0, 1]$.

We set

$$R = \bigcup_{q\in[0,1]} B(q, (\beta - 1)/2),$$

where $B(q, r)$ denotes the open ball on $\mathbb{C}$ centered at q with the radius $r > 0$. We denote $z \in R$ by $z = q + re^{i\theta}$, $q \in [0, 1]$, $0 \leqq r < (\beta - 1)/2$ and $0 \leqq \theta < 2\pi$. Then

$$\begin{aligned}
\|f(z)\| &\leqq C \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^n} \sum_{\omega_1\cdots\omega_n} |z|^{\sum_{i=1}^n \omega_i} |1 - z|^{n-\sum_{i=1}^n \omega_i} \\
&= C \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^n} (|z| + |1 - z|)^n \\
&\leqq C \cdot \sum_{n=1}^{\infty} \frac{1}{\beta^n} (1 + 2r)^n \\
&< C \cdot \sum_{n=1}^{\infty} \alpha^n,
\end{aligned}$$

where $\alpha = (1 + 2r)/\beta < 1$. Hence, we see that the series $f(z)$ converges absolutely and locally uniformly on R.

(1) From Lemma 4.3(3), we get the assertion by taking $X = \mathbb{C}$ and $c_{\omega_1\cdots\omega_n} = \chi_S(T_{\beta,\omega_1\cdots\omega_n}(1))$ in the above argument.

(2) We set

$$\rho_{\beta,p} = B^0(\beta, p)\rho^1_{\beta,p} + B^0(\beta, 1 - p)\rho^0_{\beta,p}.$$

For the function $\rho^1_{\beta,p}$, we get that the map $p \mapsto \rho^1_{\beta,p}$ is analytic by taking $X = L^1(\lambda)$ and $c_{\omega_1\cdots\omega_n} = \chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]}$ in the above argument. Since the function $p \mapsto B^0(\beta, p)$ is analytic, we get that the map $p \mapsto B^0(\beta, p)\rho^1_{\beta,p}$ is analytic. In the same way, we have that $p \mapsto B^0(\beta, 1 - p)\rho^0_{\beta,p}$ is also analytic. Combining these results, we get the statement.

(3) We obtain the statement in the same way as in the statement (2).

□

THEOREM 5.3. *Let $\beta \in (1, +\infty) \setminus \mathbb{N}$ and $A \in \mathcal{B}(J_\beta)$. Then each of the following functions can be extended to an analytic function on some region in $\mathbb{C}$.*

(1) *The function $p \mapsto h_{\beta,p}$ from $[0, 1]$ into $L^1(\lambda)$.*

(2) *The function $p \mapsto \mu_{\beta,p}(A)$ from $[0, 1]$ into $\mathbb{R}$.*

Proof. By taking $A = J_\beta$ in Lemma 5.2(3), the function $p \mapsto F(\beta, p)$ is analytic in some region in $\mathbb{C}$ including $[0, 1]$. By the definition of the normalizing constant $F(\beta, p)$, we have that $F(\beta, p) > 0$ for $p \in [0, 1]$. Therefore, we can take a region $R' \subset \mathbb{C}$ such that the function $p \mapsto F(\beta, p)$ has no zero in R'. Therefore, the statement (1) follows from Lemma 5.2(2) and the statement (2) follows from Lemma 5.2(3). □

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Finally, we investigate the continuity of the density function $h_{\beta,p}$ as a function of β. Let us denote by EP_β the set of all end points of E_k and S_k, that is,

$$EP_\beta = \left\{0, \frac{1}{\beta}, \dots, \frac{[\beta]}{\beta}\right\} \cup \left\{\frac{[\beta]}{\beta-1} - \frac{[\beta]}{\beta}, \dots, \frac{[\beta]}{\beta-1} - \frac{1}{\beta}, \frac{[\beta]}{\beta-1}\right\}.$$

We define the set $\mathcal{F}$ by

$$\mathcal{F} = \left\{\beta \in (1, +\infty) \setminus \mathbb{N} \;\middle|\; \begin{array}{l} 1 \in EP_\beta \text{ or there exists } n \in \mathbb{N} \text{ and} \\ \omega_1 \cdots \omega_n \in \{0, 1\}^n \text{ such that } T_{\beta,\omega_1\cdots\omega_n}(1) \in EP_\beta \end{array}\right\}.$$

We note that the set $\mathcal{F}$ is an analog of the Parry numbers for the random β-transformation. We recall that β is called a Parry number if 1 has a finite orbit under the β-transformation (e.g., [**15**]). The properties of the Parry numbers and their analog for a generalization of the β-transformation were investigated, e.g., in [**12**, **15**, **19**].

PROPOSITION 5.4. *The set $\mathcal{F}$ has the following properties.*

(1) *The set $\mathcal{F}$ is countable.*

(2) *The set $\mathcal{F}$ is dense in* $(1, +\infty)$.

(3) *If $\beta \in \mathcal{F}$, then β is a Perron number, i.e. an algebraic integer $\beta > 1$ of which the absolute values of the other conjugates are less than β.*

Proof. (1) Let $\beta \in \mathcal{F}$. Then β is an algebraic integer since 1 has a β-expansion of the form

$$1 = \sum_{i=1}^{n} \frac{a_i}{\beta^i}$$

or

$$1 = \sum_{i=1}^{n} \frac{a_i}{\beta^i} + \frac{[\beta]}{\beta^n(\beta-1)},$$

where $n \in \mathbb{N}$ and $a_i \in \{0, 1, \dots, [\beta]\}$ for $i = 1, \dots, n$. Since the set of all algebraic integers is countable, so is the set $\mathcal{F}$.

(2) If β is a simple Parry number, i.e. there exists $n \in \mathbb{N} \cup \{0\}$ such that $T^n_{\beta,1}(1) \in \{0, 1/\beta, \dots, [\beta]/\beta\}$, then $\beta \in \mathcal{F} \cup \mathbb{N} \setminus \{1\}$. Since the set of all simple Parry numbers is dense in $(1, +\infty)$ (see [**15**, Theorem 5]), so is the set $\mathcal{F}$.

(3) Let $\beta \in \mathcal{F}$. Then 1 has a β-expansion of the form

$$1 = \sum_{i=1}^{n} \frac{a_i}{\beta^i} \tag{5.1}$$

or

$$1 = \sum_{i=1}^{n} \frac{a_i}{\beta^i} + \frac{1}{\beta^n} \cdot \frac{[\beta]}{(\beta-1)}, \tag{5.2}$$

where $n \in \mathbb{N}$ and $a_i \in \{0, 1, \dots, [\beta]\}$ for $i = 1, \dots, n$. We set

$$f(z) = \sum_{n=1}^{\infty} a_n z^n,$$

where $a_{k+n} = 0$ for $k \in \mathbb{N}$ if 1 has a β-expansion of the form (5.1) and $a_{k+n} = 1$ for $k \in \mathbb{N}$ if 1 has a β-expansion of the form (5.2).

We will show that $1 - f(z)$ has no zero in $|z| \leqq 1/\beta$ except $z = 1/\beta$ as a rational function on $\mathbb{C}$. By definition, it is clear that $f(1/\beta) = 0$. Since $a_i \geqq 0$ for $n \in \mathbb{N}$,

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we have that for $z \in \mathbb{C}$ with $|z| \leqq 1/\beta$,

$$|f(z)| \leqq \sum_{i=1}^{\infty} a_i |z|^i \leqq \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} = 1. \tag{5.3}$$

In particular, we have that $|f(z)| < 1$ for $z \in \mathbb{C}$ with $|z| < 1/\beta$. Suppose that $z = e^{\sqrt{-1}\theta}/\beta$, $\theta \in [0, 2\pi)$. Then

$$f(z) = \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} \cos n\theta + \sqrt{-1} \cdot \sum_{i=1}^{\infty} \frac{a_i}{\beta^i} \sin n\theta.$$

Since $Re(f(z)) < 1$ for $\theta \in (0, 2\pi)$, we have that $1 - f(z) \neq 0$ for $z = e^{\sqrt{-1}\theta}/\beta$, $\theta \in (0, 2\pi)$.

As $1 - f(z)$ has no zero in $|z| < 1/\beta$ except $z = 1/\beta$, we obtain that if $z_0 \in \mathbb{C} \setminus \{0\}$ is the solution of the equation

$$1 = \sum_{i=1}^{n} \frac{a_i}{z^i}$$

or

$$1 = \sum_{i=1}^{n} \frac{a_i}{z^i} + \frac{[\beta]}{z^n(z-1)},$$

then $z_0 = \beta$ or $|z_0| < \beta$. Hence, β is a Perron number. □

LEMMA 5.5. *Let $\beta_0 \in (1, +\infty) \setminus \mathcal{F} \cup \mathbb{N}$. Then we have*

$$\lim_{\beta \to \beta_0} d_\beta^i(1) = d_{\beta_0}^i(1),$$

$$\lim_{\beta \to \beta_0} d_\beta^i(T_{\beta,\omega_1 \cdots \omega_n}(1)) = d_\beta^i(T_{\beta_0,\omega_1 \cdots \omega_n}(1))$$

and

$$\lim_{\beta \to \beta_0} T_{\beta,\omega_1 \cdots \omega_n}(1) = T_{\beta_0,\omega_1 \cdots \omega_n}(1)$$

for $i \in \{0, 1\}$, $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$.

Proof. By the assumption $\beta_0 \in (1, +\infty) \setminus \mathcal{F} \cup \mathbb{N}$, there exists an interval

$$I \in \{E_0, \ldots, E_{[\beta_0]}, S_1, \ldots, S_{[\beta_0]}\}$$

such that $1 \in \mathrm{int}(I)$, where $\mathrm{int}(I)$ denotes the interior of I. Set $\mathrm{int}(I) = (a(\beta_0), b(\beta_0))$. Since $a(\beta)$ and $b(\beta)$ are continuous as a function of β, there exists $\delta_0 > 0$ such that $1 \in (a(\beta), b(\beta))$ whenever $|\beta - \beta_0| < \delta_0$. This shows that $\lim_{\beta \to \beta_0} d_\beta^i(1) = d_{\beta_0}^i(1)$ for $i \in \{0, 1\}$. Since $T_{\beta,1}(1)$ is a polynomial function on $(\beta - \delta_0, \beta + \delta_0)$ as a function of β, we have that $T_{\beta,1}(1) \to T_{\beta_0,1}(1)$ as $\beta \to \beta_0$ for $i \in \{0, 1\}$. We also have that $T_{\beta,0}(1) \to T_{\beta_0,0}(1)$ as $\beta \to \beta_0$ in the same way.

By the assumption $\beta_0 \in (1, +\infty) \setminus \mathcal{F} \cup \mathbb{N}$, there exist intervals

$$I_0, I_1 \in \{E_0, \ldots, E_{[\beta_0]}, S_1, \ldots, S_{[\beta_0]}\}$$

such that $T_{\beta,1}(1) \in \mathrm{int}(I_1)$ and $T_{\beta,0}(1) \in \mathrm{int}(I_0)$. In the same way as before, we have that there exists $\delta_1 > 0$ such that

$$\lim_{\beta \to \beta_0} d_\beta^i(T_{\beta,j}(1)) = d_{\beta_0}^i(T_{\beta_0,j}(1))$$

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and

$$\lim_{\beta\to\beta_0} T_{\beta,ij}(1) = T_{\beta_0,ij}(1)$$

for $ij \in \{0, 1\}^2$. We can get the desired result inductively. □

LEMMA 5.6. *Let $p \in (0, 1)$ and $A \in \mathcal{B}(J_\beta)$. Then the following functions are continuous everywhere except on $\mathcal{F}$.*

(1) *The function $\beta \mapsto B^0(\beta, p)$ from $(1, +\infty) \setminus \mathbb{N}$ into $\mathbb{R}$.*
(2) *The function $\beta \mapsto \rho_{\beta,p}$ from $(1, +\infty) \setminus \mathbb{N}$ into $L^1(\lambda)$.*
(3) *The function $\beta \mapsto \int_A \rho_{\beta,p}\, d\lambda_\beta$ from $(1, +\infty) \setminus \mathbb{N}$ into $\mathbb{R}$.*

Proof. (1) Let $\beta_0 \in (1, +\infty) \setminus \mathcal{F} \cup \mathbb{N}$ and set $\epsilon_0 = \min\{\langle\beta_0\rangle, 1 - \langle\beta_0\rangle\}/2 > 0$. Then the power series $B^0(\beta, p)$ whose variable is β converges absolutely and locally uniformly on the interval $(\beta_0 - \epsilon_0, \beta_0 + \epsilon_0)$ since the power series in the definition of $B^0(\beta, p)$ satisfies the following inequality:

$$\frac{1}{\beta^n} \sum_{\substack{\omega_1\cdots\omega_n;\\ \omega_n=0}} d^0_\beta(T_{\beta,\omega_1\cdots\omega_{n-1}}(1)) m_{1-p}([\omega_1 \cdots \omega_n]^n_1) \leqq \frac{p[\beta]}{\beta^n} \leqq \frac{p[\beta_0]}{\gamma_0^n}$$

for $n \in \mathbb{N}$ and $\beta \in (1, +\infty) \setminus \mathbb{N}$ with $|\beta - \beta_0| < \epsilon_0$, where $\gamma_0 = \beta_0 - \epsilon_0 > 1$. Therefore, by Lemma 5.5, we obtain the assertion.

(2) Let $\beta_0 \in (1, +\infty) \setminus \mathcal{F} \cup \mathbb{N}$. By Lemma 5.5, for $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$, we have

$$\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]}(x) \to \chi_{[0,T_{\beta_0,\omega_1\cdots\omega_n}(1)]}(x)$$

as $\beta \to \beta_0$ for λ-a.e. $x \in [0, +\infty)$. Therefore, by the Lebesgue bounded convergence theorem, we get

$$\chi_{[0,T_{\beta,\omega_1\cdots\omega_n}(1)]} \to \chi_{[0,T_{\beta_0,\omega_1\cdots\omega_n}(1)]}$$

as $\beta \to \beta_0$ in $L^1(\lambda)$. Since l_β is continuous as a function of β, we have

$$\begin{aligned}\chi_{[l_\beta 1,[\beta]/(\beta-1)]} &= \chi_{[0,1]} \circ l_\beta \\ &\to \chi_{[0,1]} \circ l_{\beta_0} \\ &= \chi_{[l_{\beta_0}1,[\beta_0]/(\beta_0-1)]}\end{aligned}$$

as $\beta \to \beta_0$ in $L^1(\lambda)$ and

$$\begin{aligned}\chi_{[T_{\beta,\omega_1\cdots\omega_n}(l_\beta 1),[\beta]/(\beta-1)]} &= \chi_{[0,T_{\beta,\tilde{\omega}_1\cdots\tilde{\omega}_n}(1)]} \circ l_\beta \\ &\to \chi_{[0,T_{\beta_0,\tilde{\omega}_1\cdots\tilde{\omega}_n}(1)]} \circ l_{\beta_0} \\ &= \chi_{[T_{\beta_0,\omega_1\cdots\omega_n}(l_{\beta_0}1),[\beta_0]/(\beta_0-1)]}\end{aligned}$$

as $\beta \to \beta_0$ in $L^1(\lambda)$ for $n \in \mathbb{N}$ and $\omega_1 \cdots \omega_n \in \{0, 1\}^n$.

Since $B^0(\beta, p) + B^0(\beta, 1 - p) \leqq 2$ from Lemma 4.3(1) and (2), we get that $\rho_{\beta,p}$ converges absolutely and locally uniformly on the interval $(\beta_0 - \epsilon_0, \beta_0 + \epsilon_0)$ as a function of β in $L^1(\lambda)$ in the same way as in the statement (1). Therefore, we get the assertion.

(3) The statement easily follows from the statement (2) and $\lambda = ([\beta]/(\beta - 1)) \cdot \lambda_\beta$. □

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THEOREM 5.7. *Let* $p \in (0, 1)$ *and* $A \in \mathcal{B}(J_\beta)$. *Then the following functions are continuous everywhere except on* $\mathcal{F}$.

(1) *The function* $\beta \mapsto h_{\beta,p}$ *from* $(1, +\infty) \setminus \mathbb{N}$ *into* $L^1(\lambda)$.

(2) *The function* $\beta \mapsto \mu_{\beta,p}(A)$ *from* $(1, +\infty) \setminus \mathbb{N}$ *into* $\mathbb{R}$.

Proof. Since $F(\beta, p) > 0$ for $(\beta, p) \in ((1, +\infty) \setminus \mathbb{N}) \times [0, 1]$ and by Lemma 5.6(3), the function $\beta \mapsto F(\beta, p)$ has no zero on $\beta \in (1, +\infty) \setminus \mathbb{N}$ and is continuous except on $\mathcal{F}$. Therefore, we get the statement (1) by combining the above fact and Lemma 5.6(2). We also get the statement (2) by combining the above fact and Lemma 5.6(3). □

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