Math. Proc. Camb. Phil. Soc. (1987), **102**, 71
Printed in Great Britain

Unstable Adams operations on classifying spaces

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(*Received* 23 *May* 1986; *revised* 23 *September* 1986)

The Adams operations $\{\psi^k\}$ considered as self-maps of BU have the property that $(\psi^k)^*_{2n} = k^n \cdot \mathrm{Id}_{2n}$ on $H^{2n}(BU; \mathbb{Q})$. Sullivan in his M.I.T. notes [**12**] constructed ψ^k-type self-maps of $BU(n)$ in the case in which k is prime to $n!$. Later work by Friedlander [**5**] and Wilkerson [**14**] gave constructions for all other compact connected Lie groups G, subject to the condition that k be prime to the order of the Weyl group $W(G)$.

In the non-realizability direction, G. Cooke and others used real K-theory to eliminate maps such as ψ^2 on BS^3. Recently G. Mislin [**10**] has produced a striking classification of the self-maps of BS^3, namely that all maps of type ψ^k for a given k are homotopic. His work, as does ours, uses work of W. Dwyer [**4**] and Miller and Wilkerson [**9**] characterizing maps of classifying spaces of p-groups into BG. Using this work of Dwyer in a different way, this note confirms the conjecture of Sullivan on the non-existence of certain ψ^k on BG:

THEOREM 1. *Let G be a compact connected Lie group. Suppose a non-zero integer k is divisible by a prime p which divides the order of the Weyl group. Then there is no self-map of the p-localized classifying space $BG_{(p)}$ such that the induced endomorphism of $H^*(BG_{(p)}; \mathbb{Q})$ is multiplication by k^n in degree $2n$ for each n.*

The proof involves first Dwyer's theorem.

THEOREM 0. *If π is a finite p-group and G is a compact Lie group, then*

$$\pi_0(\mathrm{map}(B\pi, BG)) \simeq \mathrm{Hom}(\pi, G)/G\text{-conjugation}.$$

The other main element in the proof is the construction of specific test p-groups in G. For odd primes p, this requires work of Tits [**13**] on the 'extended' Weyl group.

An easy consequence of Theorem 1 together with some results of Adams and Mahmud [**2**] gives a fairly complete classification of the self-maps of BG, up to action on $H^*(BG; \mathbb{Q})$, where G is a compact simply-connected simple Lie group.

§1. *Some elementary representation theory*

Let $R(\pi)$ denote the complex representation ring of a finite group π. The Adams operations $\{\psi^k\}$ are defined on $R(\pi)$ for any non-negative integer k ([**11**], §9·1). If χ is the character of a representation of π, then $\psi^k\chi(u) = \chi(u^k)$ for any $u \in \pi$. Suppose χ is irreducible. If k is prime to the order of the group π, then $\psi^k\chi$ is also an irreducible character of π. On the other hand, if k is not prime to $|\pi|$, then $\psi^k\chi$ may be only virtual.

LEMMA 2. *Let χ be the character of a faithful complex representation ρ of a finite group π. If there exists a complex representation ρ_k with character $\psi^k\chi$, then* $\mathrm{Ker}\,\rho_k = \{u \in \pi \mid u^k = 1\}$.

https://doi.org/10.1017/S0305004100067050 Published online by Cambridge University Press

Proof. Recall that if γ is a complex representation of a finite group, then $a \in \operatorname{Ker}\gamma$ if and only if $\chi_\gamma(a) = \chi_\gamma(1)$. Suppose now that $u \in \operatorname{Ker}\rho_k$. Then $\psi^k\chi(u) = \psi^k\chi(1)$ and hence $\chi(u^k) = \chi(1)$. Since ρ is a complex representation of a finite group, we see that $u^k \in \operatorname{Ker}\rho$. Since ρ is faithful, it follows that $u^k = 1$. Conversely, if $u^k = 1$, then $\psi^k\chi(u) = \psi^k\chi(1)$. Therefore $u \in \operatorname{Ker}\rho_k$ since ρ_k is a complex representation of π. ∎

The following lemma will enable us to find a subtorus T^{p-1} in the compact connected Lie group G on which a cyclic subgroup Z/p of the Weyl group $W(G)$ acts by the reduced regular representation.

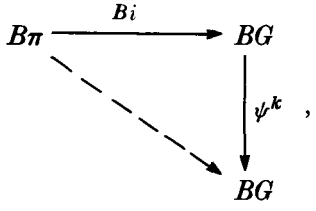
LEMMA 3. *Any faithful integral representation of Z/p has an invariant submodule of rank $p-1$ on which a generator b of Z/p acts with minimal polynomial $b^{p-1}+b^{p-2}+\ldots+1$.*

Proof. Suppose M is the integral representation space of Z/p, then $f = \Sigma_{i=0}^{p-1} b^i$ acts on $M \otimes \mathbb{Q}$. Since f is irreducible over $\mathbb{Q}$, the module $\operatorname{Ker}f$ is a vector space over the field $\mathbb{Q}[b]/f$. Since $b^{p-1} = (b-1)(b^{p-1}+b^{p-2}+\ldots+1)$ acts as zero and $b \neq 1$, we see that $\operatorname{Ker}f \neq 0$. Consequently, we can take a rank one $\mathbb{Q}[b]/f$-summand of $\operatorname{Ker}f$ as the $(p-1)$-dimensional invariant subspace V of $M \otimes \mathbb{Q}$ on which the generator b acts with $\Sigma_{i=0}^{p-1} b^i = 0$. Therefore the submodule $V \cap M$ has the desired property. ∎

§2. *Proof of Theorem* 1

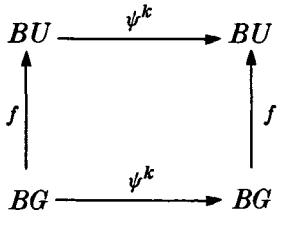
Let X be a space whose loop space ΩX is rationally homotopic to a finite CW-complex. An unstable Adams operation of type ψ^k on X is defined as a self-map of X such that the induced endomorphism of $H^*(X;\mathbb{Q})$ is multiplication by k^n in degree $2n$ for each n. In general, it is not known whether such a map ψ^k is unique up to homotopy [**10**].

The main part of our theorem is to show that if k is not prime to the order of the Weyl group, then ψ^k does not exist as a self-map of the classifying space BG. The idea of the proof is to construct a test p-group $\pi \xrightarrow{i} G$ such that, in the diagram



the composite map $\psi^k \circ Bi$ cannot be realized by a homomorphism.

Proof of Theorem 1. First we will prove the unlocalized version of our theorem. We claim that for any map $f: BG \to BU$, the following diagram



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commutes up to homotopy. The composite maps $\psi^k \circ f$ and $f \circ \psi^k$ can be recognized as elements of $\tilde{K}(BG; \mathbb{Z})$. By the Atiyah–Hirzebruch spectral sequence, one can show that the canonical map $\tilde{K}(BG; \mathbb{Z}) \to \tilde{K}(BG; \mathbb{Q})$ is injective. Since $(\psi^k)^* = k^n \cdot \mathrm{Id}_{2n}$ on $H^{2n}(BG; \mathbb{Q})$, the Chern character tells us that $\psi^k \circ f = f \circ \psi^k$ in $\tilde{K}(BG; \mathbb{Q})$. Consequently, $\psi^k \circ f \simeq f \circ \psi^k$.

Suppose ψ^k is defined as a self-map of BG. If τ is a unitary representation of the compact connected Lie group G, then the diagram

$$\begin{array}{ccc} BU & \xrightarrow{\psi^k} & BU \\ \uparrow & & \uparrow \\ BU(m) & & BU(m) \\ {\scriptstyle B\tau}\uparrow & & \uparrow{\scriptstyle B\tau} \\ BG & \xrightarrow{\psi^k} & BG \end{array}$$

must commute up to homotopy. Now suppose ρ is a representation of a finite p-group π into G. Then the Dwyer's theorem implies that the composition $\psi^k \circ B\rho$ should be realized as another representation of π. Recall that the complex representation ring $R(\pi)$ is embedded in the K-theory $K(B\pi)$ since π is a finite p-group. Consequently, if χ is the character of ρ and $k > 0$, then the character of the representation $\Omega(\psi^k \circ B\rho)$ into G is $\psi^k \chi$ ([1], p. 69). In order to prove the non-existence of ψ^k on BG, it suffices to find a representation ρ of π such that $\psi^k \chi$ is only virtual. Using this argument and the formula $\psi^k \psi^l = \psi^{kl}$, we will show the non-existence of ψ^k when k is not prime to $|W(G)|$.

We first prove that if an odd prime p divides $GCD(k, |W(G)|)$, then ψ^k does not exist as a self-map of BG. We need to construct a test p-group π of BG. Let T^n be a maximal torus of G. If NT^n is the normalizer of T^n and $W = W(G)$, then we claim that for an odd prime p the pull-back of $T^n \to NT^n \to W$ and an inclusion $\mathbb{Z}/p \to W$ is a semi-direct product $T^n \rtimes \mathbb{Z}/p$. Tits([13], proposition 6) shows, in fact, that there is a subgroup $N_{\mathbb{Z}}$ of NT^n such that the diagram

$$\begin{array}{ccccc} & & n & & \\ \oplus\, \mathbb{Z}/2 & \longrightarrow & N_{\mathbb{Z}} & \longrightarrow & W \\ {\scriptstyle j}\downarrow & & \downarrow & & \| \\ T^n & \longrightarrow & NT^n & \longrightarrow & W \end{array}$$

commutes. Hence the extension $T^n \to NT^n \to W$ corresponds to an element $x \in H^2(W, T^n)$ such that $2x = 0$. Consequently, the extension induced by the inclusion $\mathbb{Z}/p \to W$ corresponds to the zero element in $H^2(\mathbb{Z}/p, T^n)$ since p is odd. We now see that the pull-back is $T^n \rtimes \mathbb{Z}/p$.

Lemma 3 says that there is a subtorus of rank $p-1$ on which the subgroup $\mathbb{Z}/p$ of W acts by the reduced regular representation. Using the canonical inclusions $\mathbb{Z}/p^j \to S^1$, we obtain the faithful representations into G:

$$\rho_j : (\mathbb{Z}/p^j)^{p-1} \rtimes \mathbb{Z}/p \to T^n \rtimes \mathbb{Z}/p \to G.$$

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Suppose χ_j is the character of ρ_j. We will show that if $k = d \cdot p^m$ where $m \geqslant 1$ and $(d, p) = 1$, then the character $\psi^k \chi_{m+1}$ is only virtual. If $(a, b) \in (\mathbb{Z}/p^j)^{p-1} \rtimes \mathbb{Z}/p$ where $a \in (\mathbb{Z}/p^j)^{p-1}$ and $b \in \mathbb{Z}/p$, then $(a, b)^{p^m} = (\sum_{i=0}^{p^m-1} b^i(a), 1)$. Here the identity element of the semi-direct product is denoted by $(0, 1)$. By Lemma 2 we see that

$$\operatorname{Ker} \psi^k \chi_{m+1} = \{(a, b) \in (\mathbb{Z}/p^{m+1})^{p-1} \rtimes \mathbb{Z}/p \mid b \neq 1, p^m a = 0\},$$

since

$$(a, b)^p = \begin{cases} (0, 1) & \text{if} \quad b \neq 1 \\ (pa, 1) & \text{if} \quad b = 1. \end{cases}$$

Consequently $|\operatorname{Ker} \psi^k \chi_{m+1}| = (p-1)p^{m+1} + p^m$. If $\psi^k \chi_{m+1}$ were the character of a representation, the cardinality of the kernel would be a prower of p. Since the character is only virtual, under the given condition for k we see the non-existence of ψ^k.

We next prove that if 2 divides $GCD(k, |W(G)|)$, then ψ^k does not exist as a self-map of BG. Since $|W(G)| \geqslant 2$, the compact connected Lie group G is non-abelian. It is well known that either $SO(3)$ or S^3 is a subgroup of G. Our test 2-groups are dihedral groups D_{2^n} and quaternion groups Q_{2^n} of order 2^n. There are faithful representations $D_{2^n} \to SO(3)$ and $Q_{2^n} \to S^3$. Consequently, we can find a faithful representation $D_{2^n} \to G$ or $Q_{2^n} \to G$. (This result can be obtained also using a $\mathbb{Z}/2$-action on a subtorus of G and group cohomology.) Let χ_n be the character of the above representation. One can show that $\psi^k \chi_n$ is only virtual if $k = d \cdot 2^{n-2}$ where $n \geqslant 4$ and d is odd. The argument is parallel to that for $p > 2$, using an appropriate presentation of our 2-group. It follows that ψ^{4k} does not exist for any $k > 0$. But if $f: BG \to BG$ has type ψ^{2k}, then $f \circ f$ has type ψ^{4k^2}, and the above result implies the non-existence of ψ^k whenever 2 divides $GCD(k, |W(G)|)$.

Finally, we will prove the non-existence of ψ^k on the p-localized classifying space $BG_{(p)}$ when a prime p divides $GCD(k, |W(G)|)$. Note that if $k = d \cdot p^m$ with $(d, p) = 1$, then there exists $\psi^{1/d}$ on $BG_{(p)}$ since d is a unit in the ring of p-adic integers $\hat{\mathbb{Z}}_p$. By [**14**], corollary 1·5, we have ψ^{p^m} on $BG_{\mathbf{p}-p}$. As usual $X_{\mathbf{p}-p}$ denotes localization away from the prime p. Suppose ψ^k were defined on $BG_{(p)}$. From the fibre square

$$\begin{array}{ccc} BG & \xrightarrow{\beta} & BG_{\mathbf{p}-p} \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ BG_{(p)} & \longrightarrow & BG_{(0)} \end{array}$$

we see that the maps $\psi^{1/d} \circ \psi^k \circ \alpha: BG \to BG_{(p)}$ and $\psi^{p^m} \circ \beta: BG \to BG_{\mathbf{p}-p}$ would force the existence of ψ^{p^m} on BG. This contradiction completes the proof of our theorem.

§3. *Self-maps of BG*

We first make a remark on the two exotic self-equivalences of $(BG_2)_{\mathbf{p}-3}$ and $(BF_4)_{\mathbf{p}-2}$ constructed by Friedlander [**5**]. The composition of each of the maps with itself becomes an unstable Adams operation; namely ψ^3 on $(BG_2)_{\mathbf{p}-3}$ and ψ^2 on $(BF_4)_{\mathbf{p}-2}$

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([2], §2). By Theorem 1 we see that without localization there are no maps which correspond to the exotic self-maps, since $|W(G_2)| \equiv 0 \bmod 3$ and $|W(F_4)| \equiv 0 \bmod 2$.

In [6], Hubbuck stated the following converse to the result of Sullivan and others. 'Let G be a compact simply-connected simple Lie group. If G is not the exceptional Lie group G_2, then up to outer automorphism all self-maps of BG are unstable Adams operations.' A proof of this statement for the classical groups is in [7]. The proof for the exceptional Lie groups is apparently unpublished. Hubbuck's result (for all exceptional Lie groups) admits another proof which makes use of the relationship between the non-zero admissible maps $G \dashrightarrow G$ and the automorphisms of the Dynkin diagram of G ([2], proposition 2·13). Recall that the identity automorphism of the Dynkin diagram corresponds to the class of admissible maps which may be realized as unstable Adams operations. We sketch the proof for the exceptional Lie groups. If $G = E_7$ or E_8, then there are no non-trivial automorphisms of the Dynkin diagram. If $G = E_6$, then the two automorphisms correspond to ψ^1 and ψ^{-1}. If $G = G_2$ or F_4, then there is just one non-trivial automorphism, which corresponds to the exotic map. Consequently, one can show that if G is an exceptional Lie group, then all self-maps of BG are unstable Adams operations, because the exotic maps are not realizable without localization. This classification is only at the level of induced maps on rational cohomology, but recent work of Mislin [10] suggests that this may in fact be a classification up to homotopy class.

The author would like to thank W. Dwyer, D. Gluck, G. Mislin and C. Wilkerson for their help and suggestions. He also thanks the referee for a simplification of the original proof of Theorem 1.

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