# From dawn to dusk*

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*Introduction*

It used to be the case in some jurisdictions that a rhythmic knell of the Angelus bell would mark the onset of dawn, noon and the passing of a day at dusk. Between these times there is daylight whose duration varies from place to place and from day to day and which can be predicted either exactly or approximately. An illuminating problem concerning the number of daylight hours at a winter solstice in London was posed recently and answered in the Problem Corner of *The Mathematical Gazette* [1]. It was shown, for example, that a calculation of daylight hours rested strictly upon the numerical solution to a transcendental trigonometric equation. Related references to earlier works in the *Gazette* involving a point source Sun were also given.

The purpose of this note is multifold. First, it is to point out that the above-cited equation might be viewed also as a "Sun-ray-to-Earth tangency condition". Such a condition was developed earlier by the author in a publication that is now defunct [2], and so for completeness the steps necessary to establish the condition will be produced again here. Second, it will be evident from the manner of its derivation that the governing equation is valid at all orbit points, not just at a given solstice. Starting at a solstice, it is possible to solve the equation using a numerical procedure after which the duration of daylight can be calculated. Thereafter it is possible to secure similarly a solution for each successive day until the year is completed. Third, it will be shown that it is possible to determine also a simple, alternative approximate solution to the governing equation, the results from which compare well with those obtained via the more precise numerical method, the accuracy being only marginally impaired. Finally, the results from both exercises will be compared with known results [3], which are believed to be accurate for the year 2017.

*Establishing the condition*

To establish the condition, a suitable geometry is required for the Earth as it orbits the Sun. Here it will be assumed that the orbit is circular and that the Earth is illuminated by a point source image Sun such as that shown in Figure 1.

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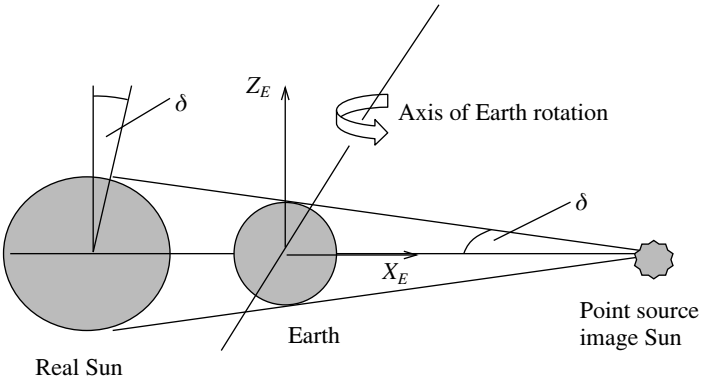



FIGURE 1: A sideways schematic as seen by an observer on the Ecliptic at a Winter Solstice in the Northern Hemisphere, showing the angle $\delta$ as it appears also in [1], and, right-handed axis as they relate to the Earth.

With reference to this figure, tangent light rays common to the real Sun and the Earth are shown meeting at a point source (image Sun). It can be appreciated that illumination by the image Sun is the complement of illumination by the real Sun. That is to say, the duration of darkness on the Earth under the real Sun will correspond to the duration of daylight under the image Sun. If, as in [1], the diameter of the Earth is 7927 miles and that of the real Sun is 864950 miles, and if the distance between their respective centres is 93000000 miles, then the half cone angle $\delta$ shown in the picture is such that $\sin\delta = 0.00460765$ and the ratio of the Earth's radius, $r$ say, to the distance $D$ from the centre of the Earth to the image Sun is simply $\frac{r}{D} = \sin\delta$, as evidenced also in [1].

Thus focusing now on illumination under the image Sun, a schematic such as that shown in Figure 2 may be constructed. In this figure, which is a bird's eye view of the ecliptic, the real Sun is shown with a fixed axis set, one axis passing through a solstice point and the other passing through an equinox point. The angle $\nu\,(= \Omega t)$ is an Earth orbit angle that is zero at the solstice and $\Omega = \dfrac{\frac{1}{12}\pi}{365.25}$ is the orbit rotation rate of the Earth around the Sun; $t$ denotes the time parameter in hours and it is taken to be zero at noon on the winter solstice (21st December). The year here is taken to comprise 365.25 days, as in [1].

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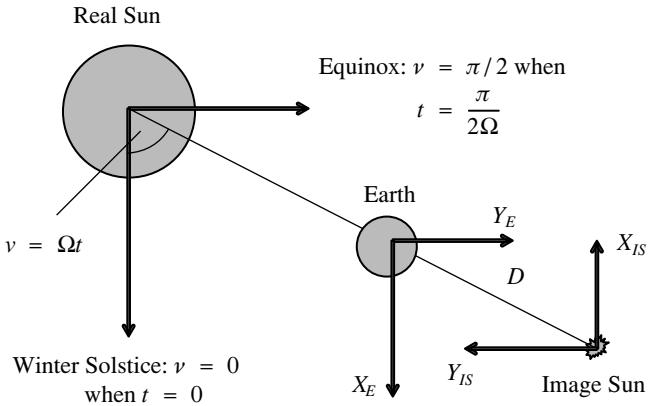



FIGURE 2: A bird's eye schematic of the orbit geometry on the ecliptic. $D$ denotes the distance between the centre of the Earth and the image Sun; $\Omega\left[=\frac{\frac{1}{12}\pi}{365.25}\right]$ denotes the orbital rotation rate of the Earth around the Sun and $t$ is the time in hours.

The image Sun and the Earth are shown also with fixed orientation, right-handed axes sets $[X_{IS}, Y_{IS}, Z_{IS}]$ and $[X_E, Y_E, Z_E]$, respectively. These are parallel to the fixed axes through the Sun and their orientation does not change as the Earth orbits the Sun. The orbit angle for the Earth with respect to a solstice point is denoted by $\nu$, as shown, and a rotated axis set $[X_{ER}, Y_{ER}, S_{ER}]$ for the Earth is shown also in Figure 3.

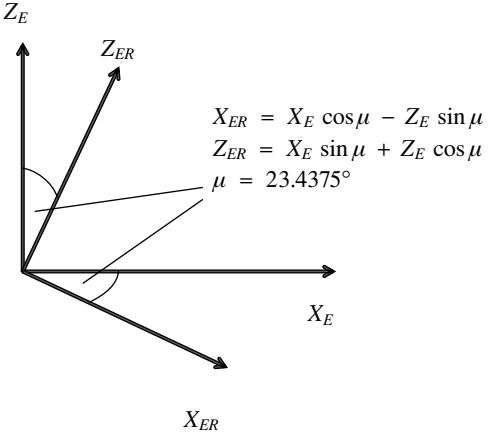



FIGURE 3: Axis rotation formulae

This set is obtained simply by rotating the earlier Earth axis set through an angle $\mu\,(\approx 23.4375°)$ about the $Y_E$ axis; it accounts for the inclination of the Earth's polar axis to the ecliptic. These two axes sets are related via the usual axis rotation formulae shown also in the figure. Lastly, it is necessary

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to define angles of interest for a point $P$ on an Earth latitude circle. These are shown in Figure 4.

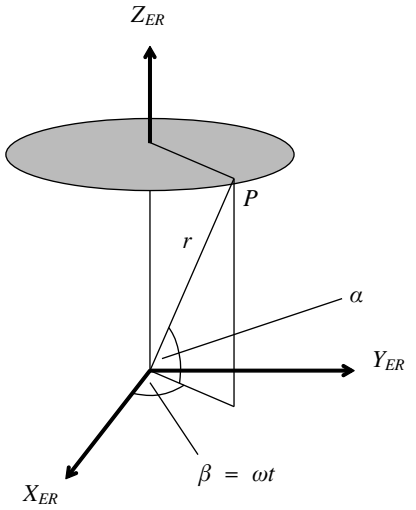



FIGURE 4: A point $P$ on a latitude circle on the Earth (radius $r$) and angles of interest; $\alpha$ denotes an angle of latitude and $\beta (= \omega t)$ denotes an angle of longitude; $\omega = \frac{1}{12}\pi$ denotes the angular rotation rate of the Earth about its axis and $t$ is the time in hours

Here the latitude (elevation) angle is denoted by $\alpha$, which is about 51.5° at London UK, and the longitude (azimuth) angle is denoted by $\beta (= \omega t)$, where $\omega = \frac{1}{12}\pi$ is the angular rotation rate of the Earth about its axis and $t$ is, as before, the time in hours starting at zero at noon on the solstice.

It might now be appreciated that a light ray, or vector, from the image Sun to the point $P$ will make some angle with an Earth radius vector to the same point. There are two tangent points on the Earth's latitude circle for such a ray and daylight ensues between them under illumination from the image; latitude points on the dark side of illumination under the image will be facing daylight via the real Sun. Thus it is important to determine when tangency occurs. To this end, it will be necessary first to reference to a common system the coordinates of the point $P$ on the latitude circle and the image Sun. Then, working in the $[X_{ER}, Y_{ER}, Z_{ER}]$ system it is a simple matter to express from Figure 4 the coordinates of the point $P$ as

$$\begin{aligned} X_{ER}^{P} &= r \cos\alpha \cos\beta, \\ Y_{ER}^{P} &= r \cos\alpha \sin\beta, \\ Z_{ER}^{P} &= r \sin\alpha. \end{aligned} \tag{1}$$

For the coordinates of the image Sun in this system it is first necessary to write down its coordinates in the $[X_E, Y_E, Z_E]$ system, after which axis rotation formulae can be employed to determine its coordinates in the

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system of interest. The coordinates of the image Sun in the $[X_E, Y_E, Z_E]$ system are, from Figure 2,

$$\begin{aligned} X_E^{IS} &= D\cos\nu, \\ Y_E^{IS} &= D\sin\nu, \\ Z_E^{IS} &= 0. \end{aligned} \tag{2}$$

If $(X_{ER}^{IS}, Y_{ER}^{IS}, Z_{ER}^{IS})$ denotes the coordinates of the image Sun in the rotated reference frame, the axis rotation formulae can be employed to write

$$\begin{aligned} X_{ER}^{IS} &= X_E^{IS}\cos\mu - Z_E^{IS}\sin\mu, \\ Y_{ER}^{IS} &= Y_E^{IS}, \\ Z_{ER}^{IS} &= X_E^{IS}\sin\mu + Z_E^{IS}\cos\mu. \end{aligned} \tag{3}$$

The coordinates of the image Sun in the rotated reference frame are, from (2) and (3),

$$\begin{aligned} X_{ER}^{IS} &= D\cos\nu\cos\mu, \\ Y_{ER}^{IS} &= D\sin\nu, \\ Z_{ER}^{IS} &= D\cos\nu\sin\mu. \end{aligned} \tag{4}$$

Now that the image Sun and the latitude point have been assigned coordinates in a common frame (equations (1) and (4)) it is a simple matter to write down the components of the vector joining them. Using a column vector notation, this is simply

$$\begin{bmatrix} D\cos\nu\cos\mu - r\cos\alpha\cos\beta \\ D\sin\nu - r\cos\alpha\sin\beta \\ D\cos\nu\sin\mu - r\sin\alpha \end{bmatrix}. \tag{5}$$

The dot product of this vector and the radius vector to the latitude point $P$ (its components are given in equation (1)) can be formed and it will be zero at a point of tangency. Accordingly, from equations (1) and (5), it follows that at such a point

$$\begin{aligned} &r\cos\alpha\cos\beta\left[D\cos\nu\cos\mu - r\cos\alpha\cos\beta\right] \\ &\quad + r\cos\alpha\sin\beta\left[D\sin\nu - r\cos\alpha\sin\beta\right] \\ &\quad + r\sin\alpha\left[D\cos\nu\sin\mu - r\sin\alpha\right] = 0 \end{aligned}$$

which, after replacing $\frac{r}{D}$ by $\sin\delta$, reduces to the expression

$$\begin{aligned} &\cos\alpha\cos\beta\cos\nu\cos\mu + \cos\alpha\sin\beta\sin\nu + \sin\alpha\cos\nu\sin\mu \\ &= \sin\delta\left[\cos^2\alpha\cos^2\beta + \cos^2\alpha\sin^2\beta + \sin^2\alpha\right]. \end{aligned}$$

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This simplifies to the result

$$\cos\beta\,\cos\mu\,\cos\nu\,+\,\sin\beta\,\sin\nu\,+\,\tan\alpha\,\sin\mu\,\cos\nu\,-\,\frac{\sin\delta}{\cos\alpha}\,=\,0. \quad (6)$$

Equation (6) is the "Sun-ray-to-Earth tangency condition", albeit one due to an image Sun. It is the equation that governs the duration of daylight (or darkness).

It is of interest to compare the above equation with that given for the problem in [1]. It can be appreciated that the format there is identical to the one here if one reconciles the use of the different notations as follows: in equation (6), replace $\alpha$ by $(\pi\,-\,\theta)$ and $\mu$ by $\phi$ (as before, $\beta$ and $\nu$ are $\omega t$ and $\Omega t$ respectively). In effect, the above result determines the longitude of points on the Earth at a given date and latitude when the sun is just setting or rising. This is the converse of finding the time of rising or setting at a specific point.

From the manner in which it has been derived, it should be appreciated that equation (6) is valid not just at solstice points but also at all other points of orbit.

*A precise solution to the governing equation*

The parameters of interest in the governing equation ($\beta$ and $\nu$) are not independent but are related by the time parameter $t$. Specifically, the relationship between these parameters is $\nu\,=\,\varepsilon\beta$, where $\varepsilon\,=\,1\,/\,365.25$. The governing equation is then of the form $f\,(\beta)\,=\,0$, where

$$f\,(\beta)\,=\,\cos\beta\,\cos\varepsilon\beta\,+\,\frac{\sin\beta\,\sin\varepsilon\beta}{\cos\mu}\,+\,\tan\alpha\,\tan\mu\,\cos\varepsilon\beta\,-\,\frac{\sin\delta}{\cos\alpha\,\cos\mu}.$$

With a view to solving this equation numerically using a Newton-Raphson method, say, the derivative is required and this is

$$f'(\beta) = -\sin\beta\cos\varepsilon\beta\left(1-\frac{\varepsilon}{\cos\mu}\right)-\cos\beta\sin\varepsilon\beta\left(\varepsilon-\frac{1}{\cos\mu}\right)-\varepsilon\tan\alpha\tan\mu\sin\varepsilon\beta.$$

In the usual notation, the iterative method is then

$$\beta^{(N\,+\,1)}\,=\,\beta^{(N)}\,-\,\frac{f\,(\beta^{(N)})}{f'\,(\beta^{(N)})}$$

for $N\,=\,0,\,1,\,2,\,\ldots$, where $\beta^{(N)}$ denotes the $N$ th iterate. The procedure is particularly well suited to manipulations using a spreadsheet. Starting values are required to kick-start the business and these may be obtained from the 'excellent and very simple approximation' given in [1], which implies starting values of $\beta^{(0)}\,=\,-0.99440217$ and $\beta^{(0)}\,=\,0.99440217$, one for sunrise and the other for sunset at points of tangency when $\nu\,=\,0$. Using these starting values the iterative method yields, after about three iterations, the solutions $\beta_1\,=\,-1.007002123$ and $\beta_2\,=\,+1.007002123$ from which the daylight time duration is $\frac{12\,(\beta_2\,-\,\beta_1)}{\pi}\,=\,7.69292955$ hours, which is given

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as 7 hours 41 minutes 35 seconds in [1]. To determine the duration of daylight for the next day, useful starting values in the iterative procedure can be obtained by adding $2\pi$ to the $\beta$-values obtained from the day before. The starting values for the second day are thus $\beta_1 = -1.007002123 + 2\pi = 5.276183184$ and $\beta_2 = 1.007002123 + 2\pi = 7.290187430$. Again, after about three iterations, the $\beta$-values can be found to be $\beta_1 = 5.294910671$ and $\beta_2 = 7.309143169$ respectively from which it follows that daylight duration for the second day is

$$\frac{12\,(\beta_2 - \beta_1)}{\pi} = 7.693801406 \text{ hours},$$

which translates into 7 hours 41 minutes 38 seconds. The process can be repeated with ease to cover the whole year.

*An alternative approximate solution to the governing equation*

If for the moment one assumes that the variables of interest $\beta$ and $\nu$ in equation (6) are independent, it is possible to obtain a relatively simple approximation for the duration of daylight. To this end, two solutions for $\beta$ ($\beta_1$ and $\beta_2$) can be obtained from equation (6) for any specified orbit angle $\nu$, provided of course the latitude circle is seen by the Sun, image or otherwise as the case may be. One of these solutions will give a $\beta$-value at sunrise and the other will give its value at sunset. The difference between these two angles can be formed and divided by $2\pi$ (360°) after which it is multiplied by 24 hours to provide the rough estimate for the duration of either daylight or darkness. The steps involved require the use of half-angle formulae for the $\beta$-terms in equation (6) and lead to a quadratic equation for $\tan\frac{1}{2}\beta$. Specifically, writing $\tau = \tan\frac{1}{2}\beta$ so that $\cos\beta = \dfrac{1-\tau^2}{1+\tau^2}$ and $\sin\beta = \dfrac{2\tau}{1+\tau^2}$ it is possible to establish from equation (6) the equation

$$\left(\frac{1-\tau^2}{1+\tau^2}\right)\cos\mu\cos\nu + \left(\frac{2\tau}{1+\tau^2}\right)\sin\nu + \tan\alpha\sin\mu\cos\nu - \frac{\sin\delta}{\cos\alpha} = 0$$

which reduces to the quadratic form $A\tau^2 + B\tau + C = 0$ in $\tau$, with a solution given by

$$\tau = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

where

$$A = -\cos\mu + \tan\alpha\sin\mu - \frac{\sin\delta}{\cos\alpha\cos\nu},$$

$$B = 2\tan\nu,$$

$$C = A + 2\cos\mu.$$

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The difference between the roots $\beta_1$ and $\beta_2$ can be shown to be

$$\beta_1 - \beta_2 = 2 \tan^{-1} \frac{\sqrt{B^2 - 4AC}}{A + C}$$

and the hours of interest are accordingly

$$\frac{12}{\pi}(\beta_2 - \beta_1) \text{ or } \frac{12}{\pi}[24 - (\beta_2 - \beta_1)]$$

in some order. If the sign of the above discriminant becomes negative no further useful information can be gleaned; this will be the case in the polar regions at times when perpetual darkness ensues. This places a constraint upon the angles for illumination to occur. It is a straightforward but somewhat laborious task to show that the discriminant in this instance is positive if $\cos \nu$ lies between the two values

$$(-1) \frac{\cos(\alpha + \delta)}{\sin \mu} \text{ and } \frac{\cos(\alpha - \delta)}{\sin \mu}.$$

As an illustration of the usefulness of the approximate approach, a procedure can be triggered at the solstice point where $\nu = 0$, with inputs as per Problem 98L [1] to produce the result $(\beta_2 - \beta_1) = 2.00798854$ and hence a daylight duration of 7.66995124 hours (i.e. 7 hours, 40 minutes, 12 seconds). This approximate result compares well with the precise one of 7 hours, 41 minutes, 35 seconds; it is an improvement on the earlier, cited 'excellent and very simple approximation' of 7 hours 36 minutes. The process can be continued likewise to the next day, where $\nu = 0 + \frac{2\pi}{365.25}$, which produces the result $(\beta_2 - \beta_1) = 7.670816593$ hours (i.e. 7 hours 40 minutes 15 seconds). It can be repeated for each day of the year, but care in the business has to be exercised in choosing the correct branch of the inverse tangent function as $\nu$ transitions through the odd multiples of $\frac{1}{2}\pi$.

It is instructive to display curves for the daylight hours versus number of days obtained from the above calculations for the approximate and precise scenarios. These are shown in Figure 5.

It can be seen that there is good agreement between the two curves. The scale does not allow for a clear distinction and the aspect of a difference between them will be addressed later when comparisons are made with available data. For the moment, suffice it to say that such a difference was found from spreadsheet values to lie in the range (2.66 ±1.15) minutes.

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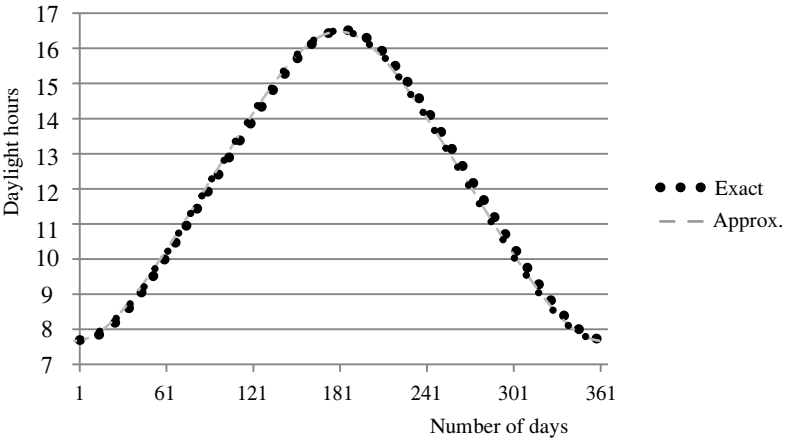



FIGURE 5: Precise and approximate values starting and finishing at the solstice

*A comparison of results with known data*

Data for duration of daylight at the London latitude can be extracted from [3] and day-to-day differences with the exact and approximate formulations above can be plotted to produce the results shown in Figure 6.

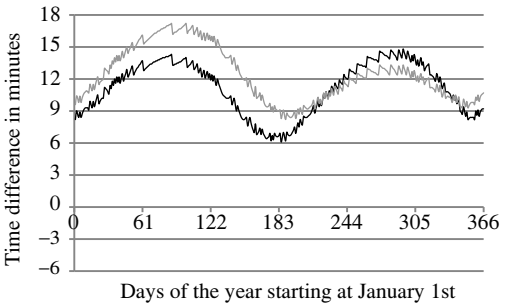



FIGURE 6: Time differences between available data and predictions based on the precise numerical method (black curve) and the approximate method (light grey curve)

In this Figure, the solstice is assumed to occur on 21st December and it is possible to assess the merits of the different formulations discussed above. Evidently, they compare reasonably well, but both are at variance with known data. At worst, the variation in the case of the approximate method is about seventeen minutes (at an equinox), which implies an error in time of no more than about 2.5%. At best it is about 8 minutes at a solstice, which similarly implies a time error of only about 1.0%. The errors are marginally less in the case of the more precise numerical results.

It is of interest to note that the agreement with known data evidenced in Figure 6 can be improved if effects due to atmospheric refraction are taken

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into account. To see this, it is necessary first a to ppreciate that the radius of the Sun subtends about 16 minutes of arc at the centre of the Earth and that the radius of the Earth subtends a similar amount at the image Sun (this is the value for $\delta$ from the expression $\sin \delta = 0.00460765$). In the absence of effects due to atmospheric refraction, the Sun's centre would appear at sunrise to lie below the horizon by this amount (16 arc minutes). However, it is generally assumed that the *geometric zenith distance of the centre of the Sun at sunrise* is 90.8333°, meaning that the Sun's centre is taken to be about $50 = (0.8333 \times 60)$ minutes of arc below the horizon at sunrise when effects due to atmospheric refraction are taken into account. This is a presumed average figure and Googling the words printed in italics will give information about it over the Internet. Thus, in effect, one can legitimately increase the value of $\delta$ in the above computations from about 16 minutes of arc to a value in the vicinity of 50 minutes of arc and re-compute the earlier sums in the spreadsheet. Specifically, it can be shown that a value of about 58 minutes of arc for $\delta$ in the precise numerical exercise and one of about 67 minutes of arc in the case of the approximate one sufficed to reduce the errors evident in Figure 6 to those shown now in Figure 7.

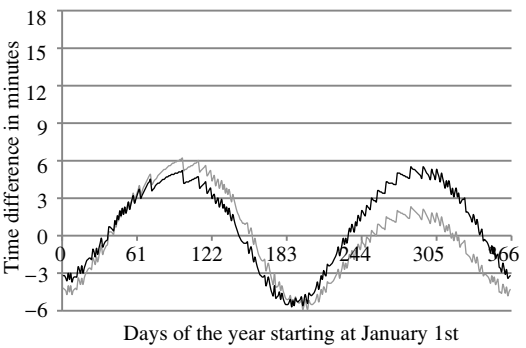



FIGURE 7: Further improved agreement between theory and practice after acknowledging effects due to atmospheric refraction

The measure of disagreement in predicted daylight hours is then reduced to about ± 5 minutes, which implies an encouraging relative error magnitude that is everywhere less than 1%.

The reader might note from Figure 7 that the improvement in time differences is based upon the assumption that the new value for $\delta$ (chosen so as to mimic effects due to atmospheric refraction) is constant throughout the year. However, it is conceivable that this might not be the case and that effects due to atmospheric refraction might vary with the seasons, in which case a harmonic component in the description of the parameter $\delta$ might be entertained to bring the above model closer to reality at the present latitude. It is likely too that refraction effects might also be latitude dependent. It is not the intention here to pursue such matters. This is left as an exercise for the reader; but it is possible to secure agreement between the model and the

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reality, concerning the latitude in which we are interested in, to within about ±2.0 minutes. Other latitudes, of course, would have to be considered on a case-by-case basis. Nonetheless, the best results that can be achieved without considering refraction-cum-latitude effects are shown in Figure 6.

The mathematical model used here is the simplest of its kind. It involves only circular orbits, not elliptic ones, nor does it acknowledge other perturbations that might affect the motion. With respect to the former, one could neglect second order effects due to the Earth's eccentricity and assume that the orbit is still circular with the Sun displaced to its usual focal point as if the orbit were elliptic. In this fashion, one can appreciate that the angle $\delta$ will vary also with the orbit position of the Earth. In the event, it was found that such a variation was not sufficient to account for any significant discrepancy in reconciling theory to practice. It was assumed then that the theory described here is limited largely by the absence of a theoretical model incorporating refraction effects at any latitude and at any time of the year.

*Conclusion*

An interest that was prompted by a problem in *The Mathematical Gazette* about duration of daylight at a solstice has been exploited to look at daylight duration throughout the year, by using some elementary mathematics to construct a simple mathematical model that could be appreciated by an enterprising sixth former or first year undergraduate mathematician. It has resulted in a simple approximate solution to a 'Sun-ray-to-Earth tangency condition' equation that leads to daylight duration results that are in error by only a few per cent. The trade off regarding the use of the approximate solution as opposed to that from the more precise method is one of simplicity of sums versus accuracy.

*Acknowledgement*

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*References*

1. M. G. Elliott, Problem Corner, Problem No. 98L, *Math. Gaz.* **99** (July 2015) pp. 373-375.
2. John D. Mahony, Henges, Heel Stones, and Analemmas, *Mathematical Spectrum*, **41** (2008/2009) pp. 11-20.
3. U.S. Naval Observatory, Duration daylight/darkness table for one year, accessed January 2018 at: aa.usno.navy.mil/data/docs/Dur_OneYear.php

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