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Conformally Kähler structures

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We establish a one-to-one correspondence between Kähler metrics in a given conformal class and parallel sections of a certain vector bundle with conformally invariant connection, where the parallel sections satisfy a set of non-linear algebraic constraints that we describe. The vector bundle captures 2-form prolongations and is isomorphic to $\Lambda^3(\mathcal{T})$, where $\mathcal{T}$ is the tractor bundle of conformal geometry, but the resulting connection differs from the normal tractor connection by curvature terms.

Our analysis leads to a set of obstructions for a Riemannian metric to be conformal to a Kähler metric. In particular, we find an explicit algebraic condition for a Weyl tensor which must hold if there exists a conformal Killing–Yano tensor, which is a necessary condition for a metric to be conformal to Kähler. This gives an invariant characterization of algebraically special Riemannian metrics of type D in dimensions higher than four.

Keywords: conformal geometry; geometric invariants; Kähler structures; prolongation of PDE

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In memory of Anna Edmunds (1969–2023)

1. Introduction

Let (M, g) be a Riemannian manifold of even dimension $n \geq 4$. Does there exist a non-zero function $\Omega : M \longrightarrow \mathbb{R}$ such that $\hat{g} = \Omega^2 g$ is Kähler? That is, does there exist a non-degenerate two-form $\hat{\omega}$, which is covariantly constant with respect to

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the Levi-Civita connection of $\hat{g}$ and such that the endomorphism $J : TM \to TM$ defined[1] by $\hat{\omega}(X, Y) = \hat{g}(X, JY)$ satisfies $J^2 = -\mathrm{Id}$?

In this paper, we shall focus on local obstructions, which arise because the conformal to Kähler problem leads to an over-determined system of PDEs of finite type [4]. We shall establish a one-to-one correspondence between Kähler metrics in a conformal class and certain (*special*) parallel sections of a vector bundle $E \to M$ of rank $n(n+1)(n+2)/6$

$$E \cong \begin{pmatrix} \Lambda^2(M) \\ \Lambda^1(M) \oplus \Lambda^3(M) \\ \Lambda^2(M) \end{pmatrix} \tag{1.1}$$

equipped with a connection that takes the form

$$\mathcal{D}\begin{pmatrix} \omega \\ K,\ \mu \\ \Sigma \end{pmatrix} = \begin{pmatrix} \nabla\omega - \mu - g \oslash K \\ \nabla K - \Sigma - P \oslash \omega, \quad \nabla\mu - g \oslash \Sigma - P \oslash \omega - C \oslash \omega \\ \nabla\Sigma - P \oslash K - A \oslash \omega - C \oslash K \end{pmatrix}. \tag{1.2}$$

Here, A =Cotton, C =Weyl, and P =Schouten are different components of the curvature tensor of g and its derivatives, and $\oslash$ indicates an algebraic operation involving contractions of various kinds (differing in each line above). The latter will be specified in Theorem 2.1 in §2. We shall say that a section $\Psi = (\omega, K, \mu, \Sigma)$ of $(E, \mathcal{D})$ is *special* if it satisfies a set of algebraic conditions

$$\mathcal{Q}(\Psi) = 0, \tag{1.3}$$

which will be specified in Theorem 4.1 in §4. It is the presence of these conditions that makes the analysis difficult. General parallel sections of $(E, \mathcal{D})$ are in one-to-one correspondence with conformal Killing–Yano tensors [13, 16, 20], and the algebraic constraints single out those conformal Killing–Yano tensors which give rise to Kähler forms.

The analysis leads to two kinds of obstructions. Those arising from reducing the holonomy of the curvature of $\mathcal{D}$ to a subgroup stabilizing a section of E, and those arising from differentiating the algebraic conditions. This second class of constraints is an overdetermined system of algebraic equations which can admit non-zero solutions if and only if the relevant Bezout resultants vanish (Theorem 3.1). The dimension of the variety of constraints (1.3) in the fibres of E is at most $(n^2+2n+4)/4$ if $n>4$. In dimension $n=4$, the constraints can be solved explicitly, reducing the rank 20 vector bundle E to a rank 10 sub-bundle

[1]This is one of many equivalent definitions. It implies that J is Hermitian and that

$$[T^{(1,0)}, T^{(1,0)}] \subset T^{(1,0)}, \quad \text{where} \quad T^{(1,0)} = \{X \in TM \otimes \mathbb{C}, J(X) = iX\}.$$

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$$E_0 = \Lambda^2_+(M) \oplus \Lambda^1(M) \oplus \Lambda^2_-(M),$$

whose section consists of a self-dual two-form, a one-form, and an anti-self-dual two-form [8] (this is isomorphic to the bundle $\Lambda^3_+(\mathbf{T})$ of self-dual tractor three-forms). That is, using the notation above, the system of $\mathcal{Q}(\Psi) = 0$ is equivalent to

$$*\omega = \omega, \quad K = *\mu, \quad *\Sigma = -\Sigma.$$

The complete set of obstructions has been constructed explicitly in this case[2] [8]. In §5, we shall link Theorem 2.1 and Theorem 4.1 with the tractor approach to conformal geometry [2], and identify the prolongation bundle E with the third exterior power of the rank $(n+2)$ tractor bundle $\mathcal{T}$ over M. The prolongation connection (1.2) differs from the standard tractor connection by the curvature terms, and some of the non-linear constraints can be encoded in algebraic conditions involving the scale tractor.

2. Prolongation of the conformal-to-Kähler system

In this section, we shall directly construct the prolongation connection (1.2) underlying the conformal-to-Kähler problem.

THEOREM 2.1. *There exists a correspondence between Kähler metrics in a given conformal class and parallel sections* $\Psi = (\omega, K, \mu, \Sigma)$ *of* $(E \to M, \mathcal{D})$ *given by* (1.1) *and* (1.2).

REMARK. *This is not a one-to-one correspondence. Every Kähler metric corresponds to a parallel section of* $(E, \mathcal{D})$*, but not all parallel sections give rise to Kähler metrics. Those that do will be characterized in Theorem* 4.1.

Proof of Theorem 2.1. Consider a Kähler structure $(\hat{g}, \hat{\omega})$, and define (g, ω) by[3]

$$\hat{g} = \Omega^2 g, \qquad \hat{\omega} = \Omega^3 \omega,$$

where Ω is a smooth positive function. The condition $\hat{\nabla}\hat{\omega} = 0$ yields

$$\nabla_a \omega_{bc} = \mu_{abc} + 2g_{a[b}K_{c]}, \tag{2.1}$$

where $\mu \in \Lambda^3(M)$ and $K \in \Lambda^1(M)$ are given by

$$\mu = -3\Upsilon \wedge \omega, \quad K = -\Upsilon \lrcorner \omega, \tag{2.2}$$

and $\Upsilon = \Omega^{-1} d\Omega$.

[2]The details of the prolongation connection on E_0 constructed in [8] differ from those in Theorem 2.1 as in four dimensions it is convenient to fix the freedom in the definition of Σ so that Σ is anti-self-dual and so $g(\omega, \Sigma) = 0$.

[3]The standard scaling in Kahler geometry would be $\hat{\omega} = \Omega^2\omega$. We attach a different conformal weight to ω to ensure that equation (2.1) is conformally invariant.

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Conversely, assume that (2.1) holds with arbitrary μ and K, for some ω such that $\omega^a{}_b\omega^b{}_c$ is pure trace. Recall the decomposition of the Riemann tensor in conformal geometry

$$R_{abcd} = C_{abcd} + P_{ac}g_{bd} - P_{bc}g_{ad} + P_{bd}g_{ac} - P_{ad}g_{bc},$$

where C_{abcd} is the Weyl curvature and

$$P_{ab} = \frac{1}{n-2}\Big(R_{ab} + \frac{R}{2(1-n)}g_{ab}\Big)$$

is the Schouten tensor. Under conformal rescalings $\hat{g} = \Omega^2 g$ we have

$$\hat{C}_{abcd} = \Omega^2 C_{abcd}, \quad \hat{P}_{ab} = P_{ab} - \nabla_a \Upsilon_b + \Upsilon_a\Upsilon_b - \frac{1}{2}|\Upsilon|^2 g_{ab}.$$

We differentiate (2.1), commute the derivatives, and use the Ricci identity

$$[\nabla_a, \nabla_b]\omega_{cd} = -R_{abc}{}^p\omega_{dp} + R_{abd}{}^p\omega_{cp}.$$

This leads to a set of algebraic conditions

$$C_{bc[a}{}^e\omega_{d]e} + C_{ad[b}{}^e\omega_{c]e} = 0, \tag{2.3}$$

and a pair of linear differential equations

$$\nabla_a K_b = P_a{}^c\omega_{bc} + \Sigma_{ab} \tag{2.4}$$

and

$$\nabla_a \mu_{bcd} = -3g_{a[b}\Sigma_{cd]} - 3P_{a[b}\omega_{cd]} - \frac{3}{2}C_{[bc|a}{}^p\omega_{p|d]}, \tag{2.5}$$

where Σ_{ab} is as yet undetermined two-form. Differentiating (2.4) and (2.5) once more gives

$$\nabla_a\Sigma_{bc} = 2P_{a[b}K_{c]} - P_a{}^e\mu_{ebc} + \frac{1}{2}A^p{}_{bc}\omega_{pa} + A^p{}_{a[b}\omega_{c]p} + C_{bca}{}^pK_p, \tag{2.6}$$

where $A_{abc} = \nabla_b P_{ca} - \nabla_c P_{ba}$ is the Cotton tensor. The system is now closed, as derivatives of all unknowns have been determined. We can combine equations (2.1), (2.4), (2.5), (2.6) into a connection (1.2), where the meaning of $\oslash$ in each slot is now clear. □

As a spin-off from the prolongation procedure, we deduce the following (well-known)

Corollary 2.2. *If a non-Kähler manifold* (M, g) *is Einstein and conformal to Kähler, then* g *admits a Killing vector.*

Proof. This follows directly from (2.4). If g is Einstein then the RHS of (2.4) is skew-symmetric, and thus K satisfies the Killing equations $\nabla_{(a}K_{b)} = 0$. □

In four dimensions, one can establish a stronger result: an ASD Einstein metric with non-zero Ricci scalar is conformal to Kähler if and only if it admits a Killing vector [7, 8].

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3. Type *D* and obstructions algebraic in Weyl tensor

If we view both C and ω as endomorphisms of Λ^2 given by

$$C(\varphi)_{ab} = C^{cd}{}_{ab}\varphi_{cd} \quad \text{and} \quad \omega(\varphi)_{ab} = \omega_{[a}{}^{c}\varphi_{b]c},$$

then the constraint (2.3) is equivalent to the commutativity of these endomorphisms. Thus, they can be diagonalized in the same basis. In [16], it was used to show that the Weyl tensor is of algebraic type D in the sense of [5, 18].

We adopt a different approach. Consider a linear map

$$\begin{aligned} &B : \Lambda^2 \to \mathcal{W} \subset \Lambda^2 \odot \Lambda^2, \quad \text{with} \\ &B(\varphi)_{bcad} := C_{bc[a}{}^{e}\varphi_{d]e} + C_{ad[b}{}^{e}\varphi_{c]e}, \end{aligned}$$

given by the LHS of equation (2.3). Here, $\mathcal{W}$ is a vector space of rank-four tensors which have the algebraic symmetries of a Weyl tensor, i.e. if $e \in \Gamma(\mathcal{W})$ then

$$e_{a[bcd]} = 0, \quad e_{abcd} = e_{[ab]cd}, \quad e_{abcd} = e_{ab[cd]},$$

and e is trace-free with respect to metric contractions. The dimension of ω is greater than that of Λ^2, and equation (2.3) implies that B has a non-empty kernel which contains a non-degenerate two-form. Therefore, the rank of B is not maximal. This leads to a set of algebraic conditions on the Weyl tensor which we shall now give.

THEOREM 3.1. *Let $X \in \Lambda^2(TM)$ be a bi-vector and let $\beta_X : \Lambda^2 \to \Lambda^2$ be given by*

$$(\beta_X)_{ab}{}^{cd} \equiv X^{ef}\beta_{efab}{}^{cd}, \quad \textit{where} \quad \beta_{bcad}{}^{ef} = C_{bc[a}{}^{e}\delta^{f}{}_{d]} + C_{ad[b}{}^{e}\delta^{f}{}_{c]}. \tag{3.1}$$

Then g *is conformal to a Kähler metric only if, for all bi-vectors* X, *det*$(\mathcal{B}) = 0$, *where*

$$\mathcal{B} = \frac{1}{N!}\begin{pmatrix} 0 & (N-1)! & 0 & \dots & 0 & 0 \\ s_2 & 0 & (N-2)! & \dots & 0 & 0 \\ s_3 & s_2 & 0 & (N-3)! & \dots & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ s_{N-1} & s_{N-2} & & & 0 & 1 \\ s_N & s_{N-1} & s_{N-2} & \dots & s_2 & 0 \end{pmatrix}, \tag{3.2}$$

and $s_k \equiv Tr(\beta_X{}^{k})$, and $k = 1, 2, \dots, N$.

Proof. Rewriting (2.3) as

$$\beta_{bcad}{}^{ef}\omega_{ef} = 0,$$

where β is given by (3.1), we deduce that for any fixed values of the pair of indices $[bc]$, the determinant of the resulting $n(n-1)/2$ by $n(n-1)/2$ traceless square matrix β must vanish. In the case $n = 4$, this leads to an invariant condition on

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the self-dual part of Weyl tensor, as explained in [8]. For any bi-vector $X \equiv X^{ab}$ consider a composition of homomorphisms

$$\Lambda^2(M) \xrightarrow{B} \omega \xrightarrow{X \lrcorner} \Lambda^2(M)$$

where the second map is a contraction. This gives a traceless homomorphism

$$(\beta_X)_{ab}{}^{cd} \equiv X^{ef}\beta_{efab}{}^{cd}.$$

To find an invariant obstruction—a tensor of rank $n(n-1)$ on TM—we shall use the Cayley–Hamilton theorem for traceless $N \times N$ matrices, where $N = n(n-1)/2$ is the dimension of $\Lambda^2(M)$. Set

$$s_k = \mathrm{Tr}(\beta_X{}^k), \quad k = 1, 2, \ldots, N$$

so that

$$s_1 = 0, \quad s_2 = X^{ab}X^{cd}\beta_{abpq}{}^{rs}\beta_{cdrs}{}^{pq}, \quad s_3 = X^{ab}X^{cd}X^{ef}\beta_{abpq}{}^{rs}\beta_{cdrs}{}^{uv}\beta_{efuv}{}^{pq}, \quad \ldots.$$

The determinant of β_X can then be expressed as the Nth Bell polynomial [3]

$$\begin{aligned} \det(\beta_X) &= \frac{1}{N!}B_N(s_1, -1!s_2, 2!s_3, \ldots, (-1)^N N!s_N) \\ &= \det(\mathcal{B}), \end{aligned} \tag{3.3}$$

where $\mathcal{B}$ is given by (3.2). □

For this to be a non-trivial obstruction, we need to show that β_X can have maximal rank (and therefore is injective) for some Weyl tensor $C_{abc}{}^d$ as it will then have maximal rank in a neighbourhood of this Weyl tensor in ω. We could show it by specifying an element of $\mathcal{E}$ at a point in M, but we can do even better, and write down a metric which gives rise to such an injective Weyl tensor. On an open set in $\mathbb{R}^6$ with coordinates (x, y, z, t, u, v) consider a metric

$$g = dx^2 + dy^2 + dz^2 + dt^2 + du^2 + dv^2 + c(t^2 + yt)dxdy + c(t^2 + tu)dudv,$$

where c is a constant, and take

$$X = 2\partial_x \wedge \partial_y + \partial_z \wedge \partial_u.$$

Evaluating the Weyl tensor of this metric at the point $(0, 0, 0, 0, 0, 0)$, and computing the obstruction (3.3) yields

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$$\begin{aligned}\mathcal{B} = &-\frac{s_3\,{s_4}^3}{1152}+\frac{{s_2}^6 s_3}{138240}-\frac{{s_2}^3{s_3}^3}{7776}+\frac{s_3\,{s_6}^2}{216}-\frac{{s_2}^5 s_5}{19200}-\frac{{s_3}^3 s_6}{972}+\frac{{s_2}^4 s_7}{2688}+\frac{{s_4}^2 s_7}{224}\\
&-\frac{s_7\,s_8}{56}-\frac{{s_2}^3 s_9}{432}+\frac{{s_3}^2 s_9}{162}-\frac{s_6\,s_9}{54}-\frac{1}{50}\,s_{10}\,s_5+\frac{s_{11}\,{s_2}^2}{88}-\frac{1}{44}\,s_{11}\,s_4\\
&-\frac{1}{36}\,s_3\,s_{12}-\frac{{s_2}^2 s_3\,s_8}{192}+\frac{s_2\,s_5\,s_8}{80}+\frac{s_3\,s_4\,s_8}{96}-\frac{{s_2}^2 s_4\,s_7}{224}-\frac{s_2\,{s_3}^2 s_7}{252}+\frac{s_2\,s_6\,s_7}{84}\\
&+\frac{s_3\,s_5\,s_7}{105}+\frac{{s_2}^3 s_3\,s_6}{864}-\frac{{s_2}^2 s_5\,s_6}{240}+\frac{s_4\,s_5\,s_6}{120}+\frac{{s_2}^3 s_4\,s_5}{960}+\frac{{s_2}^2{s_3}^2 s_5}{720}-\frac{s_2\,s_3\,{s_5}^2}{300}\\
&-\frac{s_2\,{s_4}^2 s_5}{320}-\frac{{s_3}^2 s_4\,s_5}{360}-\frac{{s_2}^4 s_3\,s_4}{4608}+\frac{{s_2}^2 s_3\,{s_4}^2}{768}+\frac{s_2\,{s_3}^3 s_4}{1296}-\\
&\frac{s_2\,s_3\,s_4\,s_6}{144}+\frac{s_{15}}{15}+\frac{{s_5}^3}{750}-\frac{1}{26}\,s_2\,s_{13}+\frac{{s_3}^5}{29160}+\frac{s_{10}\,s_2\,s_3}{60}+\frac{s_2\,s_4\,s_9}{72},\end{aligned}$$

and gives a non-zero answer

$$\mathcal{B}=\frac{9639\,c^{15}}{17592186044416}.$$

4. Nonlinear algebraic conditions

We now move to the second source of obstructions, namely the nonlinear condition $J^2=-\mathrm{Id}$. In the index notation, this is

$$\omega^a{}_b\omega^b{}_c+\frac{1}{n}|\omega|^2\delta^a{}_c=0. \tag{4.1}$$

The case $n=4$ was treated in [8], so in the Theorem below, we shall assume that $n>4$.

THEOREM 4.1. *There is a one-to-one correspondence between Kähler metrics in a conformal class and parallel sections Ψ of the vector bundle $(E,\mathcal{D})$ from Theorem 2.1 such that*

$$\mathcal{Q}(\Psi)=0, \tag{4.2}$$

where $\mathcal{Q}$ is the set of non-linear algebraic conditions given by (4.1) and

$$\mu_{abc}+\frac{3n}{|\omega|^2}\omega_{[ab}\omega^d{}_{c]}K_d=0, \tag{4.3}$$

$$\begin{aligned}\Sigma_{ab}=&-\Big(\frac{n}{2}|\omega|^{-2}|K|^2+\frac{n}{4(n-2)(n-4)}C_{cdef}\omega^{cd}\omega^{ef}\Big)\omega_{ab}\\
&+\frac{1}{2(n-4)}C_{cdab}\omega^{cd}+2n|\omega|^{-2}K_c\omega^c{}_{[b}K_{a]}\quad(n>4).\end{aligned} \tag{4.4}$$

We shall split the proof into two steps.

PROPOSITION 4.2. *Solutions ω_{bc} to the CKY equation (2.1) that also satisfy (4.1) correspond to conformally related Kähler metrics iff (4.3) holds.*

Proof. $\Rightarrow$: In this direction, the result is immediate as (2.2) implies (4.3).

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$\Leftarrow$: Assume (2.1) and (4.1) hold. We would like to deduce (2.2), as then a conformal factor can be found which turns g into a Kähler metric. Differentiating the condition (4.1) leads to

$$\omega_a{}^b\nabla_d\omega_{bc}+\omega_c{}^b\nabla_d\omega_{ba}=2\Omega^{-2}g_{ac}\Upsilon_d,$$

where we defined the positive function Ω by $|w|_g^2=n\Omega^{-2}$, and as usual $\Upsilon_a:=\Omega^{-1}\nabla_a\Omega$. Substituting (2.1) yields

$$\omega_a{}^b\mu_{dbc}+\omega_c{}^b\mu_{dba}+\omega_{ad}K_c+\omega_{cd}K_a-g_{dc}\omega_a^bK_b-g_{da}\omega_c^bK_b=2\Omega^{-2}g_{ac}\Upsilon_d. \quad (4.5)$$

From this, we find easily the second part of (2.2), as follows. Contracting (4.5) with g^{ac} and g^{dc} respectively, and taking an appropriate linear combination of the resulting two equations yields

$$(n-2)K_c\omega^c{}_a+\omega^{bc}\mu_{abc}=0, \quad (4.6)$$

which then implies

$$K_a=\omega_{ab}\Upsilon^b. \quad (4.7)$$

Thus, (4.5) now gives a stronger relation between K and μ, namely

$$\omega_a{}^b\mu_{dbc}+\omega_c{}^b\mu_{dba}+\omega_{ad}K_c+\omega_{cd}K_a-g_{dc}\omega_a^bK_b-g_{da}\omega_c^bK_b=-2g_{ac}\omega_d^bK_b,$$

or equivalently in terms of Υ

$$\omega_a{}^b\mu_{dbc}+\omega_c{}^b\mu_{dba}+3\omega_a{}^b\Upsilon_{[d}\omega_{bc]}+3\omega_c{}^b\Upsilon_{[d}\omega_{ba]}=0. \quad (4.8)$$

Now clearly $\mu_{abc}=-3\Upsilon_{[a}\omega_{bc]}$ is a solution of (4.8). Note, however, that by linearity, (4.8) only determines μ_{abc} up to the addition of three-forms which belong to the kernel of a linear operator $T:\Lambda^3\to(\Lambda^1\odot\Lambda^1)\otimes\Lambda^1$ given by

$$T(\tau)_{abc}=\omega_a{}^d\tau_{bcd}+\omega_b{}^d\tau_{acd}. \quad (4.9)$$

We can decompose Λ^3 orthogonally into parts that are trace-free and pure trace with respect to ω:

$$\Lambda^3=\mathring{\Lambda}^3\oplus\Lambda^{1,3}.$$

The kernel of T consists of 3-forms that are in $\Lambda^{3,0}\oplus\Lambda^{0,3}$ with respect to J, but all we need to know currently is that if $\tau\in\mathrm{Ker}\ (T)$ then $\omega^{ab}\tau_{abc}=0$ (as follows immediately by contracting (4.9) with g^{ab}), so $\ker(T)\subset\mathring{\Lambda}^3$.

Let us write I for the linear sub-bundle of $(\Lambda^1\odot\Lambda^1)\otimes\Lambda^1$ consisting of elements of the form

$$-3\omega_a{}^b\Upsilon_{[d}\omega_{bc]}-3\omega_c{}^b\Upsilon_{[d}\omega_{ba]}.$$

Then $T^{-1}(I)$ is a linear sub-bundle of Λ^3. On the other hand, since $\mu_{abc}=-3\Upsilon_{[a}\omega_{bc]}$ is a solution of (4.8), it is clear that $\Lambda^{1,3}\subset T^{-1}(I)$ and every element in $T^{-1}(I)$ is, pointwise, the vector sum of an element $\Lambda^{1,3}$ with an element of $\ker(T)$. Putting these things together, it is clear that we obtain unique solutions

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μ to (4.8) if we restrict to μ which are pure trace. Or in other words μs such that their trace-free part is zero: We can write the condition for μ explicitly as[4]

$$|\omega|^2\mu_{abc} - \frac{3n}{n-2}\omega_{[bc}\mu_{a]pq}\omega^{pq} = 0. \tag{4.10}$$

If this condition holds together with (4.1) then, using (4.7), there exists a unique solution to (2.1) given by (2.2). Using (4.6) we see that (4.10) is equivalent to (4.3). □

LEMMA 4.3. *For* μ *and* K *as defined in* (2.1)*, the following identities hold*

$$\mu_{abc}K^c = 0, \quad \textit{and} \quad \omega^a{}_{[b}\Sigma_{c]a} = 0. \tag{4.11}$$

Proof. In the conformal to Kähler scale we have $\mu_{abc} = -3\Upsilon_{[a}\omega_{bc]}$ and $\Upsilon_a = n|\omega|^{-2}K^d\omega_{da}$ so that

$$K^c\mu_{abc} \sim (\Upsilon_a\Upsilon_b - \Upsilon_b\Upsilon_a) = 0.$$

For the second property, which shows that Σ is Hermitian, we use that from the expression for Υ in terms of ω and K, as above, we have

$$\nabla_a\Upsilon_b = \Omega^2(2\Upsilon_a\Upsilon_b\Omega + \mathrm{P}_a{}^d\omega_{cd}\omega^c{}_b + \Sigma_{ac}\omega^c{}_b - K^c\mu_{abc} - g_{ab}|K|^2 + K_aK_b),$$

and so

$$\nabla_{[a}\Upsilon_{b]} = -\Sigma_{c[a}\omega^c{}_{b]}.$$

But of course Υ is exact, and hence closed. □

PROPOSITION 4.4. *If* $\mathrm{n}>4$ *then* Σ *is determined by* ω *and* K*, and is given by* (4.4).

Proof. Consider (2.4), (2.5) and (4.6) and compute

$$\nabla_a(\mu_{bcd}\omega^{cd}) = (n-2)\nabla_a(\omega_b{}^eK_e).$$

The RHS gives

$$(n-2)(g_{ab}|K|^2 - K_aK_b - \frac{1}{n}|\omega|^2\mathrm{P}_{ab} + \omega_b{}^e\Sigma_{ae}) \tag{4.12}$$

and the LHS gives

$$-g_{ab}\Sigma\cdot\omega - 2\Sigma_{bc}\omega^c{}_a + \mu_b{}^{cd}\mu_{acd} + \frac{2-n}{n}|\omega|^2\mathrm{P}_{ab} - \frac{1}{2}C_{cda}{}^p\omega_{pb}\omega^{cd} \tag{4.13}$$

where we have used (4.11).

[4]Note that this holds identically if $n=4$, as any element of Λ^3 in four dimensions is of the form $\tau = *\kappa$ for $\kappa\in\Lambda^1$, and

$$\omega^{ab}\tau_{abc} = \frac{1}{6}\omega^{ab}\varepsilon_{abcd}\kappa^d = \frac{1}{3}\omega_{cb}\kappa^b$$

as ω is self-dual. The RHS of the expression above never vanishes unless $\kappa=0$.

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Next use

$$\mu = 3\Upsilon_{[a}\omega_{bc]}, \quad \text{and} \quad \Upsilon_a = n|\omega|^{-2}K_b\omega^b{}_c$$

to compute

$$\mu_b{}^{cd}\mu_{acd} = 2|K|^2 g_{ab} - 2K_aK_b + (n-4)n|\omega|^{-2}K_pK_q\omega^p{}_a\omega^q{}_b.$$

Substituting this formula in (4.13), comparing with (4.12) and contracting both sides with $\omega^b{}_e$ gives

$$\frac{n-4}{n}|\omega|^2\Sigma_{ae} = -(\Sigma\cdot\omega)\omega_{ae} + \frac{1}{2n}|\omega|^2 C_{cdae}\omega^{cd} + (4-n)|K|^2\omega_{ae} + 2(n-4)K_b\omega^b{}_{[e}K_{a]}. \tag{4.14}$$

Contracting this with ω^{ae} gives an expression for $\Sigma\cdot\omega$, which we can substitute back to (4.14). This gives (4.4). □

Theorem 4.1 now follows from Proposition 4.2 and Proposition 4.4.

The non-linear conditions in Theorem 4.1 trace a variety $\mathcal{S}$ in the fibres of the prolongation bundle E. If $n>4$ then

$$\dim\mathcal{S} \leq \frac{1}{4}(n^2+2n+4). \tag{4.15}$$

To see it, note that in Theorem 4.1 both Σ and μ have been determined in terms of ω and K. Substituting the expression for Σ_{ab} into the expression (2.6) for $\nabla_a\Sigma_{bc}$ could lead to an algebraic condition only involving K and ω. We claim that, at least in the conformally flat case, this condition is an identity, and does not lead to any further constraints on K. Indeed, we can choose a flat metric g in the conformal class, in which case (4.4) and (2.6) reduce to

$$\Sigma_{ab} = -\frac{n}{2}|\omega|^{-2}|K|^2\omega_{ab} + 2n|\omega|^{-2}K_c\omega^c{}_{[b}K_{a]}, \quad \nabla_a\Sigma_{bc} = 0.$$

Substituting the first expression into the second and eliminating the derivatives of μ,ω, and K using the prolongation connection leads to an identity. Therefore, we can specify the n components of K, which are unconstrained, and the components of skew form ω which squares to a pure trace. To count those set $n=2m$, take

$$J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

and consider

$$\omega = J + \epsilon A, \quad \text{where} \quad \mathrm{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Impose $\omega^2 = -I_{2n}$. This gives $A = JAJ$, and gives $b=c, a=-d$. If ω is a two-form then $a=-a^T$, and only the skew-part of b contributes so we can take $b=-b^T$ which gives a total of $m(m-1)$ components. There is one remaining component corresponding to the choice of an overall scale of ω. Therefore, the dimension of $\mathcal{S}$ is $m(m-1)+1+2m$ which gives (4.15).

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In dimension four, where the constraints on (Σ, ω) have been solved using self-duality [8], and the bundle E has been identified with the bundle of self-dual tractor three-forms, which has rank 10. The tractor approach for the general n will be discussed in the next Section.

5. Tractors

The aim of this Section is to outline how the prolongation of the conformal Killing-Yano equations in Theorem 2.1 and the associated non-linear conditions on the parallel sections (Theorem 4.1) can be formulated in the tractor language of [2].

In this section, by a conformal manifold, we mean a manifold equipped with an equivalence class $\boldsymbol{c}$ of Riemannian metrics such that if $g, \hat{g} \in \boldsymbol{c}$ then $\hat{g} = \Omega^2 g$ for some positive function Ω. With $\mathcal{F}$ denoting the frame bundle, the bundle of densities of weight $w \in \mathbb{R}$ is the associated line bundle $\mathcal{E}[w] = \mathcal{F} \times_{\rho_w} \mathbb{R}$ where ρ_w is the representation of $GL(n, \mathbb{R})$ on $\mathbb{R}$ given by

$$\mathbb{R} \times GL(n, \mathbb{R}) \ni (x, A) \mapsto |\det(A)|^{-w/n} x \in \mathbb{R}.$$

Therefore, there exists a canonical isomorphism $\mathcal{E}[2n] \cong \otimes^2(\Lambda^n TM)$. Using the conformal structure a section $\varphi \in \Gamma(\mathcal{E}[w])$ can be identified with an equivalence class of metric function pairs $[(g, f)]$ where

$$(\Omega^2 g, \Omega^w f) \sim (g, f).$$

For any weight w, the bundle $\mathcal{E}[w]$ is oriented and we write $\mathcal{E}_+[w]$ for the positive elements.

Using this notation, it is straightforward to see that the conformal structure determines a tautological section $\boldsymbol{g}$ of $S^2 T^*M \otimes \mathcal{E}[2]$ that we term the conformal metric. Then the metrics in a conformal class correspond 1-1 with sections $\sigma \in \Gamma(\mathcal{E}_+[1])$ by the formula

$$g = \sigma^{-2} \boldsymbol{g}.$$

We call $\sigma \in \Gamma(\mathcal{E}_+[1])$ (or the corresponding $g \in \boldsymbol{c}$) a *scale*. In the following, we mainly follow [2] and [6].

5.1. The standard tractor bundle and normal tractor connection

On a conformal manifold $(M, \boldsymbol{c})$ there is, in general, no preferred connection on TM but (in dimensions $n \geq 3$) there exists a connection on a rank-2 extension

$$\mathcal{T} = \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1], \tag{5.1}$$

that we call the *conformal tractor bundle* $\mathcal{T}$. The right hand side of (5.1) gives the composition series of $\mathcal{T}$ and means that $\mathcal{E}[-1]$ is a canonical sub-bundle of $\mathcal{T}$. The quotient of $\mathcal{T}$ by this has $T^*M[1]$ as a canonical sub-bundle, and then the quotient by this is $\mathcal{E}[1]$. The tractor bundle $\mathcal{T}$ will be denoted $\mathcal{E}_A$ in the abstract index notation.

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Given $g \in \boldsymbol{c}$ the tractor bundle splits into a direct sum

$$\mathcal{T} \overset{g}{=} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1].$$

So $V_B \in \Gamma(\mathcal{E}_B)$ can be then represented by a triple

$$V_B \overset{g}{=} \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix}.$$

The *normal tractor connection* is

$$\nabla^{\mathcal{T}}_a(\sigma, \mu_b, \rho) = (\nabla_a\sigma - \mu_a,\ \nabla\mu_b + P_{ab}\sigma + \boldsymbol{g}_{ab}\rho,\ \nabla_a\rho - P_{ab}\mu^b).$$

This connection acts on tensor powers of $\mathcal{T}$, and $\nabla^{\mathcal{T}}$ preserves a conformally invariant *tractor metric* h that is given as a quadratic form by

$$\mathcal{T} \ni V = (\sigma, \mu_b, \rho) \mapsto 2\sigma\rho + \mu_b\mu^b = h(V, V).$$

In abstract indices we denote this h_{AB} and use it to raise and lower tractor indices. It is convenient to introduce the algebraic splitting operators X^A, Y^A, Z^{Ab} that encode the slots,

$$V^A = \sigma Y^A + Z^{Ab}\mu_b + X^A\rho, \tag{5.2}$$

and we will need this below. For example in the slot notation X_B is represented by $(0,\ 0,\ 1)$.

There is a conformally invariant differential operator $D : \Gamma(\mathcal{E}[1]) \to \mathcal{T}$ given, in a scale g, by

$$\sigma \mapsto D_B\sigma \overset{g}{=} (\sigma,\ \nabla_b\sigma, -\frac{1}{n}(\Delta\sigma + P\sigma)),$$

where $P := \boldsymbol{g}^{ab}P_{ab}$. Given the splittings as described this is determined by the tractor connection formula. (Alternatively, when the tractor bundle is constructed via jets, this operator actually determines the splitting of the tractor bundle into the triples [6].) It is termed a splitting operator as the composition $X^B D_B$ is the identity on $\Gamma(\mathcal{E}[1])$. If σ is a scale, then we define

$$I_B := D_B\sigma,$$

and call I_B the corresponding *scale tractor*. So $\sigma = X^A I_A$. The squared length of the scale tractor recovers a multiple of the scalar curvature of the metric $g = \sigma^{-2}\boldsymbol{g}$:

$$I^A I_A = h^{AB} I_A I_B = -\frac{1}{n(n-1)}R.$$

One reason that D_B is important is that if $V_B \in \Gamma(\mathcal{E}_B)$ is parallel for the tractor connection then $V_B = D_B\tau$ for some $\tau \in \Gamma(\mathcal{E}[1])$.

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5.2. 3-form tractors

It follows from the semi-direct composition series of $\mathcal{T}$ that the corresponding decomposition of $\Lambda^3\mathcal{T}$ is

$$\mathcal{E}_{[ABC]} = \mathcal{E}^2[3] \oplhd \left(\mathcal{E}^3[3] \oplus \mathcal{E}^1[1]\right) \oplhd \mathcal{E}^2[1], \tag{5.3}$$

where $\mathcal{E}^k[w]$ denotes $\Lambda^k(T^*M) \otimes \mathcal{E}[w]$.

3-form tractors are useful for studying the conformal Killing Yano equation. For $\sigma_{ab} \in \Gamma(\mathcal{E}_{[ab]}[3])$ let us write

$$KY(\sigma)_{abc} := \nabla_a\sigma_{bc} - \nabla_{[a}\sigma_{bc]} + \frac{2}{n-1}\boldsymbol{g}_{a[b}\nabla^p\sigma_{c]p}\,.$$

This is conformally invariant, and σ is a conformal Killing-Yano tensor if

$$KY(\sigma)_{abc} = 0. \tag{5.4}$$

Now, a choice of metric g from the conformal class, determines a splitting of the bundle $\Lambda^3\mathcal{T}$ into four components (a replacement of the $\oplhd$s with $\oplus$s is effected) so that a 3-tractor Φ can be written as a 4-tuple

$$\Phi_{ABC} \overset{g}{=} \begin{pmatrix} \sigma_{bc} \\ \nu_{abc} \quad \varphi_c \\ \rho_{bc} \end{pmatrix}$$

where $\sigma_{bc} \in \mathcal{E}^2[3]$, $\nu \in \mathcal{E}^3[3]$ and so forth.

Given $\sigma_{bc} \in \Gamma(\mathcal{E}^2[3])$ there is a conformally invariant differential splitting operator

$$L : \Gamma(\mathcal{E}^2[3]) \to \Gamma(\Lambda^3\mathcal{T}),$$

determined by the tractor connection, and given by

$$\Gamma(\mathcal{E}^2[3]) \ni \sigma_{bc} \mapsto L(\sigma) \overset{g}{=} \begin{pmatrix} \sigma_{bc} \\ \nabla_{[a}\sigma_{bc]} \quad \frac{2}{n-1}\nabla^b\sigma_{bc} \\ \frac{1}{2n}\nabla^p KY(\sigma)_{pbc} - \frac{1}{n-1}\nabla_b\nabla^p\sigma_{pc} - P_b{}^p\sigma_{pc} \end{pmatrix} \in \Gamma(\Lambda^3\mathcal{T}), \tag{5.5}$$

see [13]. Now the key importance of L is that it is related to the prolongation connection $\mathcal{D}$ of Theorem 2.1 for the conformal Killing-Yano equation (5.4). The following is a special case of Theorem 3.9 in [13].

PROPOSITION 5.1. *There is a conformally invariant connection $\mathcal{D}$ on $\Lambda^3\mathcal{T}$ with the property that*

(1) $\mathcal{D}\Phi = 0$ *implies that*

$$\Phi = L(\sigma_{ab}) \quad \textit{and} \quad KY(\sigma_{ab}) = 0;$$

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(2) *If* $KY(\sigma_{ab})=0$ *then* $\mathcal{D}(L(\sigma_{ab}))=0$.

This has the form

$$\mathcal{D}_a\Phi_{BCD}=\nabla_a\Phi_{BCD}+(\kappa\sharp\Phi)_{aBCD}$$

where ∇_a *is the normal tractor connection and* $\kappa\sharp\Phi$ *a linear action of its curvature on* Φ.

The details of $(\kappa\sharp\Phi)_{aBCD}$ will not be needed below, but, with a little translation, they can be read-off from (2.6).

We want to apply this to $\omega_{ab}\in\Gamma(\mathcal{E}^2[3])$. If we assume that ω_{ab} satisfies the conformal Killing-Yano equation then the image of (5.5) simplifies to

$$L(\omega_{ab}):=\begin{pmatrix}\omega_{bc}\\ \nabla_{[a}\omega_{bc]} & \frac{2}{n-1}\nabla^b\omega_{bc}\\ -\frac{1}{n-1}\nabla_b\nabla^p\omega_{pc}-P_b{}^p\omega_{pc}\end{pmatrix}. \tag{5.6}$$

5.3. Characterizations of conformally Kähler

For $\omega_{ab}\in\Gamma(\mathcal{E}^2[3])$ let us write

$$\sigma:=|\omega|:=\sqrt{\frac{1}{n}\omega_{ab}\omega^{ab}}, \tag{5.7}$$

where indices have been raised by the conformal metric, so if ω is non-zero then

$$\sigma\in\Gamma(\mathcal{E}_+[1])$$

is a distinguished scale determined by ω.

Proposition 5.2. *The conformal class* $\mathbf{c}$ *contains a Kähler metric iff there exists* $\omega\in\Lambda^2(M)$ *such that*

$$\omega^a{}_c\omega^c{}_b=-\sigma^2\delta^a_b, \tag{5.8}$$

and

$$X\wedge I\wedge\Phi=0, \tag{5.9}$$

where $\Phi=L(\omega)$ *and* $I:=D\sigma$, *with* σ *defined by* (5.7).

Proof. $\Leftarrow$: From (5.8) we have that σ is a scale and that $\sigma^{-3}\omega_{ab}$ is Hermitian for the metric $g:=\sigma^{-2}\boldsymbol{g}$. The Levi-Civita for g preserves σ, i.e. $\nabla^g\sigma=0$. Working on this scale, we have

$$D_A\sigma=\sigma Y_A+\frac{-\sigma P}{n}X_A.$$

Thus

$$X\wedge I\wedge\Phi=\sigma X\wedge Y\wedge\Phi.$$

So, with $\Phi=L(\omega)$, (5.9) exactly captures the condition that the $Z\wedge Z\wedge Z$-slot of (5.6) is zero, that is that $\sigma^{-3}\omega$ is closed. Thus, $\sigma^{-3}\omega$ is the Kähler form for the Kähler metric g.

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$\Rightarrow$: If $\sigma^{-3}\omega$ is a Kähler form for a metric $g = \sigma^{-2}\boldsymbol{g}$ then we have (5.8). Moreover the Levi-Civita connection of g preserves σ and $\sigma^{-3}\omega$. Thus, in particular, the latter is closed and co-closed in the scale g, and $L(\omega)$ takes the form

$$L(\omega) \stackrel{g}{=} \begin{pmatrix} & \omega_{bc} & \\ 0 & & 0 \\ -P_b{}^p\omega_{pc} & & \end{pmatrix} \tag{5.10}$$

From this, it is evident that (5.9) holds. □

REMARK. *From the last display, we see that for* $\Phi = L(\omega)$ *satisfying* (5.8) *and* (5.9) *we must also have*

$$X \lrcorner I \lrcorner \Phi = 0.$$

This is the co-closed condition.

Next, from the same display, we also see that in the case that (5.8) *and* (5.9) *hold, then the squared length of* $L(\omega)$*, i.e.*

$$\Phi^{ABC}\Phi_{ABC}$$

is a non-zero constant times the scalar curvature of the Kähler metric $g = \sigma^{-2}\boldsymbol{g}$*.*

It is well known that on a conformal structure, a metric g is Einstein iff there is a parallel tractor I_A and $g = \sigma^{-2}\boldsymbol{g}$ where $\sigma = X^A I_A \in \Gamma(\mathcal{E}_+[1])$ is nowhere zero. If a tractor I_A is parallel for the normal tractor connection then $I_A = D_A\sigma$ for some $\sigma \in \Gamma(\mathcal{E}[1])$. These results follow from the construction of the tractor connection in [2], as discussed in [11, 12].

Thus, one immediately has the following result.

PROPOSITION 5.3. *The conformal class* $\mathbf{c}$ *contains a Kähler–Einstein metric if there exists* $\omega \in \Lambda^2(M)$ *such that*

$$\omega^a{}_c\omega^c{}_b = -\sigma^2\delta^a_b,$$

and

$$X \wedge I \wedge \Phi = 0,$$

where $I = D\sigma$ *is parallel for the normal tractor connection and* $\Phi = L(\omega)$*.*

A special case is when the Einstein structure considered is Ricci flat. The length of the scale tractor is a multiple of the scalar curvature. Thus, $g = \sigma^{-2}\boldsymbol{g}$ is scalar flat iff $I := D\sigma$ is null. On the other hand, from (5.10), we see that if the Kähler scale is Ricci flat then $I \wedge L(\omega) = 0$. So we have the following result.

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PROPOSITION 5.4. *The conformal class* $\mathbf{c}$ *contains a Ricci-flat Kähler metric iff there exists* $\omega \in \Lambda^2(M)$ *such that*

$$\omega^a{}_c\omega^c{}_b = -\sigma^2\delta^a_b,$$

and

$$I \wedge \Phi = 0,$$

where $I = D\sigma$ *is parallel and null for the normal tractor connection, and* $\Phi = L(\omega)$.

REMARK 5.5. Since $\mathcal{D}\Phi = 0$ implies $\Phi = L(\omega)$ for some $\omega \in \Gamma(\mathcal{E}^2[3])$, there are (slightly weaker) variants of these propositions where we replace, in each case, the condition $\Phi = L(\omega)$ with $\mathcal{D}\Phi = 0$.

6. Outlook

We have constructed a rank $n(n+1)(n+2)/6$ vector bundle E with a connection $\mathcal{D}$ over a Riemannian manifold (M, g) of even dimension n, such that the $\mathcal{D}$-parallel sections of E belonging to a certain non-linear variety $\mathcal{S}$ in the fibres of E are in one-to-one correspondence with Kähler metrics in a conformal class of $[g]$. The construction of the connection followed from the prolongation of the conformal Killing–Yano (CKY) tensor equation [8, 13, 19], and the construction of $\mathcal{S}$ resulted from exploring the differential consequences of $J^2 = -\mathrm{Id}$, where the endomorphism $J : TM \to TM$ is the complex structure of the Kähler form.

The integrability conditions for the existence of the parallel sections in $\mathcal{S}$ imply that the conformal Weyl tensor of g is of the algebraic type-D. We have established an explicit algebraic obstruction for this, which makes the results relevant in general relativity of type-D spaces in dimensions higher than four [5, 16, 18].

The conformal Killing–Yano tensors, which underlie our work, give rise to hidden symmetries of gravitational instantons [1, 8, 10, 15, 17], as well as to first integrals of the conformal geodesics [9, 14]. The obstructions we have constructed can be of separate interest in deciding whether a given metric (Lorentzian or Riemannian) admits such hidden symmetries, or whether a conformal geodesic motion is integrable.

Finally, there is a connection with the tractor approach to conformal differential geometry [2]: the prolongation bundle E in our work can be identified with a parallel transport condition on $\Lambda^3(\mathcal{T})$, where $\mathcal{T} \to M$ is the rank-$(n+2)$ tractor bundle. It is, however, the case that the connection induced on $\Lambda^3(\mathcal{T})$ by the standard tractor connection on $\mathcal{T}$ differs from the prolongation connection $\mathcal{D}$ we have constructed on E in Theorem 2.1. It would be interesting to reformulate the non-linear algebraic conditions on the parallel sections of E in our Theorem 4.1 purely in terms of tractors. This is essentially implicit in Proposition 5.2 as $D\sigma$ and ω can each be expressed algebraically in terms of Φ (as the prolonged system is closed). However, it would be useful to find a simpler and explicit description. In [8], this problem has been solved in dimension $n = 4$, where the non-linear conditions reduce the bundle E to the rank-10 bundle $\Lambda^3{}_+(\mathcal{T})$ of self-dual tractor three-forms. The problem of finding an analogue of this remains open for $n > 4$.

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