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ON THE RELATIONSHIP BETWEEN THE MINIMUM IRREDUCIBLE CHARACTER CODEGREE AND FINITE SIMPLE GROUPS

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Abstract

It is established that the sporadic simple groups, the alternating groups and the projective special linear groups $L_2(p)$ for primes $p \geq 5$ are uniquely determined by their group orders and the minimum nontrivial codegrees of their irreducible characters.

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1. Introduction

Let G be a finite group, $\mathrm{Irr}(G)$ the set of all complex irreducible characters of G, and $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ the set of degrees of all complex irreducible characters of G. In 2000, Huppert [7] made the following conjecture.

CONJECTURE 1.1. Let S be a finite simple group and G a finite group. Then, $\mathrm{cd}(G) = \mathrm{cd}(S)$ if and only if $G = S \times A$, where A is an abelian group.

This conjecture has been verified for sporadic simple groups, alternating groups and several families of simple groups of Lie type, as documented in [1, 7, 8, 13, 15–19]. In 2012, Chen and Xu investigated the characterisation of finite simple groups using more limited character degree information. Specifically, they successfully characterised the simple K_3-groups solely by their group orders and either one or both of their largest and second-largest irreducible character degrees, as reported in [22]. Subsequently, this line of research has been further pursued (see [9, 23]).

For $\chi \in \mathrm{Irr}(G)$, Qian [14] defined the codegree of χ, denoted by $\chi^c(1)$, as $\chi^c(1) = |G : \ker\chi|/\chi(1)$, and conducted extensive research on the relationship between character codegrees and the structure of finite groups. Let $\mathrm{cod}(G)$ denote the set of codegrees

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of all irreducible complex characters of G. Based on this definition, Qian proposed the following conjecture [10, Problem 20.79]).

CONJECTURE 1.2. Let S be a finite simple group and let G be a finite group. Then, $\mathrm{cod}(G) = \mathrm{cod}(S)$ if and only if $G \cong S$.

This conjecture has been verified for sporadic simple groups, alternating groups and certain simple groups of Lie type. Of particular note are recent advances, specifically the work of Hung and Moretó, who investigated Qian's conjecture through a more general framework rather than relying on case-by-case verification for individual simple groups [5, 6]. They established the following elegant result.

THEOREM 1.3 [5]. *Let S and H be finite simple groups such that* $\mathrm{cod}(S) \subseteq \mathrm{cod}(H)$. *Then, $S \cong H$.*

Moreover, in [6], Hung and Moretó proposed and investigated the following more general conjecture. They confirmed the validity of this conjecture for alternating groups, sporadic simple groups and simple groups of the following Lie types: ${}^2B_2$, ${}^2G_2$, ${}^2F_4$, G_2, ${}^3D_4$, F_4, ${}^2E_6$, $A_{\leq 3}$, ${}^2A_{\leq 6}$, C_2, C_3 and B_3.

CONJECTURE 1.4. Let H be a nonabelian simple group and G a nontrivial finite group. Then, $\mathrm{cod}(G) \subseteq \mathrm{cod}(H)$ if and only if $G \cong H$.

Our motivation stems from investigating whether finite simple groups can be characterised using more refined character codegree information. Specifically, we aim to determine whether a simple group can be uniquely identified by its order and the minimum nontrivial irreducible character codegree. This problem is of significant theoretical interest, as supported by several established results. Let $m^c(G)$ denote the minimum nontrivial irreducible character codegree of a finite group G, and $b(G)$ denote the maximum degree among its irreducible characters. The invariant $m^c(G)$ was first introduced and studied in [5], where it is denoted by $f(G)$. For every finite simple group G, we have $m^c(G) = |G|/b(G)$. A number of researchers, including Xu, Jiang, Yan and Chen [9, 22, 23], have studied the interplay between the order of a simple group, the maximum degree of its irreducible characters and its structural properties. Their work has shown that certain simple groups can be uniquely determined by their order and the maximum irreducible character degree, although in some cases, additional data—such as the second or third largest character degrees—are required. However, there are notable examples of nonisomorphic groups that share both the same order and the same maximum irreducible character degree. For instance, the Mathieu group M_{22} and the direct product $T \times M_{11}$ possess identical orders and equal largest irreducible character degrees, where T is a Frobenius group with an elementary abelian kernel of order 2^3 and a cyclic complement of order 7. Similarly, the alternating group A_6 shares both its order and its largest irreducible character degree with the groups $S_3 \times A_5$ and $Z_3 \rtimes S_5$. These examples highlight that while there exists an inverse relationship between $b(G)$ and $m^c(G)$ in simple groups, the question of whether a

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simple group can be uniquely characterised by its order and $m^c(G)$ remains both meaningful and challenging.

In this paper, we establish that all sporadic simple groups, all alternating groups and the projective special linear groups $L_2(p)$ for primes $p \geq 5$ are uniquely determined by their group orders and the minimum nontrivial irreducible character codegrees.

2. Characterisation of $L_2(p)$

LEMMA 2.1 [5, Lemma 2.1(5)]. *If G is abelian, then $\mathrm{cod}(G) = \pi_e(G)$, where $\pi_e(G)$ is the set of orders of the elements of G.*

LEMMA 2.2. *Let G be a finite group. If $m^c(G) > \max(\pi(G))$, then $G = G'$ and, thus, G is nonsolvable.*

The following lemma is a direct corollary of the preceding lemma.

LEMMA 2.3 [5, Proposition 7.1]. *Let H be a simple group of Lie type. If $n = 7$ or $n \geq 9$ and the order of the alternating group A_n divides $|H|$, then $m^c(H) > m^c(\mathrm{A}_n)$.*

LEMMA 2.4 [5, Lemma 6.1]. *Let H be a simple group of Lie type defined over a field of characteristic p. Then, the following statements hold:*

(1) $|H|_p < |H|_{p'}/p - 1$;
(2) $|H|_{p'} < |H|_p^2$, *thus,* $m^c(H) < |H|_p^2$;
(3) *if H is not of type A_1, then every prime divisor of $|H|_{p'}$ is less than $\sqrt{|H|_{p'}}$.*

LEMMA 2.5 [4, Theorem 12]. *Let S be a simple group. Then, $\sqrt[3]{|S|} < b(S)$.*

LEMMA 2.6 [11, Theorem 5.1]. *Let S and T be two nonisomorphic finite simple groups of the same order. Then, S and T are either $L_3(4)$ and A_8 or $B_n(q)$ and $C_n(q)$, where $n \geq 3$ and q is an odd prime power.*

THEOREM 2.7. *Let G be a group and let $H = L_2(p)$, where $p \geq 5$ is a prime. Then, $G \cong H$ if and only if $|G| = |H|$ and $m^c(G) = m^c(H)$.*

PROOF. It suffices to prove the sufficiency. We have $|L_2(p)| = p(p-1)(p+1)/2$ and $\mathrm{cd}(H) = \{1, p, p-1, p+1, (p+\varepsilon)/2\}$, where $\varepsilon = (-1)^{(p-1)/2}$ and $p \geq 5$ is a prime. Thus, $|G| = p(p-1)(p+1)/2$ and $m^c(G) = m^c(L_2(p)) = p(p-1)/2$.

Note that p is the largest prime divisor of $|G|$. Since $p < p(p-1)/2 = m^c(G)$, it follows from Lemma 2.2 that $G = G'$ and, hence, G is nonsolvable. Therefore, G has a maximal normal subgroup M such that $G/M \cong S$ and $m^c(G) \leq m^c(S)$, where S is a nonabelian simple group. The proof proceeds in three steps.

Step 1. We prove that S is not a sporadic group. Suppose, to the contrary, that S is a sporadic group.

Assume $p \mid |S|$. Note that $|S| \mid |G| = \frac{1}{2}p(p-1)(p+1)$ and p is the largest prime divisor of $|S|$. Thus, for any prime $r \mid |S|$ with $r < p$, we have $r \mid (p-1)(p+1)$. However, by Table 1, for every sporadic group except M_{11} and M_{12}, there exists

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TABLE 1. Sporadic groups except M_{11} and M_{12}.

Group	p	r	Group	p	r
J_1	19	11	M_{22}	11	7
J_2	7	5	HS	11	7
J_3	19	17	M_{24}	23	7
M^cL	11	7	He	17	7
Ru	29	13	Suz	13	11
$O'N$	31	19	Co_3	23	7
Co_2	23	7	Fi_{22}	13	11
HN	19	11	Ly	67	37
Th	31	19	Fi_{23}	23	17
Co_1	23	13	J_4	43	37
Fi'_{24}	29	23	B	47	31
M	71	59	M_{23}	23	7

a prime r such that $r \nmid (p-1)(p+1)$. If $S = M_{11}$ ($|M_{11}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$) or M_{12} ($|M_{12}| = 2^6 \cdot 3^3 \cdot 5 \cdot 11$), then $p = 11$, so $|G| = 2^2 \cdot 3 \cdot 5 \cdot 11$. Hence, $|G| < |S|$, which is a contradiction.

Now, assume $p \nmid |S|$. In this case, $|S| \mid \frac{1}{2}(p-1)(p+1)$. Then, $p \geq \sqrt{2|S|+1} > \sqrt{2|S|}$ and, thus, $m^c(G) = \frac{1}{2}p(p-1) > |S| - \sqrt{2|S|}$. By Lemma 2.5, $\sqrt[3]{|S|} < b(S)$. Since $|S|/b(S)$ is a codegree, it follows that $m^c(S) \leq |S|/b(S) \leq |S|^{2/3}$. Consider the function $f(x) = x - \sqrt{2x} - x^{2/3}$ for $x > 2^6$. This function is strictly increasing and satisfies $f(2^6) > 0$, so $f(|S|) > 0$ for all sporadic groups S. That is, $|S| - \sqrt{2|S|} - |S|^{2/3} > 0$, which implies $m^c(G) > |S| - \sqrt{2|S|} > |S|^{2/3} \geq m^c(S)$, again being a contradiction.

Therefore, S is not a sporadic group.

Step 2. When S is an alternating group, we prove that $S = A_5$ and $G = S$.

Assume $S = A_n$ with $n = 7$ or $n \geq 9$. Since $|S| \mid |L_2(p)|$, it follows from Lemma 2.3 that $m^c(S) < m^c(L_2(p))$. Therefore, $m^c(G) \leq m^c(S) < m^c(L_2(p))$, which is a contradiction.

When $S = A_5$, we have $m^c(S) = 12$. Since $m^c(G) = \frac{1}{2}p(p-1) \leq m^c(S)$, it follows that $p = 5$ and $|G| = 2^2 \cdot 3 \cdot 5$. Comparing orders immediately yields $G = S = A_5$.

When $S = A_6$, we have $m^c(S) = 36$. Since $m^c(G) = \frac{1}{2}p(p-1) \leq m^c(S)$, we conclude that $p = 5$ or $p = 7$. Hence, $|G| = 2^2 \cdot 3 \cdot 5$ or $2^3 \cdot 3 \cdot 7$, both of which are less than $|S|$, which is a contradiction.

When $S = A_8$, we have $m^c(S) = 2^5 \cdot 3^2$. Since $m^c(G) = \frac{1}{2}p(p-1) \leq m^c(S)$, the possible values of p are $5, 7, 11, 13, 17, 19$ or 23. Checking each case individually, we find that $|S| \nmid |G|$, again being a contradiction.

Therefore, $G = S = A_5$ when S is an alternating group.

Step 3. When S is a simple group of Lie type, we prove that $S \cong L_2(p)$ and $G = S$.

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Assume that S is a simple group of Lie type defined over a field of characteristic r. We divide the proof into two cases: $r = p$ and $r \neq p$.

Case 1. Assume $r = p$. Then, $|S| = p|S|_{p'}$ and $|S|_{p'} \mid |G|_{p'} = \frac{1}{2}(p+1)(p-1)$. If $|S|_{p'} < \frac{1}{2}(p+1)(p-1)$, then $|S|_{p'} \leq \frac{1}{4}(p+1)(p-1)$. However, since $m^c(G) = \frac{1}{2}p(p-1) \leq m^c(S) \leq \mathrm{cod}(\mathrm{St}) = |S|_{p'} \leq \frac{1}{4}(p+1)(p-1)$, where St denotes the Steinberg character, this leads to a contradiction. Therefore, $|S|_{p'} = \frac{1}{2}(p+1)(p-1)$. It follows that $|S| = |G| = |L_2(p)|$. By Lemma 2.6, we conclude that $S \cong L_2(p)$ and, hence, $G = L_2(p)$.

Case 2. Assume $r \neq p$. By [12, 21], considering the degree of the Steinberg character $\mathrm{St}(1)_r$ of S, we have $\mathrm{St}(1)_r = |S|_r$ and $\mathrm{cod}(\mathrm{St})_r = |S|_{r'}$. Thus, $m^c(G) \leq m^c(S) \leq |S|_{r'}$. We proceed with the proof by distinguishing whether S is of type A_1 or not.

Case 2.1. Assume S is not of type A_1. Since $|S| \mid \frac{1}{2}p(p-1)(p+1)$ and $(p-1, p+1) = 2$, it follows that $|S|_r \mid p-1$ or $|S|_r \mid p+1$.

First, assume $|S|_r \mid p-1$. If $|S|_r < p-1$, then $|S|_r \leq \frac{1}{2}(p-1)$. By Lemma 2.4,

$$\tfrac{1}{2}p(p-1) = m^c(G) \leq m^c(S) \leq |S|_{r'} \leq |S|_r^2 \leq \tfrac{1}{4}(p-1)^2,$$

which implies that $-1 \leq p \leq 3$, a contradiction. Therefore, $|S|_r = p-1$. It follows that $|S|_{r'} \mid \frac{1}{2}p(p+1)$.

If $|S|_{r'} = \frac{1}{2}p(p+1)$, then $|S| = |L_2(p)|$ and, by Lemma 2.6, $S \cong L_2(p)$, which is of type A_1, contrary to the assumption for this case. If $|S|_{r'} < \frac{1}{2}p(p+1)$, then $|S|_{r'} \leq \frac{1}{4}p(p+1)$. Since $\frac{1}{2}p(p-1) = m^c(G) \leq |S|_{r'}$, we deduce $\frac{1}{2}p(p-1) \leq \frac{1}{4}p(p+1)$ and, hence, $0 \leq p \leq 3$, again a contradiction.

Second, assume $|S|_r \mid p+1$. If $|S|_r < p+1$, then $|S|_r \leq \frac{1}{2}(p+1)$. By Lemma 2.4,

$$\tfrac{1}{2}p(p-1) = m^c(G) \leq m^c(S) \leq |S|_{r'} \leq |S|_r^2 \leq \tfrac{1}{4}(p+1)^2,$$

which implies $2 - \sqrt{5} \leq p \leq 2 + \sqrt{5}$, a contradiction. Therefore, $|S|_r = p+1$. It follows that $|S|_{r'} \mid \frac{1}{2}p(p-1)$. Since $\frac{1}{2}p(p-1) = m^c(G) \leq |S|_{r'}$, we deduce $|S|_{r'} = \frac{1}{2}p(p-1)$. Hence, $|S| = |L_2(p)|$. By Lemma 2.6, $S \cong L_2(p)$, which is of type A_1, again contrary to the assumption in this case.

Case 2.2. Assume S is of type A_1. Let $S = L_2(r^n)$. If $r \neq 2$, then we have $m^c(S) = \frac{1}{2}r^n(r^n-1)$. Since $m^c(G) \leq m^c(S)$, it follows that $p \leq r^n$. Hence, $|S| \geq |L_2(p)|$ and, therefore, $|S| = |L_2(p)|$ and $G = S = L_2(p)$.

If $r = 2$, then $|S| = 2^n(2^n-1)(2^n+1)$ and $m^c(S) = 2^n(2^n-1)$. Clearly, $2^n \mid p-1$ or $2^n \mid p+1$.

When $2^n \mid p-1$, we claim that $2^n = p-1$. Otherwise, $2^n \leq \frac{1}{2}(p-1)$ and

$$m^c(S) \leq \tfrac{1}{2}(p-1)(\tfrac{1}{2}(p-1)-1) < \tfrac{1}{2}p(p-1) = m^c(G),$$

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which is a contradiction. Thus, $2^n = p - 1$. Since $|S| \mid \frac{1}{2}p(p-1)(p+1)$, it follows that $|S|_{2'} \mid \frac{1}{2}p(p+1)$. We claim that $|S|_{2'} = \frac{1}{2}p(p+1)$. Otherwise, $|S|_{2'} \leq \frac{1}{4}p(p+1)$ and

$$m^c(G) = \tfrac{1}{2}p(p-1) \leq m^c(S) \leq |S|_{2'} \leq \tfrac{1}{4}p(p+1),$$

which implies $0 \leq p \leq 3$, a contradiction. Therefore, $|S|_{2'} = \frac{1}{2}p(p+1)$. Together with $2^n = p - 1$, we obtain $|G| = |S| = |L_2(p)|$, which contradicts Lemma 2.6.

When $2^n \mid p + 1$, we claim that $2^n = p + 1$. Otherwise, $2^n \leq \frac{1}{2}(p+1)$ and

$$m^c(S) \leq \tfrac{1}{2}(p+1)(\tfrac{1}{2}(p+1) - 1) < \tfrac{1}{2}p(p-1) = m^c(G),$$

which is a contradiction. Since $2^n = p + 1$,

$$|S| = 2^n(2^n - 1)(2^n + 1) = (p+1)p(p+2) > \tfrac{1}{2}p(p-1)(p+1) = |G|,$$

again a contradiction.

Therefore, $G = S = L_2(p)$ when S is a simple group of Lie type. □

3. Characterisation of the alternating group

In this section, we demonstrate that the group order and the minimum nontrivial codegree are sufficient to characterise the alternating groups. We begin by recalling several properties of the alternating groups.

LEMMA 3.1 [2, Theorem 3.7]. *Suppose that* $\frac{1}{2}n! = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$, *where* $n \geq 5$ *and* $p_1, p_2, \ldots, p_s$ *are distinct primes such that* $p_1^{\alpha_1} > p_i^{\alpha_i}$ *for* $i = 2, \ldots, s$. *Then:*

(1) $p_1 = 2$ *for* $n \neq 5, 6, 7, 9$;
(2) $p_1 = 3$ *for* $n = 6, 7, 9$;
(3) $p_1 = 5$ *for* $n = 5$.

LEMMA 3.2.

(1) *Let* $S = A_n$ *with* $n \geq 5$. *Then,* $m^c(S) > \sqrt{n!}/2$.
(2) *Let* $m < n$ *be positive integers. Then,* $m^c(A_m) < m^c(A_n)$.
(3) *Let* S *be a sporadic simple group such that* $|S| \mid |A_n|$. *Then,* $m^c(S) < m^c(A_n)$.

PROOF. The lemma follows straightforwardly from [5, Lemma 5.4(1), Proposition 8.1 and the second paragraph of the proof of Proposition 8.2]. □

LEMMA 3.3. *Let* S *be a simple group of Lie type such that* $|S| \mid |A_{39}|$. *Then,* S *is either a group contained in the list in* [3], *or one of* $L_2(11^3)$, $L_2(13^2)$ *and* $L_2(17^2)$.

PROOF. Assume $S \cong A_n(q)$. Since $q^{n(n+1)/2} \parallel |S| \mid |A_{39}| = 2^{34} \cdot 3^{18} \cdot 5^8 \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37$, we have $\frac{1}{2}n(n+1) \leq 34$, so $n \leq 7$. Therefore, if S is not a group listed in [3], then S must be one of the following simple groups of Lie type: $A_1(q)$ with $q > 125$, $A_2(q)$ with $q > 31$, $A_3(q)$ with $q > 11$, $A_4(q)$ with $q > 11$, $A_5(q)$ with $q > 5$, $A_6(q)$ with $q > 3$ or $A_7(q)$ with $q > 2$.

If $S = A_1(q)$ with $q > 125$, given that $|S|$ divides $|A_{39}|$, it follows that q must be one of 2^t for $7 \leq t \leq 34$, 3^t for $5 \leq t \leq 18$, 5^t for $4 \leq t \leq 8$, 7^t for $3 \leq t \leq 5$, 11^3, 13^t for $2 \leq t \leq 3$, 17^2 or 19^2.

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For these values of q, checking the orders of $S = A_1(q)$ such that $|S| \mid |A_{39}|$, we find that S must be one of the following: $A_1(11^3)$ (with order $2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11^3 \cdot 19 \cdot 37$), $A_1(13^2)$ (with order $2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 13^2 \cdot 17$) or $A_1(17^2)$ (with order $2^5 \cdot 3^2 \cdot 5 \cdot 17^2 \cdot 29$).

If $S = A_2(q)$ with $q > 31$, then q^3 divides one of 2^{34}, 3^{18}, 5^8, 7^5, 11^3 or 13^3. Hence, q must be one of 2^t for $5 \le t \le 11$ or 3^t for $4 \le t \le 6$. By computing the orders of $S = A_2(q)$ for these cases, we conclude that $|S| \nmid |A_{39}|$, which is a contradiction.

Using the same approach, we obtain contradictions for $S = A_3(q)$ with $q > 11$, $A_4(q)$ with $q > 11$, $A_5(q)$ with $q > 5$, $A_6(q)$ with $q > 3$ and $A_7(q)$ with $q > 2$.

Now, assume $S \cong {}^2A_n(q)$. From $q^{n(n+1)/2} \parallel |S| \mid |A_{39}|$ and the fact that the largest exponent of a prime power dividing $|A_{39}|$ is 34, it follows that $n \le 7$. Therefore, if S is not listed in [3], then S must be one of ${}^2A_2(q)$ with $q > 32$, ${}^2A_3(q)$ with $q > 11$, ${}^2A_4(q)$ with $q > 11$, ${}^2A_5(q)$ with $q > 5$ or ${}^2A_6(q)$ with $q > 3$.

If $S = {}^2A_2(q)$ with $q > 32$, then q^3 divides one of 2^{34}, 3^{18}, 5^8, 7^5, 11^3 or 13^3. Thus, q is either 2^t for $6 \le t \le 11$ or 3^t for $4 \le t \le 6$. Computing the orders of $S = {}^2A_2(q)$ for these values yields $|S| \nmid |A_{39}|$, which is a contradiction.

For $S = {}^2A_3(q)$ with $q > 11$, ${}^2A_4(q)$ with $q > 11$, ${}^2A_5(q)$ with $q > 5$ or ${}^2A_6(q)$ with $q > 3$, the same method leads to a contradiction.

Finally, suppose S is any other simple group of Lie type not yet considered. By the same argument as above, all such S satisfying $|S| \mid |A_{39}|$ are contained in the list of [3]. □

Since $|A_8| = |L_3(4)| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ and $70 = b(A_8) > b(L_3(4)) = 64$, it follows that $m^c(A_8) = 2^5 \cdot 3^2 < m^c(L_3(4)) = 3^2 \cdot 5 \cdot 7$. Further, we have the following lemma.

LEMMA 3.4. *Let S be a simple group of Lie type. If $S \neq L_3(4)$ and $|S| \mid |A_n|$, then $m^c(S) < m^c(A_n)$, $n \ge 5$. If $S = L_3(4)$, then $m^c(S) < m^c(A_n)$, $n \ge 9$.*

PROOF. Assume S is defined over a field of characteristic p. By Lemma 2.4, $m^c(S) < |S|_p^2 \le |A_n|_p^2$. When $n > 10$, we have $m^c(S) < |A_n|_2^2 \le 2^{2n-2}$ by Lemma 3.1. By Lemma 3.2, $m^c(A_n) \ge \sqrt{n!}/2$. Therefore, the result holds if $\frac{1}{2}\sqrt{n!} \ge 2^{2n-2}$. This inequality is verified using MATLAB and holds for all $n \ge 40$. It remains to consider the case $5 \le n \le 39$.

Through direct computation, we obtain Table 2, which lists all simple groups of Lie type whose orders divide $|A_n|$, but do not divide $|A_{n-1}|$ for $5 \le n \le 39$. Upon inspection, for each simple group of Lie type $S \neq L_3(4)$ such that $|S| \mid |A_n|$ and $|S| \nmid |A_{n-1}|$, we find that $m^c(S) < m^c(A_n)$. Thus, the conclusion follows from Lemma 3.2. For $S = L_3(4)$, [3] shows that $3^2 \cdot 5 \cdot 7 = m^c(S) < m^c(A_9) = 2^3 \cdot 3 \cdot 5 \cdot 7$. In conclusion, by Lemma 3.2, $m^c(S) < m^c(A_n)$ for all $n \ge 9$. □

THEOREM 3.5. *Let G be a finite group and let $H = A_n$ with $n \ge 5$. Then, $G \cong H$ if and only if $|G| = |H|$ and $m^c(G) = m^c(H)$.*

PROOF. It suffices to prove the sufficiency.

We first show that G is simple. Assume, for a contradiction, that G is not simple and let M be a maximal normal subgroup of G. Then, $M \neq 1$ and $G/M \cong S$, where S is a

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TABLE 2. Simple groups of Lie type S satisfying $|S| \mid |A_n|$, but $|S| \nmid |A_{n-1}|$.

A_n	S	A_n	S
A_5	$L_2(5)(A_5)$	A_{23}	$L_2(23)$
A_6	$L_2(9)(A_6)$	A_{24}	$L_5(3)\ O_{10}^-(2)$
A_7	$L_2(7)\ L_2(8)$	A_{25}	None
A_8	$L_3(4)\ L_4(2)(A_8)$	A_{26}	$L_2(13^2)$
A_9	$U_3(3)\ U_4(2)$	A_{27}	$O_8^+(3)$
A_{10}	None	A_{28}	$S_4(7)\ F_4(2)$
A_{11}	$L_2(11)$	A_{29}	$L_2(29)$
A_{12}	$S_6(2)$	A_{30}	None
A_{13}	$L_2(13)\ L_3(3)\ L_2(25)\ L_2(27)$ $Sz(8)\ U_3(4)\ L_2(64)$	A_{31}	$L_2(31)\ L_2(32)\ L_3(5)\ L_2(125)$ $L_5(2)\ G_2(5)\ L_4(5)\ L_6(2)$ $L_3(25)\ O_{10}^+(2)\ L_5(4)\ S_{10}(2)$
A_{14}	$L_2(49)\ U_5(2)$	A_{32}	$O_{12}^+(2)\ O_{12}^-(2)\ L_6(4)$
A_{15}	$U_3(5)\ U_4(3)\ G_2(3)\ L_4(3)\ L_3(9)$	A_{33}	$L_3(11)\ L_6(3)$
A_{16}	${}^2F_4(2)'\ O_8^+(2)\ {}^3D_4(2)\ G_2(4)\ S_4(8)$	A_{34}	$O_8^+(4)\ L_2(17^2)$
A_{17}	$L_2(17)\ L_2(16)\ S_4(4)\ O_8^-(2)$ $L_4(4)\ U_4(4)\ L_3(16)$	A_{35}	None
A_{18}	$U_6(2)$	A_{36}	None
A_{19}	$L_2(19)\ U_3(8)$	A_{37}	$L_2(37)\ U_3(11)\ R(27)\ U_3(27)\ L_2(11^3)$
A_{20}	$S_4(5)\ U_4(5)\ S_8(2)$	A_{38}	None
A_{21}	$L_3(7)\ S_6(3)\ O_7(3)$	A_{39}	None
A_{22}	$S_6(4)\ U_4(8)$		

simple group. If S is nonabelian, it follows from $|S| \mid |A_n|$ and $|S| < |A_n|$ that $m^c(S) < m^c(A_n)$ by Lemmas 3.2 and 3.4. Since $m^c(G) \le m^c(S)$, we obtain $m^c(G) < m^c(A_n)$, which is a contradiction. If S is abelian, then $m^c(S)$ is prime and

$$m^c(S) \le \max \pi(A_n) \le n \le \frac{\sqrt{n!}}{2} < m^c(A_n),$$

again a contradiction. Therefore, G must be simple.

Since $|G| = |H|$ and G is simple, it follows from Lemma 2.6 that either $G \cong H$ or $G = L_3(4)$ and $H = A_8$. If $G = L_3(4)$ and $H = A_8$, then

$$m^c(A_8) = 2^5 \cdot 3^2 < 3^2 \cdot 5 \cdot 7 = m^c(G),$$

which is a contradiction. Hence, $G \cong H = A_n$ and the theorem follows. □

4. Characterisation of a sporadic simple group

LEMMA 4.1 [20, Lemma 2.14]. *Let S be a sporadic simple group and T a nonabelian simple group such that $|T| \mid |S|$.*

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(1) *If $S = M$, then T is either a group contained in the list in* [3]*, or one of A_n for $26 \leq n \leq 32$, $L_2(2^{10})$ or $L_2(13^2)$, where M denotes the Monster group.*
(2) *If $S \neq M$, then T is a group contained in the list in* [3].

LEMMA 4.2. *Let S be a sporadic simple group and let p be the largest prime divisor of $|S|$. Then, $p < m^c(S)$.*

PROOF. Let $b = b(S)$ be the largest irreducible character degree of S. Note that $b^2 < |S|$, so $b < |S|/b = m^c(S)$. Moreover, by [3], $p < b(S)$ for the largest prime divisor p of $|S|$. Therefore, $p < m^c(S)$ and the lemma follows. □

THEOREM 4.3. *Let G be a finite group and let H be a sporadic simple group. Then, $G \cong H$ if and only if $|G| = |H|$ and $m^c(G) = m^c(H)$.*

PROOF. It is sufficient to prove the sufficiency. First, it follows from Lemmas 4.2 and 2.2 that $G = G'$ and G is nonsolvable. We claim that G is simple. Suppose, for contradiction, that G is not simple. Then, there exists a maximal nontrivial normal subgroup M of G such that $G/M = S$, where S is a nonabelian simple group. In the subsequent arguments, we shall repeatedly apply the fact that $|S|$ divides $|H|$ and that $|S| < |H|$.

Case 1. Assume that H is a Mathieu group. By Lemma 4.1 and [3], if $H = M_{11}$, then S is one of A_5, A_6 or $L_2(11)$. If $H = M_{12}$, then S is one of

$$A_5,\ A_6,\ L_2(11) \text{ or } M_{11}. \tag{4.1}$$

If $H = M_{22}$, then S is one of

$$A_5,\ L_2(7),\ A_6,\ L_2(8),\ L_2(11),\ A_7,\ A_8,\ L_3(4) \text{ or } M_{11}. \tag{4.2}$$

If $H = M_{23}$, since $|M_{12}|$ divides $|M_{23}|$, S is either one of the groups listed in (4.1), or one of M_{22} or $L_2(23)$. If $H = M_{24}$, since $|M_{22}|$ divides $|M_{24}|$, S is either one of the groups listed in (4.2), or one of M_{22}, $L_2(23)$ or M_{23}.

According to [3], we always have $m^c(S) < m^c(H)$. Since $m^c(G) \leq m^c(S)$, it follows that $m^c(G) < m^c(H)$, which leads to a contradiction.

Case 2. Assume that H is a Janko group. By Lemma 4.1 and [3], if $H = J_1$, then S is one of A_5, $L_2(7)$ or $L_2(11)$. If $H = J_2$, then S is one of

$$A_5,\ L_2(7),\ A_6,\ L_2(8),\ A_7,\ U_3(3),\ A_8 \text{ or } L_3(4). \tag{4.3}$$

If $H = J_3$, then S is one of

$$A_5,\ A_6,\ L_2(17),\ L_2(19),\ L_2(16) \text{ or } U_4(2).$$

If $H = J_4$, then S is one of

$$A_5, L_2(7),\ A_6,\ L_2(8),\ L_2(11),\ A_7,\ U_3(3),\ M_{11},\ L_2(23),\ L_2(29),\ L_2(31),$$
$$A_8,\ L_3(4),\ L_2(32),\ L_2(43),\ M_{12},\ M_{22},\ L_5(2),\ M_{23} \text{ or } M_{24}.$$

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According to [3], we always have $m^c(S) < m^c(H)$. Since $m^c(G) \leq m^c(S)$, it follows that $m^c(G) < m^c(H)$, which leads to a contradiction.

Case 3. Assume that H is one of the following sporadic simple groups: HS, M^cL, Suz, Ly, He, Ru, $O'N$, Th or HN. By Lemma 4.1 and [3], if $H = HS$, then S is one of

$$A_5,\ L_2(7),\ A_6,\ L_2(8),\ L_2(11),\ A_7,\ A_8,\ L_3(4),\ M_{11},\ U_3(5) \text{ or } M_{22}. \tag{4.4}$$

If $H = M^cL$, then S is one of

$$\begin{gathered} A_5,\ L_2(7),\ A_6,\ L_2(8),\ L_2(11),\ A_7,\ U_3(3),\ A_8,\ L_3(4),\ M_{11}, \\ U_4(2),\ M_{12},\ U_3(5),\ A_9,\ M_{22},\ J_2,\ A_{10},\ U_4(3) \text{ or } A_{11}. \end{gathered} \tag{4.5}$$

If $H = Suz$, since $|M_{12}|$, $|M_{22}|$ and $|J_2|$ divide $|Suz|$, then S is either one of the groups listed in (4.1), (4.2) and (4.3), or one of

$$\begin{gathered} A_n\ (9 \leq n \leq 13),\ L_2(13),\ L_3(3),\ L_2(25),\ L_2(27),\ U_4(2),\ Sz(8),\ U_3(4),\ M_{12},\ L_2(64), \\ M_{22},\ J_2,\ S_6(2),\ U_4(3),\ G_2(3),\ L_4(3),\ U_5(2),\ {}^2F_4(2)',\ L_3(9),\ O_8^+(2) \text{ or } G_2(4). \end{gathered} \tag{4.6}$$

If $H = Ly$, since $|M^cL|$ divides $|Ly|$, then S is either one of the groups listed in (4.5), or one of

$$L_2(31),\ L_2(32),\ L_3(5),\ L_2(125),\ M^cL \text{ or } G_2(5).$$

If $H = He$, since $|J_2|$ divides $|He|$, then S is either one of the groups listed in (4.3), or one of

$$L_2(17),\ L_2(16),\ L_2(49),\ J_2 \text{ or } S_4(4). \tag{4.7}$$

If $H = Ru$, since $|J_2|$ divides $|Ru|$, then S is either one of the groups listed in (4.3), or one of

$$\begin{gathered} L_2(13),\ L_3(3),\ L_2(25),\ L_2(27),\ L_2(29),\ Sz(8), \\ U_3(4), U_3(5),\ L_2(64),\ J_2,\ G_2(4)\ \text{ or } {}^2F_4(2)'. \end{gathered} \tag{4.8}$$

If $H = O'N$, since $|M_{22}|$ divides $|O'N|$, then S is either one of the groups listed in (4.2), or one of

$$\begin{gathered} L_2(19),\ U_3(3),\ L_2(31),\ U_4(2),\ L_2(32),\ M_{12}, \\ J_1,\ A_9,\ M_{22},\ S_6(2),\ L_3(7) \text{ or } U_3(8). \end{gathered} \tag{4.9}$$

If $H = Th$, since $|J_2|$ divides $|Th|$, then S is either one of the groups listed in (4.3), or one of

$$\begin{gathered} L_2(19),\ L_3(3),\ L_2(25),\ L_2(27),\ L_2(31),\ U_4(2),\ Sz(8),\ L_2(49), \\ U_3(4),\ U_3(5),\ A_9,\ L_2(64),\ L_3(5),\ J_2,\ L_2(125),\ S_4(4),\ S_6(2), \\ A_{10},\ U_4(3), G_2(3),\ U_3(8),\ L_4(3), L_5(2),\ {}^2F_4(2)',\ L_3(9),\ O_8^+(2), \\ {}^3D_4(2),\ G_2(4),\ S_4(8),\ O_7(3),\ S_6(3) \text{ or } L_6(2). \end{gathered} \tag{4.10}$$

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If $H = HN$, since $|M^cL|$ and $|HS|$ divide $|HN|$, then S is either one of the groups listed in (4.1), (4.2), (4.3), (4.4) and (4.5), or one of

$$L_2(19),\ U_4(2),\ J_1,\ S_6(2),\ U_3(8),\ U_5(2),\ HS,\ O_8^+(2),\ A_{12} \text{ or } M^cL. \tag{4.11}$$

According to [3], we always have $m^c(S) < m^c(H)$. Since $m^c(G) \le m^c(S)$, it follows that $m^c(G) < m^c(H)$, which leads to a contradiction.

Case 4. Assume that H is a Conway group. By Lemma 4.1 and [3], if $H = Co_3$, since $|M^cL|$, $|M_{24}|$ and $|HS|$ divide $|Co_3|$, then S is either one of the groups listed in (4.2), (4.3), (4.4) and (4.5), or one of

$$L_2(23),\ S_6(2),\ M_{23},\ U_5(2),\ HS,\ A_{12},\ M_{24} \text{ or } M^cL. \tag{4.12}$$

If $H = Co_2$, since $|M^cL|$, $|M_{24}|$ and $|HS|$ divide $|Co_2|$, then S is either one of the groups listed in (4.2), (4.3), (4.4), (4.5) and (4.12), or one of

$$O_8^+(2) \text{ or } U_6(2). \tag{4.13}$$

If $H = Co_1$, since $|Co_3|$, $|Suz|$ and $|Co_2|$ divide $|Co_1|$, then S is either one of the groups listed in (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.12) and (4.13), or one of

$$\begin{aligned} &L_2(16),\ L_2(49),\ S_4(5),\ {}^3D_4(2),\ S_4(8),\ S_6(3),\ O_7(3),\ U_6(2),\\ &U_4(5),\ A_n\ (14 \le n \le 16),\ Suz,\ Co_3,\ Co_2 \text{ or } Fi_{22}. \end{aligned} \tag{4.14}$$

According to [3], we always have $m^c(S) < m^c(H)$. Since $m^c(G) \le m^c(S)$, it follows that $m^c(G) < m^c(H)$, which leads to a contradiction.

Case 5. Assume that H is one of the sporadic simple groups Fi_{22}, Fi_{23} or Fi'_{24}. By Lemma 4.1 and [3], if $H = Fi_{22}$, since $|Suz|$, $|M_{22}|$ and $|J_2|$ divide $|Fi_{22}|$, then S is either one of the groups listed in (4.1), (4.2), (4.3) and (4.6), or one of

$$S_6(3),\ O_7(3),\ U_6(2) \text{ or } Suz. \tag{4.15}$$

If $H = Fi_{23}$, since $|M_{24}|$, $|Fi_{22}|$ and $|J_2|$ divide $|Fi_{23}|$, then S is either one of the groups listed in (4.1), (4.2), (4.3), (4.6) and (4.15), or one of

$$\begin{aligned} &L_2(17),\ L_2(16),\ L_2(23),\ S_4(4),\ M_{23},\ O_8^-(2),\\ &M_{24},\ L_4(4),\ L_3(16),\ S_8(2),\ O_8^+(3) \text{ or } Fi_{22}. \end{aligned} \tag{4.16}$$

If $H = Fi'_{24}$, since $|Fi_{23}|$ and $|He|$ divide $|Fi'_{24}|$, then S is either one of the groups listed in (4.1), (4.2), (4.3), (4.6), (4.7), (4.15) and (4.16), or one of

$$L_2(49),\ {}^3D_4(2),\ He,\ S_4(8),\ A_{14},\ O_{10}^-(2) \text{ or } Fi_{23}. \tag{4.17}$$

According to [3], we always have $m^c(S) < m^c(H)$. Since $m^c(G) \le m^c(S)$, it follows that $m^c(G) < m^c(H)$, which leads to a contradiction.

Case 6. Assume that H is either the Monster group M or the Baby Monster group B. By Lemma 4.1 and [3], if $H = B$, since $|M^cL|$, $|M_{24}|$, $|Fi_{23}|$ and $|Co_1|$ divide $|B|$, then S

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is either one of the groups listed in (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15) and (4.16), or one of

$$L_2(32),\ L_2(47),\ J_3,\ U_4(4),\ G_2(5),\ L_4(5),\ L_3(25),\ S_6(4),\ O_{10}^+(2),$$
$$O_{10}^-(2),\ U_4(8),\ A_n\ (17 \le n \le 20),\ L_5(4),\ HN,\ F_4(2),\ S_{10}(2),\ Th,$$
$$Fi_{23},\ Co_1,\ O_{12}^+(2),\ O_{12}^-(2),\ L_6(4),\ {}^2E_6(2) \text{ or } S_{12}(2).$$

If $H = M$, since $|B|$, $|Fi'_{24}|$, $|O'N|$ and $|Ru|$ divide $|M|$, then S is either one of the groups listed in (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), (4.7), (4.8), (4.10), (4.11), (4.12), (4.13), (4.14), (4.15), (4.16) and (4.17), or one of

$$A_n\ (21 \le n \le 32),\ L_2(2^{10}),\ L_2(13^2),\ L_2(71),\ L_2(81),\ Sz(32),$$
$$S_4(7),\ S_4(9),\ Ru,\ L_5(3),\ O'N,\ L_4(7),\ O_8^-(3),\ L_4(9),\ U_5(4),$$
$$O_7(5),\ S_6(5),\ L_6(3),\ O_9(3),\ S_8(3),\ U_6(4) \text{ or } O_{10}^+(3).$$

According to [3], we always have $m^c(S) < m^c(H)$. Since $m^c(G) \le m^c(S)$, it follows that $m^c(G) < m^c(H)$, which leads to a contradiction.

In summary, G must be simple. By Lemma 2.6, it follows that $G \cong H$ and, hence, the theorem holds. □

By the proof of Theorem 4.3, we obtain the following corollary.

COROLLARY 4.4. *Let G be a group and H a sporadic simple group. If $|G|$ divides $|H|$ and $|G| < |H|$, then $m^c(G) < m^c(H)$.*

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