Bull. Aust. Math. Soc. **105** (2022), 286–295
doi:10.1017/S0004972721000447

ON GROUPS WHOSE SUBGROUPS ARE EITHER MODULAR OR CONTRANORMAL

FAUSTO DE MARI

(Received 17 April 2021; accepted 28 April 2021; first published online 18 June 2021)

Abstract

A subgroup H of a group G is said to be *contranormal* in G if the normal closure of H in G is equal to G. In this paper, we consider groups whose nonmodular subgroups (of infinite rank) are contranormal.

2020 *Mathematics subject classification*: primary 20E15; secondary 20F19.

Keywords and phrases: locally soluble group, contranormal subgroup, modular subgroup, permutable subgroup, infinite rank.

1. Introduction

Let G be a group. Associated to any subgroup H of G, there is the *normal closure* of H in G,

$$H^G = \langle H^x : x \in G \rangle,$$

which represents the smallest normal subgroup of G containing H, and the *core* of H in G,

$$H_G = \bigcap_{x \in G} H^x,$$

which is the largest G-invariant subgroup of H. Clearly, H is a normal subgroup of G if and only if $H = H_G = H^G$, or equivalently, the *normaliser* of H in G,

$$N_G(H) = \{x \in G : H^x = H\},$$

is equal to the group G. The subgroup H is called *contranormal* if $H^G = G$, *core-free* if $H_G = \{1\}$ and *self-normalising* if $H = N_G(H)$. The properties contranormal, core-free and self-normalising define subgroups that are antipodes of normal subgroups in some way. There are papers devoted to the study of groups in which each subgroup has either a property χ or a property that is antagonistic to χ. When χ is the property for a subgroup to be normal, this point of view is adopted in [2, 14] where groups for which every subgroup is either normal or self-normalising are considered, or in [17] which describes groups whose nonnormal subgroups are core-free. In [22], and more

© 2021 Australian Mathematical Publishing Association Inc.

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

recently in [18], attention has also been given to groups whose nonnormal subgroups are contranormal and a description of these groups has been obtained. It turns out, in particular, that if G is a locally soluble group whose nonnormal subgroups are contranormal, then either all subgroups of G are normal or $G = G'\langle g\rangle$ where $\langle g\rangle$ is a contranormal p-subgroup, for some prime number p, and all subgroups of $G'\langle g^p\rangle$ are normal in G.

A subgroup M of a group G is said to be *modular* if $\langle M, X\rangle \cap Y = \langle X, M \cap Y\rangle$ for all subgroups X, Y of G such that $X \leq Y$ and $\langle M, X\rangle \cap Y = \langle M, X \cap Y\rangle$ if $M \leq Y$; therefore the modular subgroups of G are exactly the modular elements of the lattice $\mathcal{L}(G)$ of all subgroups of G. Clearly every normal subgroup is modular, but modular subgroups need not be normal; moreover, a projectivity (that is, an isomorphism between subgroup lattices) maps any normal subgroup onto a modular subgroup, thus modularity may be considered as a lattice generalisation of normality.

Many papers deal with groups which satisfy restrictions related to modularity (see, for example, [6, 7, 9–12]). In particular in [7], and more recently in [12], groups whose subgroups are either modular or self-normalising have been considered. The aim here is to adopt a similar point of view and investigate groups whose nonmodular subgroups are contranormal. The following result is obtained.

THEOREM 1.1. *Let G be a group whose nonmodular subgroups are contranormal and such that $G \neq G'$. Assume that G does not have modular subgroup lattice. Then $G = G'\langle g\rangle$ where $\langle g\rangle$ is a contranormal p-subgroup, for some prime number p, and all subgroups of $G'\langle g^p\rangle$ are modular in G. Moreover, $G' = [G', G]$ and G' has no proper subgroups of finite p-power index.*

An immediate consequence of Theorem 1.1 is that nonperfect torsion-free groups whose subgroups are either modular or contranormal are actually abelian. Moreover, we show that if the group G of the statement of Theorem 1.1 does not have modular subgroup lattice and G is either nonperiodic or a locally finite p-group, then $p = 2$ (see Propositions 2.6 and 2.7). We also prove that any locally soluble group whose nonmodular subgroups are contranormal is soluble (see Proposition 2.1).

In recent years, the influence on a locally soluble group of the behaviour of the subgroups of infinite rank has been investigated (see, for instance, [4–6, 8, 12]). Recall that a group G is said to have *finite (Prüfer) rank r* if every finitely generated subgroup of G can be generated by r elements and r is the least such integer. It is proved in [8] that if G is a locally soluble group of infinite rank whose proper subgroups of infinite rank are either normal or contranormal, then all subgroups of G are either normal or contranormal. A corresponding result was obtained in [12] for locally soluble groups whose subgroups of infinite rank are either modular or self-normalising. Here a similar result is obtained.

THEOREM 1.2. *Let G be a locally soluble group of infinite rank such that any nonmodular subgroup of infinite rank is contranormal. Then all subgroups of G are either modular or contranormal.*

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

Recall that a subgroup H of a group G is said to be *permutable* (or *quasinormal*) if $HK = KH$ for every subgroup K of G. It is well known that a subgroup is permutable if and only if it is modular and ascendant, and that any modular subgroup of a locally nilpotent group is always permutable. Groups in which all subgroups are permutable are called *quasihamiltonian*. It turns out that nonperiodic groups with modular subgroup lattice are quasihamiltonian and that the class of all quasihamiltonian groups coincides with that of all locally nilpotent groups with modular subgroup lattice. We also obtain similar results to those stated in the main theorems above, concerning groups whose nonpermutable subgroups are contranormal (see Theorems 2.5 and 2.9).

Most of our notation is standard and can be found in [20]. We refer to the monograph [21] for results concerning modular or permutable subgroups and modular lattices.

2. Proofs

Let G be a group whose nonmodular subgroups are contranormal and let N be any proper normal subgroup of G. Then every subgroup of G/N is either modular or contranormal; moreover, since $H^G \leq N \neq G$ whenever $H \leq N$, every subgroup of N is a modular subgroup of G. We make use of this remark many times in our arguments without further mention.

PROPOSITION 2.1. *Let G be a locally soluble group whose nonmodular subgroups are contranormal. Then G is soluble of derived length at most* 3.

PROOF. Assume by contradiction that $G^{(3)} \neq \{1\}$. Since any section of G with modular subgroup lattice is metabelian (see [21, Theorem 2.4.21]), the derived subgroup G' cannot be a proper subgroup and hence $G = G'$. Let M be the join of all cyclic modular subgroups of G; clearly M is a normal subgroup of G, so that either $G = M$ or M is metabelian. Note that if $x \in G \setminus M$, then $\langle x \rangle$ is not modular in G and so $\langle x \rangle^G = G$, thus the factor G/M is simple. Since any simple locally soluble group has prime order (see [20, Part 1, Corollary 1, page 154]), $G = M \neq \{1\}$. Let N be any proper normal subgroup of G. Since $N \neq G = M$, there exists a nontrivial cyclic subgroup X which is modular in G such that $X \not\leq N$. Since G is locally soluble, any modular subgroup of G is actually permodular in the sense of [21] (see also [23]) and hence $(XN/N)^{G/N}$ contains a nontrivial abelian normal subgroup of G/N (see [21, Lemma 6.4.6]). It follows that G is hyperabelian (see [20, Part 1, Lemma 1.22]). Let

$$\{1\} = G_0 < \cdots < G_\alpha < \cdots < G_\tau = G$$

be an ascending normal series with abelian factors and let α be the least ordinal such that $G_\alpha^{(3)} \neq \{1\}$. Since any proper normal subgroup of G is metabelian, it follows that $G = G_\alpha$ and G_β is metabelian for all $\beta < \alpha$. If α is a limit ordinal, then

$$G = G_\alpha = \bigcup_{\beta<\alpha} G_\beta$$

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

is metabelian; thus $\alpha = \delta + 1$. Then G_δ is metabelian and $G/G_\delta = G_\alpha/G_\delta$ is abelian, hence $G^{(3)} = \{1\}$. This final contradiction completes the proof. □

Recall that a *power automorphism* of a group G is an automorphism of G leaving every subgroup of G invariant. The following lemma suggests that properties of power automorphisms can be used in the study of groups whose nonmodular subgroups are contranormal.

LEMMA 2.2. *Let G be a group whose nonmodular subgroups are contranormal and let N be any proper normal subgroup of G. If T is the largest periodic normal subgroup of N and $x \in N$, then $\langle x\rangle T$ is normal in G.*

PROOF. All subgroups of N are modular in G, hence T coincides with the set of all elements of finite order of N (see [21, Lemma 2.4.8]) and N/T is abelian (see [21, Lemma 2.4.9]). Clearly T is a normal subgroup of G and so, replacing G by G/T, we can suppose that N is torsion-free abelian. Thus $\langle x\rangle^G$ is likewise torsion-free abelian and $\langle x\rangle$ is a permutable subgroup of G. If a is any element of finite order of G, then

$$\langle x, a\rangle \cap \langle x\rangle^G = (\langle x\rangle\langle a\rangle) \cap \langle x\rangle^G = \langle x\rangle(\langle a\rangle \cap \langle x\rangle^G) = \langle x\rangle$$

and hence $\langle x\rangle$ is normalised by a. Consider now an element of infinite order b of G. If $\langle x\rangle \cap \langle b\rangle = \{1\}$, then b normalises $\langle x\rangle$ (see [21, Lemma 5.2.7]). Suppose that $\langle x\rangle \cap \langle b\rangle \neq \{1\}$. Then $\langle x\rangle \cap \langle b\rangle$ is a central subgroup of finite index of $\langle x, b\rangle = \langle x\rangle\langle b\rangle$, hence $[x, b]$ is an element of finite order of $\langle x\rangle^G$ (see [20, Part 1, Theorem 4.12]) and so $[x, b] = 1$. Therefore $\langle x\rangle$ is a normal subgroup of G. □

LEMMA 2.3. *Let $G = G'\langle g\rangle$ be a group whose nonmodular subgroups are contranormal, such that $G \neq G'$. If G/G' is finite, then g has finite order.*

PROOF. Clearly it can be assumed that G is not periodic, so that also G' is not periodic. Since G' is a proper subgroup of G, all subgroups of G' are modular in G and hence G' is metabelian (see [21, Theorem 2.4.21]); thus G is soluble. Let T be the largest periodic normal subgroup of G. Replacing G by G/T, it can be assumed that $T = \{1\}$. Thus G' is torsion-free abelian (see [21, Theorem 2.4.11]) and, by Lemma 2.2, G acts as a group of power automorphisms on G'. Since $G = \gamma_i(G)\langle g\rangle$ for every positive integer i (see [20, Part 1, Lemma 2.22]) and $G \neq \langle g\rangle$, the group G is not nilpotent and hence G cannot act trivially on G'. Therefore $x^g = x^{-1}$ for every $x \in G'$ (see [21, Theorem 1.5.7]). Since G/G' is finite, there exists a positive integer n such that $g^n \in G'$. Then $g^n = (g^n)^g = g^{-n}$ and hence g has finite order. □

PROOF OF THEOREM 1.1. Assume that G contains some nonmodular subgroup, so that there exists a cyclic nonmodular subgroup $\langle g\rangle$ of G. Thus $\langle g\rangle^G = G$ and $G = G'\langle g\rangle$.

Suppose that $\langle g\rangle = XY$ where X and Y are proper subgroups and $\langle g\rangle \cap G' \leq X \cap Y$. If $G = XG'$ then $\langle g\rangle = XG' \cap \langle g\rangle = X(G' \cap \langle g\rangle) = X$, which is not possible; hence $G \neq XG'$, and similarly also $G \neq YG'$. Since XG' and YG' are normal subgroups of G, it follows that X and Y are modular subgroups of G and hence $\langle g\rangle = XY$ is likewise

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

modular in G. This contradiction proves that $\langle g\rangle/G' \cap \langle g\rangle \simeq G'\langle g\rangle/G' = G/G'$ is a cyclic p-group for some prime p (see [13, Corollary 2.4, page 156]), hence $\langle g\rangle$ is finite by Lemma 2.3. Write $g = g_1g_2$ where g_1 is a p-element, g_2 is a p'-element and $[g_1, g_2] = 1$. Then $g_2 \in G'$ and hence $\langle g_2\rangle$ is modular in G. Therefore $\langle g_1\rangle$ is not modular in G; in particular, $\langle g_1\rangle^G = G = G'\langle g_1\rangle$. Since $\langle g_1\rangle$ has prime-power order, its subgroup lattice is a chain so that, as $g_1 \notin G'$, we have $\langle g_1\rangle \cap G' \leq \langle g_1^p\rangle$. If $G = G'\langle g_1^p\rangle$, then $\langle g_1\rangle = G'\langle g_1^p\rangle \cap \langle g_1\rangle = \langle g_1^p\rangle(G' \cap \langle g_1\rangle) = \langle g_1^p\rangle$, which is not possible; thus $G'\langle g_1^p\rangle$ is a proper normal subgroup of G and hence all subgroups of $G'\langle g_1^p\rangle$ are modular in G.

Let $L = [G', G]$. Since $G'/L \leq Z(G/L)$ and G/G' is cyclic, the factor G/L is abelian and hence $G' = L$. Now, assume that H is a proper subgroup of G' such that the index $|G' : H|$ is a p-number. Since G/G' is a cyclic p-group, the set $\{H^x : x \in G\}$ is finite and the index $|G : H^x| = |G : H|$ is a p-number for each $x \in G$, therefore G/H_G is a finite p-group. Hence G/H_G is nilpotent and so it has no proper contranormal subgroups, thus $G/H_G = \langle gH_G\rangle$ is abelian. Therefore $G' = H_G$, which is not possible because $H \neq G'$. This contradiction proves that G' has no subgroups of p-power index and the theorem is proved. □

COROLLARY 2.4. *Let G be a torsion-free group whose nonmodular subgroups are contranormal and such that $G \neq G'$. Then G is abelian.*

PROOF. By Theorem 1.1, G has modular subgroup lattice and so G is abelian (see [21, Lemma 2.4.9]). □

THEOREM 2.5. *Let G be a group whose nonpermutable subgroups are contranormal and such that $G \neq G'$. Assume that G is not quasihamiltonian. Then $G = G'\langle g\rangle$ where $\langle g\rangle$ is a contranormal p-subgroup, for some prime number p, and all subgroups of $G'\langle g^p\rangle$ are permutable in G. Moreover, $G' = [G', G]$ and G' has no proper subgroups of finite p-power index.*

PROOF. The result can be obtained by replacing modular by permutable subgroups in the proof of Theorem 1.1. □

Note that if the group G in Theorem 1.1 is not periodic (and does not have modular subgroup lattice), then G' is a nonperiodic group with modular subgroup lattice. Moreover, if T is the subgroup consisting of all elements of finite order of G' (recall that the set of all elements of finite order of a group with modular subgroup lattice is a subgroup by [21, Lemma 2.4.8]), then G'/T is a locally cyclic group (see [21, Theorem 2.4.11]) but G'/T cannot be infinite cyclic because G' has no subgroups with index a power of p. Further information on the nonperiodic group G is given in the following result.

PROPOSITION 2.6. *Let G be a group whose nonmodular subgroups are contranormal and such that $G \neq G'$. Assume that G is not periodic and that the subgroup lattice of G is not modular. Then $G = G'\langle g\rangle$ where $\langle g\rangle$ is a contranormal 2-subgroup and the subgroup $G'\langle g^2\rangle$ is nilpotent.*

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

PROOF. Application of Theorem 1.1 gives $G = G'\langle g\rangle$ where $\langle g\rangle$ is a contranormal p-subgroup (p prime) and all subgroups of $M = G'\langle g^p\rangle$ are modular in G. Let T be the subgroup consisting of all elements of finite order of M. Then T and M/T are abelian (see [21, Theorem 2.4.11]); in particular, M/T is contained in the centraliser $H/T = C_{G/T}(M/T)$. Therefore $M \leq H \leq G$ and, since $|G : M| = p$ is prime, either $G = H$ or $M = H$. If $G = H$, then $M/T \leq Z(G/T)$ so that, since G/M is cyclic, we have $G/T = Z(G/T)$ and hence G' is periodic. This contradiction proves that $M = H$. On the other hand, G acts as a group of power automorphisms on the torsion-free abelian factor M/T by Lemma 2.2 and this implies that $|G : M| \leq 2$ (see [21, Theorem 1.5.7]). Therefore $p = 2$ and

$$G' \leq K_1 := C_G(M/T) \cap C_G(G/M).$$

Any subgroup X of T is normal in M (see [21, Lemma 2.4.8]) and G/M is finite, hence X is a permutable subgroup of G having finitely many conjugates; if $X^{g_1}, \ldots, X^{g_t}$ are the conjugates of X in G and $H = \langle X, g_1, \ldots, g_t\rangle$, then $X^G = X^H$ and so the index $|X^G : X|$ is finite (see [21, Lemma 6.2.8]). It follows that there exist G-invariant subgroups N and D of T, with N finite and $N \leq D$, such that G acts as a group of power automorphisms on the factors T/D and D/N (see [1, Theorem 2.11]). Since the group of all power automorphisms is abelian (see [21, Theorem 1.5.1]),

$$G' \leq K_2 := C_G(D/N) \cap C_G(T/D).$$

Thus G' is contained in $K = K_1 \cap K_2$ and so K has finite index in G. Since N is finite, the centraliser $C_G(N)$ has finite index in G and hence also $S = C_G(N) \cap K$ has finite index in G. Moreover, since S stabilises the normal series

$$\{1\} \leq N \leq D \leq T \leq M \leq G,$$

the subgroup S is nilpotent (see [15]). Then $S \cap M$ is a normal nilpotent subgroup of finite index of M and hence $M = (S \cap M)E$ by a suitable finitely generated subgroup E. Note that E is a nilpotent permutable subgroup of G. In fact, since M is nonperiodic and all its subgroups are modular in G, the subgroup M is locally nilpotent (see [21, Theorems 2.4.11 and 6.2.10]) and so all subgroups of M are permutable in G. On the other hand, since G is nilpotent-by-finite, E/E_G is subnormal in G/E_G (see [3, Theorem 2.6]). Thus E is subnormal in M and hence $M = (S \cap M)E$ is nilpotent. □

Further information can be obtained for locally soluble primary groups. In particular, the next result proves that if G is a locally finite p-group whose nonmodular subgroups are contranormal and p is an odd prime, then G has modular subgroup lattice.

PROPOSITION 2.7. *Let G be a locally nilpotent group whose nonmodular subgroups are contranormal, and assume that the subgroup lattice of G is not modular. Then $G = G'\langle g\rangle$ is a 2-group, the derived subgroup G' is abelian divisible and $x^g = x^{-1}$ for every element x of G'.*

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

PROOF. The group G is soluble by Proposition 2.1 and so, by Theorem 1.1, $G = G'\langle g\rangle$ where $\langle g\rangle$ is a contranormal p-subgroup for some prime number p, and all subgroups of $G'\langle g^p\rangle$ are modular in G (and so even permutable since G is locally nilpotent). If T is the subgroup consisting of all elements of finite order of G, then $g \in T$; hence $G = \langle g\rangle^G \leq T$ and so $G = T$ is periodic. Since in a nilpotent group there are no proper contranormal subgroups, the group G is infinite; in particular, G' is infinite.

Consider a normal subgroup N of G such that G/N is not abelian, and let X/N be a subgroup of prime order of $(G/N)'$. If $X/N \lhd G/N$ then $X/N \leq Z(G/N)$ since G/N is locally nilpotent; otherwise, since X/N is permutable in G/N by Theorem 1.1, we have $X/N \leq Z_2(G/N)$ (see [21, Theorem 5.2.12]). Therefore each factor of G has nontrivial centre and hence G is hypercentral. It follows that G contains an abelian normal divisible subgroup V of finite index such that $V \leq G'$ (see [16, Corollaries 3.2 and 3.4]). Then G/V is nilpotent and so $G/V = \langle gV\rangle$. Therefore $G' \leq V$. Thus $G' = V$ and G' is a divisible abelian group.

Assume, by contradiction, that G contains some element of prime order q with $q \neq p$. Since $G = G'\langle g\rangle$, the subgroup G' contains a Sylow q-subgroup Q of G. Write $G' = Q \times R$ where R is a q'-subgroup. Clearly $G/R = QR/R \times \langle g\rangle R/R$ and so $\langle g\rangle R/R$ is not contranormal in G, which is not possible since $\langle g\rangle^G = G$. This contradiction proves that G is a p-group.

Let x be any element of G'. Then there exists $P \leq G'$ such that P is a Prüfer p-group and $x \in P$ (see [13, Theorem 2.7 and Corollary 2.8, pages 136–137]). Since P is permutable in G, it is even normal in G (in fact, if $y \in G$, then $\langle P, y\rangle = P\langle y\rangle$ contains P as a subgroup of finite index and hence P must be normalised by y); thus $\langle x\rangle$ is likewise a normal subgroup of G. This proves that all subgroups of G' are normal in G. Then G acts as a group of power automorphisms on G' and so the order of the factor $G/C_G(G')$ either is 2 or divides $p - 1$ (see [19, page 29]). On the other hand, $G'\langle g^p\rangle$ is abelian (see [21, Corollary 2.4.15]) and has index p in $G = G'\langle g\rangle$, so that, since G is not nilpotent, $G'\langle g^p\rangle = C_G(G')$ and hence $G/C_G(G')$ has order p. Therefore $p = 2$ and $x^g = x^{-1}$ for all $x \in G'$; in particular, G is a 2-group. □

Next consider groups with restrictions on subgroups of infinite rank.

LEMMA 2.8. *Let G be a locally soluble group of infinite rank whose proper nonmodular subgroups of infinite rank are contranormal, and let N be any proper normal subgroup of infinite rank of G. Then every subgroup of N is modular in G.*

PROOF. Let L be any proper normal subgroup of G of infinite rank. If H is any subgroup of infinite rank of L, then $H^G \leq L \neq G$ and hence H is a modular subgroup of L. Therefore L has modular subgroup lattice (see [6, Theorem A]); in particular, the subgroup lattice of N is modular.

Assume first that for some proper normal subgroup of infinite rank L of G, the factor G/L is periodic (and hence locally finite). If X is any finitely generated subgroup of G, then $X/X \cap L \simeq XL/L$ is finite; hence $X \cap L$ is a finitely generated subgroup of L and so, since L has modular subgroup lattice, $X \cap L$ is polycyclic (see [21, Lemma 2.1.7]).

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

It follows that X has finite rank. Thus all finitely generated subgroups of G have finite rank. Let E be any finitely generated subgroup of N and consider any element g of G. Since N has modular subgroup lattice, it is soluble (see [21, Theorem 2.4.21]) and so N contains an abelian subgroup A of infinite rank (see, for instance, [20, Part 2, page 89]). As $\langle E, g\rangle$ is finitely generated, $\langle E, g\rangle$ has finite rank and so it can be assumed that $A \cap \langle E, g\rangle = \{1\}$. Then A and $\langle A, E\rangle$ are subgroups of infinite rank of N, so that they are modular subgroups of G. Hence

$$\langle A, E\rangle \cap \langle E, g\rangle = \langle E, A \cap \langle E, g\rangle\rangle = E$$

is a modular subgroup of $\langle E, g\rangle$. Therefore E is a modular subgroup of G (see [23, Corollario B]), and so it follows that all subgroups of N are modular in G in this case. Thus it can be assumed that for every proper normal subgroup of infinite rank L of G the factor G/L is not periodic; in particular, G/N is a nonperiodic locally soluble group having no subgroups of finite index. Therefore, since N has infinite rank, all subgroups of G/N are either modular or contranormal and it follows from Proposition 2.1 and Theorem 1.1 that G/N has modular subgroup lattice. Hence G/N is generated by its elements of infinite order (see [21, Lemma 2.4.8]) and so all modular subgroups of G contained in N are actually normal in G (see [11, Lemma 6]). Therefore all subgroups of infinite rank of N are G-invariant. Let K be any subgroup of finite rank of N. Since N is a soluble group of infinite rank, it contains an abelian subgroup of infinite rank and so also two abelian subgroups of infinite rank A_1 and A_2 such that $\langle A_1, A_2\rangle = A_1 \times A_2$ and $(A_1 \times A_2) \cap K = \{1\}$. Then A_1 and A_2 are normal in G, and hence also the subgroups of infinite rank A_1K and A_2K (of N) are normal in G. Therefore $K = A_1K \cap A_2K$ is normal in G and the lemma is proved. □

Proof of Theorem 1.2. Since simple locally soluble groups have prime order (see [20, Part 1, Corollary 1, page 154]), G contains a proper normal subgroup N of infinite rank (see [8, Lemma 2.2]). Then all nonmodular subgroups of G/N are contranormal and, by Proposition 2.1, G/N is soluble. On the other hand, the subgroup N has modular subgroup lattice by Lemma 2.8 and hence it is soluble (see [21, Theorem 2.4.21]); therefore G itself is soluble.

Let H be any subgroup of finite rank of G such that $H^G \neq G$. Assume for a contradiction that H is not a modular subgroup of G; in particular, by Lemma 2.8, H^G also has finite rank. Then G/H^G is a soluble nontrivial group and it follows that $G'H^G \neq G$. Therefore Lemma 2.8 implies that $G'H^G$ has finite rank, hence $G/G'H^G$ is an abelian group of infinite rank and so there exists a proper subgroup of infinite rank K of G such that $G'H^G \leq K$. Therefore, again by Lemma 2.8, all subgroups of K are modular in G. Thus H is a modular subgroup of G, and this contradiction concludes the proof. □

Theorem 2.9. *Let G be a locally soluble group of infinite rank such that any nonpermutable subgroup of infinite rank is contranormal. Then all subgroups of G are either permutable or contranormal.*

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

PROOF. By Theorem 1.2, all subgroups of G are either modular or contranormal; in particular, G is soluble by Proposition 2.1.

Let N be any proper normal subgroup of G of infinite rank. Then all subgroups of N of infinite rank are permutable in G and so N is quasihamiltonian (see [6, Corollary A1]). It follows that all subgroups of N are ascendant in G (see [21, Theorem 6.2.10]) and hence, by Lemma 2.8, all subgroups of N are permutable in G. Using this remark instead of Lemma 2.8 and replacing modular by permutable subgroups, the same argument as in the proof of Theorem 1.2 can now be used to prove that if H is any subgroup of finite rank of G such that $H^G \neq G$, then H is a permutable subgroup. □

References

[1] C. Casolo, 'Groups with finite conjugacy classes of subnormal subgroups', *Rend. Semin. Mat. Univ. Padova* **81** (1989), 107–149.

[2] G. Cutolo, 'A normalizer condition on systems of subgroups', *Boll. Unione Mat. Ital. A (7)* **3** (1989), 215–223.

[3] G. Cutolo and A. Leone, 'Permutable subgroups and the Maier–Schmid theorem for nilpotent-by-finite groups', *J. Algebra* **546** (2020), 723–733.

[4] U. Dardano and F. De Mari, 'On groups with all proper subgroups finite-by-abelian-by-finite', *Arch. Math. (Basel)* (to appear). Published online (17 February 2021).

[5] U. Dardano and F. De Mari, 'On groups in which subnormal subgroups of infinite rank are commensurable with some normal subgroup', *Int. J. Group Theory* (to appear). Published online (12 April 2021).

[6] M. De Falco, F. de Giovanni and C. Musella, 'A note on groups of infinite rank with modular subgroup lattice', *Monatsh. Math.* **176** (2015), 81–86.

[7] M. De Falco and C. Musella, 'A normalizer condition for modular subgroups', in: *Advances in Group Theory, Napoli, 2002* (eds. F. De Giovanni and M. L. Newell) (Aracne, Rome, 2002), 163–172.

[8] A. V. De Luca and G. di Grazia, 'Groups of infinite rank with normality conditions on subgroups with small normal closure', *Bull. Aust. Math. Soc.* **94** (2016), 48–53.

[9] F. De Mari, 'Groups satisfying weak chain conditions on non-modular subgroups', *Comm. Algebra* **46** (2018), 1709–1715.

[10] F. De Mari, 'Groups with restricted non-permutable subgroups', *J. Algebra Appl.* **20**(4) (2021), 2150062.

[11] F. De Mari, 'Groups with all subgroups either modular or soluble of finite rank', *J. Algebra* **570** (2021), 416–436.

[12] F. De Mari, 'Groups with many modular or self-normalizing subgroups', *Comm. Algebra* **49**(6) (2021), 2356–2369.

[13] L. Fuchs, *Abelian Groups* (Springer, Cham, 2015).

[14] G. Giordano, 'Gruppi con normalizzatori estremali', *Matematiche (Catania)* **26** (1971), 291–296.

[15] P. Hall, 'Some sufficient conditions for a group to be nilpotent', *Illinois J. Math.* **2** (1958), 787–801.

[16] L. A. Kurdachenko, P. Longobardi and M. Maj, 'On the structure of some locally nilpotent groups without contranormal subgroups', Preprint, 2020, arXiv:2006.02345v1.

[17] L. A. Kurdachenko, A. A. Pypka and I. Y. Subbotin, 'On the structure of groups whose non-normal subgroups are core-free', *Mediterr. J. Math.* **16**(6) (2019), 136.

[18] L. A. Kurdachenko, A. A. Pypka and I. Y. Subbotin, 'On groups with normal, contranormal or core-free subgroups', *Adv. Group Theory Appl.* **10** (2020), 83–125.

[19] D. J. S. Robinson, 'Groups in which normality is a transitive relation', *Math. Proc. Camb. Phil. Soc.* **60** (1964), 21–38.

[20] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups* (Springer, Berlin, 1972).

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press

[21] R. Schmidt, *Subgroup Lattices of Groups* (de Gruyter, Berlin, 1994).
[22] I. Y. Subbotin, 'Groups with alternatively normal subgroups', *Izv. Vyssh. Uchebn. Zaved. Mat.* **3** (1992), 86–88.
[23] G. Zacher, 'Una relazione di normalità sul reticolo dei sottogruppi di un gruppo', *Ann. Mat. Pura Appl. (4)* **131** (1982), 57–73.

FAUSTO DE MARI,
Università degli Studi di Napoli Federico II,
80126, Naples, Italy
e-mail: fausto.demari@unina.it

https://doi.org/10.1017/S0004972721000447 Published online by Cambridge University Press