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Trace spaces of directed tori with rectangular holes

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Raussen (2010) gave the trace space, which corresponds to the executions of parallel non-looped, non-branching processes as a prod-simplicial complex derived from a poset. The connected components represent equivalent executions. For looped processes, the state space is a torus and the trace space is a disjoint union of trace spaces of deloopings. In the current paper, we develop the index poset for the trace space of the deloopings from the once delooped case. When just one process is looped, the index poset is generated as words in a regular language. We also construct a corresponding automaton.

1. Introduction

Dijkstra provided a simple model of concurrency in which a set of processes interact through shared objects (Dijkstra 1968). These objects are guarded by semaphores allowing only a certain number of processes access at the same time. The geometric model is a product of directed graphs with 'holes' corresponding to the restrictions on the shared objects.

An execution is a directed path, which is non-decreasing in each coordinate, from an initial point p to a final point q. The set of directed paths from p to q with the compact open topology is the path space. Two executions are equivalent if they are in the same connected component of this space or, equivalently, of the space of traces, which is the quotient of the paths space under increasing reparametrisation. The verification of properties for all executions is reduced to the verification for one execution from each component.

Raussen (2010) considered the special case of non-looped, non-branching processes. There is an algorithm for calculating the space of directed traces from an initial point to a final point. The resulting finite representation of the trace space is in terms of a prod-simplicial complex whose geometric realisation is homotopy equivalent to the trace space. This has allowed the calculation of components and a representative of each. The representatives are then used for verification – see Fajstrup *et al.* (2012). The prod-simplicial complex is built from a poset – the *schedules*.

Example 1.1. The geometric model for the PV-program $P_aV_aP_bV_b|P_bV_bP_aV_a$ with two shared resources a and b, each allowing at most one process access at a time, is a square with two square 'holes'. An execution is a path from the lower left-hand corner, $(0,0)$ to the upper right-hand corner $(1,1)$ that is non-decreasing in both coordinates and avoids the holes. The holes are sets of states, where both processes have access to one

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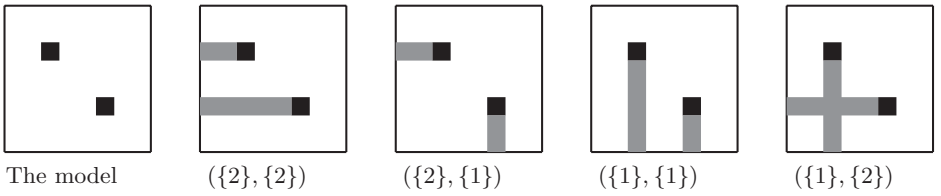


Fig. 1. $P_aV_aP_bV_b|P_bV_bP_aV_a$. The geometric model and the four schedules.

of the resources; each hole is a conflict – see Figure 1. The Raussen algorithm provides a description of the space of executions using *schedules*: for each hole, decide whether executions *do not go under* or *do not go over* that hole. In this case, there are 4 possible schedules. The grey areas indicate these schedules – paths satisfying the schedules are not allowed to enter the grey (or black) areas. Note that the last schedule cannot be satisfied by any execution. This schedule is *not allowed* and this is recognised by the formation of a deadlock at the intersection of the extended rectangles, that is, there are no non-decreasing paths from $(0,0)$ to $(1,1)$ obeying this schedule. The three *allowed schedules* are each satisfied by a contractible set of executions – see Proposition 3.2. The elements in the index poset are the allowed schedules. The partial order in this case is the discrete order.

For higher-dimensional examples, the geometric model is an n-dimensional cube. The conflicts at shared resources correspond to n-rectangular holes, the *forbidden areas*. A schedule is a non-empty set of processes for each conflict-rectangle. These processes are not allowed to access the corresponding resource before at least one of the others has released it. Let the ith conflict be the n-rectangle

$$R^i = \times_{k=1}^{n}]a_k^i, b_k^i[.$$

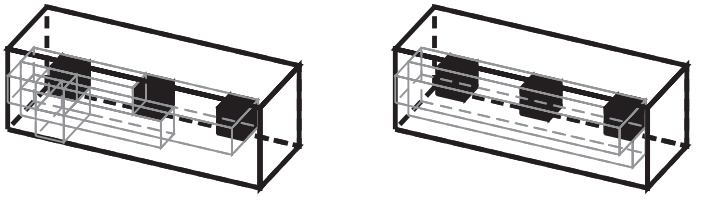
The geometric model of the schedule 'the jth process does not access the ith resource until the others have released it' is then a requirement that the execution paths are not allowed to enter the rectangle

$$R_j^i = [0, b_1^i[\times \ldots [0, b_{j-1}^i[\times]a_j^i, b_j^i[\times [0, b_{j+1}^i[\times \ldots \times [0, b_n^i[,$$

which is the rectangle R^i extended to the lower boundary along all coordinates except the jth. Again, as in the 2-dimensional case above, if a schedule is prescribed for each resource, either the corresponding trace space is contractible, or it is empty, and this may be seen by a deadlock check. The *allowed* schedules are those for which the extended rectangles do not produce a deadlock.

Example 1.2. The geometric model of the program $P_aV_aP_aV_aP_aV_a|P_aV_a|P_aV_a$ is a 3-cube with 3 holes (drawn as a 3-rectangle in Figure 2). In Figure 2(a), the three black cubes are the 'holes'. The grey transparent rectangles are the extensions corresponding to the schedule $\tilde{J} = (\{1\}, \{2\}, \{3\})$, which requires: rectangle R^1 to be extended to 0 in coordinate-directions 2 and 3, that is, *not direction 1* ; rectangle R^2 in the *not 2* directions; and rectangle R^3 in the *not 3* directions. This schedule is not allowed, since the extended

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a. $\tilde{J} = (\{1\}, \{2\}, \{3\})$ b. $J = (\{2,3\}, \{2,3\}, \{2,3\})$

Fig. 2. $P_aV_aP_aV_aP_aV_a|P_aV_a|P_aV_a$. Two schedules.

rectangles intersect to give a deadlock at the lower corner of R^1, that is, there are no non-decreasing paths from $(0,0,0)$ to $(1,1,1)$. The forbidden region is

$$R_1^1 \cup R_2^2 \cup R_3^3.$$

A general schedule, Definition 3.1, is a set of restrictions for each forbidden rectangle. In Example 1.2, the schedule $(\{2,3\}, \{2,3\}, \{2,3\})$ means that there are two requirements at each forbidden rectangle: 'Process 2 does not access the corresponding resource until at least one of the others has released it'; and the same requirement applies to process 3. That is, neither process 2 nor process 3 access the resource until process 1 has released it. The geometric model of this is shown in Figure 2b. All the forbidden areas are extended by two grey rectangles: one extended below in the 'not 2' directions and the other in the 'not 3' directions. The extended forbidden region is

$$\bigcup_{i=1}^{3} (R_2^i \cup R_3^i).$$

This schedule is allowed, as may be seen from the picture – an execution in which process 1 finishes before the others begin satisfies the schedule.

The finite model of the trace space is given by an *index poset*. The elements are the allowed schedules and the partial order is the subset relation: in Example 1.2, the schedule

$$\tilde{J} = (\{2\}, \{3\}, \{2\})$$

is a subset of

$$J = (\{2,3\}, \{2,3\}, \{2,3\})$$

since

$$\tilde{J}_1 = \{2\} \subset \{2,3\} = J_1,$$

and, similarly, $\tilde{J}_i \subset J_i$ for $i = 2,3$. See Raussen (2010) for the construction of the trace space from the poset.

For processes with loops, the directed traces from the initial to the final point may be considered as traces in deloopings, that is, in non-looped space. The total trace space is a disjoint union of the trace spaces of the deloopings – see Proposition 2.13. And the algorithm in Raussen (2010) applies to each delooping. However, there is an obvious periodicity arising from the loops, and it is this periodicity that is made clear here.

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We restrict consideration to the case of n processes, P_j $j \in [1:n]$, where each process is a loop $P_j = T_j^*$. We do not allow nested loops or branching, or even $P_j = S.T^*.U$, that is, that one process is a concatenation of non-looped and looped parts. However, it should be possible to develop these cases along the same lines. For such a program

$$\mathcal{P} = T_1^* | T_2^* \dots | T_n^*$$

and a vector

$$M = (m_1, \dots, m_n) \in \mathbb{N}^n,$$

the M delooping of $\mathcal{P}$ is

$$\mathcal{P}^M = T_1^{m_1} | T_2^{m_2} \dots | T_n^{m_n}$$

where

$$T^k = T.T. \dots .T,$$

is T repeated k times. Theorem 4.16 gives the index poset of $\mathcal{P}^M$ in terms of the index poset of $\mathcal{P}^{(1,\dots,1)} = T_1 | T_2 \dots | T_n$ and a deadlock check in the space X^M corresponding to $\mathcal{P}^M$.

In particular, Theorem 5.8, when only one process is delooped, that is, $m_r = m$ and $m_i = 1$ for $i \neq r$, gives the index poset as words in a regular language. The alphabet $\mathcal{A}$ consists of allowed schedules, for $T_1 | T_2 \dots | T_n$. A word of length m gives a schedule for $T_1 | T_2 \dots | T_r^m \dots | T_n$. A word of length m is allowed if it corresponds to an allowed schedule for $T_1 | T_2 \dots | T_r^m | \dots | T_n$. The allowed words form a *regular language*.

By Theorem 5.8, the language is recognised by an automaton with at most

$$(2^n - 1)^l + 1$$

states, where l is the number of shared objects. This automaton outputs the index poset generating the trace spaces for deloopings in one direction. Examples 4.5 and 4.6 show that for letters $a, b, c \in \mathcal{A}$, it may be that aa is not allowed. All two letter words $ab, ba, ac, \cdots$ are allowed and abc is allowed, but bac is not allowed. Hence, there is a fairly complicated interference between local schedules, and we should not expect a simple algorithm to generate the trace space for X^M for a general M. We explored the periodicity as developed here in Fajstrup *et al.* (2012) to build an automaton for general deloopings.

2. Processes with loops

The geometric model for n looped processes in parallel is a torus with rectangular holes.

Definition 2.1 (notation). We use the following conventions:

— For integers $l \leqslant m$,

$$[l : m] = \{n \in \mathbb{Z} | l \leqslant n \leqslant m\}.$$

— For integer vectors $V = (v_1, \dots, v_n)$ and $W = (w_1, \dots, w_n)$, let

$$[V : W] = \{K = (k_1, \dots, k_n) | k_i \in [v_i : w_i]\}.$$

— e_j is the jth unit vector.

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— For points $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, we define the closed n-interval

$$[a, b] = \{x \in \mathbb{R}^n | (a_i \leqslant x_i \leqslant b_i\},$$

and, similarly, $]a, b[$, $[a, b[$ and $]a, b]$.
— $I = [0, 1]$ is the unit interval.

Definition 2.2. A d-space is a topological space Y and a subset $\vec{P} \subset Y^I$, the dipaths, such that:

— All constant paths are in $\vec{P}$.
— If $\mu, \nu \in \vec{P}$, $\nu(1) = \mu(0)$, then the concatenation $\mu \star \nu \in \vec{P}$.
— For $\alpha : I \to I$ non-decreasing and $\mu \in \vec{P}$, we have $\mu \circ \alpha \in \vec{P}$.

A map $f : X \to Y$ where $(X, \vec{P})$ and $(Y, \vec{Q})$ are d-spaces is a *dimap* if it is continuous and $f(\vec{P}) \subset \vec{Q}$. A *dihomeomorphism* is a bijective dimap such that the inverse is a dimap.

Example 2.3. We have:

— $\vec{I}$ is the d-space $[0, 1]$ with dipaths the non-decreasing paths.
— $\vec{I}^n$ is the n-cube with coordinate-wise non-decreasing dipaths.
— I is a d-space with only the constant dipaths.
— For a vector $M = (m_1, \ldots, m_n)$, let

$$\vec{I}^M = \times_{i=1}^n [0, m_i]$$

with coordinate-wise non-decreasing dipaths.

Definition 2.4. For $p, q \in Y$ and Y a d-space, the *space of dipaths* $\vec{P}(Y)(p, q)$ is the set of dipaths starting in p and ending in q. The topology is the compact open topology. The *trace space* $\vec{T}(Y)(p, q)$ is the quotient of the path space under non-decreasing reparametrisation – see Raussen (2007).

Remark 2.5. In the following, d-spaces are state spaces and dipaths are (partial) execution paths.

Definition 2.6. Consider n parallel non-looped processes, T_j, $j \in [1 : n]$, and l resources such that each resource is an $n - 1$ semaphore allowing simultaneous access by $n - 1$ but not n processes, and such that process j wants access to resource i in the interval $]a^i_j, b^i_j[\subset [0, 1]$.

The geometric model, the *state space*, of $T_1 | T_2 | \ldots | T_n$ is $X = \vec{I}^n \setminus F$, where $F = \cup_{i=1}^l R^i$, $R^i =]a^i, b^i[$ and $a^i_j, b^i_j \in]0, 1[$ for all $i \in [1 : l]$ and all $j \in [1 : n]$. An execution is a path $\gamma \in \vec{P}(X)(0, 1)$. Two executions γ_0 and γ_1 are equivalent if they are *dihomotopic*, that is, there is a dimap $H : \vec{I} \times I \to X$ such that:

— $H(t, 0) = \gamma_0(t)$ and $H(t, 1) = \gamma_1(t)$.
— $H(0, s)$ and $H(1, s)$ are constant.

Executions have dihomotopic execution paths if and only if they give the same output for the same input – see Fajstrup *et al.* (2006). Since non-decreasing reparametrisation is a dihomotopy, dipaths are equivalent, if they have the same trace.

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Remark 2.7. This is the case studied in Raussen (2010).

We will use this setup, except that each process T_j is now a loop.

Definition 2.8 (geometry of loops). Consider n parallel processes $\mathcal{P} = T_1^*|T_2^*|\dots|T_n^*$, that is, each of them is a loop, with conflicts as above. Let $X = \vec{I}^n \setminus F$, where $F = \cup_{i=1}^{l} R^i$ is as above. The *state space* $\bar{X}$ for $\mathcal{P}$ is the quotient $\bar{X} = X/\sim$ under the equivalence relations

$$(x_1, \dots, x_{i-1}, 0, x_{i+1} \dots, x_n) \sim (x_1, \dots, x_{i-1}, 1, x_{i+1} \dots, x_n) \qquad i = 1, \dots, n.$$

This space has the quotient topology. It is an n-torus with rectangular holes.

A dipath is a continuous path $\gamma : I \to \bar{X}$ such that γ is *locally non-decreasing* in the following sense:

For each coordinate $\gamma_i : I \to \vec{I}/(0 \sim 1)$, let

$$\gamma_i^{-1}(0) = \cup_{j=0}^{m_i} [t_j^i, s_j^i],$$

where $t_0^i = 0$ and $s_{m_i}^i = 1$. Then γ is locally non-decreasing if for all i, k the restriction $\gamma_i :]s_k^i, t_{k+1}^i[\to \vec{I}$ is non-decreasing. A dipath is an *execution* if $\gamma(0) = [0]$ and $\gamma(1) = [1]$, where $[0]$ is the equivalence class of $(0, \dots, 0) \in X$ in $\bar{X}$, and similarly for $[1]$.

An execution γ of $\mathcal{P} = T_1^*|T_2^*|\dots|T_n^*$ is a concurrent execution of n loops. Let $M = (m_1, m_2, \dots, m_n)$ where m_i is the number of times γ traverses T_i. Then γ is also an execution of the non-looped program $T_1^{m_1}|T_2^{m_2}\dots|T_n^{m_n}$ where $T^k = T.T.\dots.T$, is T repeated k times. Moreover, an execution of $T_1^{m_1}|T_2^{m_2}\dots|T_n^{m_n}$ may be considered as an execution of $\mathcal{P}$. This motivates the following definition.

Definition 2.9 (geometry of deloopings). With the above notation, for a vector

$$M = (m_1, \dots, m_n) \in \mathbb{N}^n,$$

the M delooping of $\mathcal{P}$ is

$$\mathcal{P}^M = T_1^{m_1}|T_2^{m_2}\dots|T_n^{m_n}$$

where $T^k = T.T.\dots.T$, is T repeated k times. The *state space X^M of the M delooping* is the cube $\vec{I}^M \setminus F^M$ where

$$F^M = \cup\{R^{i,K} | i \in [1 : l], K \in [0 : M - 1]\},$$

where

$$(M-1)_i = m_i - 1$$

and

$$R^{i,K} =]a^i + K, b^i + K[.$$

Executions are order-preserving paths $\gamma : \vec{I} \to X^M$ with $\gamma(0) = 0, \gamma(1) = M$.

We define the *projection* $\Pi_M : X^M \to \bar{X}$ by

$$\Pi_M(y_1, \dots, y_n) = [(\hat{y}_1, \dots, \hat{y}_n)],$$

where $\hat{y}_i \in [0, 1[$ is the fractional part of y_i and $[-]$ denotes the equivalence class in $\bar{X}$.

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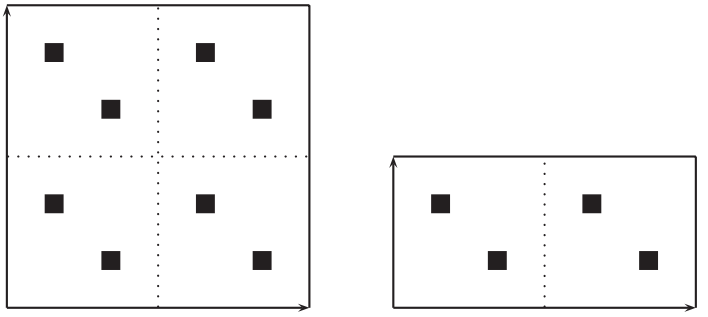


Fig. 3. The $(2,2)$ and $(2,1)$ deloopings of $(P_aV_aP_bV_b)^*|(P_bV_bP_aV_a)^*$

We define

$$X^{M,K} = X^M \cap [K, K+1].$$

Lemma 2.10. With the above notation, let $\phi_K : X \to X^M$ be $\phi(x) = x + K$. Then ϕ_K induces a di-homeomorphism $\phi_K : X \to X^{M,K}$

Proof. The map $\phi(y) = y - K$ is clearly a dimap and an inverse. □

Example 2.11. Let

$$\mathcal{P} = (P_aV_aP_bV_b)^*|(P_bV_bP_aV_a)^*.$$

Then X, the geometric model of

$$P_aV_aP_bV_b|P_bV_bP_aV_a,$$

is, as in Figure 1, a square with two off-diagonal holes. $\bar{X}$ is obtained by gluing the right- and left-hand sides of the square X, which gives a cylinder with two holes, and then gluing the upper and lower sides, the ends of the cylinder, to produce a torus.

The deloopings $X^{(2,2)}$ and $X^{(2,1)}$ are shown in Figure 3. They are 2×2 copies of X and 2×1 copies of X.

The 8 forbidden rectangles in $X^{(2,2)}$ are labelled (j, K), where $j \in \{1,2\}$ denotes one of the rectangles in X and $K \in \{(0,0),(1,0),(0,1),(1,1)\}$ indicates which of the 4 copies of X this rectangle belongs to.

Lemma 2.12. The projection $\Pi_M : X^M \to \bar{X}$ is continuous and maps execution paths to execution paths.

Proof. For each coordinate, this is the covering map $t \to e^{i2\pi t}$ from $\mathbb{R}$ to S^1, so Π_M is a restriction of the covering map from $\mathbb{R}^n$ to the torus. Hence, it is continuous. A directed path $\gamma : \vec{I} \to X^M$ is increasing in each coordinate. We have

$$(\Pi_M \circ \gamma)_j^{-1}(1) = \gamma_j^{-1}(\mathbb{N}).$$

On a connected component C of $\vec{I} \setminus \gamma_j^{-1}(\mathbb{N})$, the integer component of $\gamma_j(t)$ is constant. Hence, for $t_1, t_2 \in C$, we have $\gamma_j(t_1) \leqslant \gamma_j(t_2)$ if and only if this holds for the fractional part. □

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There is a bijection between executions of $\mathcal{P}$ and executions of *all* possible deloopings.

Proposition 2.13. The trace space $\vec{T}(\bar{X})([0],[0])$ is a disjoint union of trace spaces of the deloopings

$$\vec{T}(\bar{X})([0],[0]) = \bigsqcup_{M\in\mathbb{N}^n} \Pi_M(\vec{T}(X^M)(0,M)).$$

Proof. Let $\gamma : \vec{I} \to \bar{X}$, with $\gamma(0) = [0] = \gamma(1)$, and let $i : \bar{X} \to I^n/\sim$ be the inclusion. Now, $i\circ\gamma$ represents a class $[\gamma] \in \pi_1(I^n/\sim)$. Let $e_r \in I^n$ be the rth unit vector. The *directed loops* $\mu_r(t) = [te_r]$ are then generators of

$$\pi_1(I^n/\sim) = \oplus_{r=1}^n \mathbf{Z}.$$

With these generators, let

$$M = (m_1,\ldots,m_n) \in \oplus_{r=1}^n \mathbb{Z}$$

be the homotopy class represented by γ. Since γ is directed, all m_i are positive and there is a dipath $\tilde{\gamma} : \vec{I} \to X^M$ such that $\pi\circ\tilde{\gamma} = \gamma$. That is, M is the index of the corresponding delooping.

If $\mu : \vec{I} \to \bar{X}$ is dihomotopic to γ, then $i\circ\mu$ is homotopic to $i\circ\gamma$, and the delooping corresponding to μ is X^M. Hence, the trace spaces corresponding to different deloopings are disconnected. □

Remark 2.14. If X^M and $X^{M'}$ satisfy $m_i \leqslant m_i'$ for all i, then there is an obvious inclusion $X^M \subset X^{M'}$. This induces a partial order on the deloopings. The universal dicovering of $\bar{X}$ defined in Fajstrup (2003) is the colimit of all X^M over this partial order. Equivalently, it is the union of all X^M.

3. Traces and delooping

Raussen (2010) gives the trace spaces in the non-looped case as the geometric realisation of a prod-simplicial complex. This provides a description of $\vec{T}(X^M)(0,M)$ up to homotopy, and there is an algorithm for determining this prod-simplicial complex and calculating the topological invariants of the trace space. However, in the case of a delooping, there is an obvious periodicity. This may be exploited to simplify the algorithm in this case. See Raussen (2010) for a full definition of the prod-simplicial complex. Here, we will just focus on the index poset, which is used to build the prod-simplicial complex, and periodicity. As explained in the introduction, the index poset is a set of *schedules* and the partial order is the subset relation.

Examples of schedules were given in Examples 1.1 and 1.2. A schedule is a requirement for each 'hole', which represents a conflict at a resource, that a set of processes do not access the resource before at least one of the others has released it. If a schedule is too restrictive, there are no executions satisfying it – geometrically, there are no traces. If there are rectangles for which nothing is required, the corresponding space of traces is not contractible. We will now give the definition and the corresponding geometric model.

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Definition 3.1. A *schedule* for $X = I^n \setminus F$, with $F = \cup_{i=1}^{l} R^i$ is an l-tuple of sets $J = (J_1, \ldots, J_l)$, where $J_i \subset [1 : n]$ and $J_i \neq \varnothing$. That is, all rectangles are extended. The corresponding restricted space X_J is $X \setminus F_J$, where

$$F_J = \cup_{i=1}^{l} \cup_{\alpha \in J_i} R^i_\alpha$$

and

$$R^i_\alpha = [0, b^i_1[\times \ldots [0, b^i_{\alpha_{i-1}}[\times]a^i_\alpha, b^i_\alpha[\times [0, b^i_{\alpha+1}[\times \ldots \times [0, b^i_n[,$$

the rectangle R^i is extended to the lower boundary along all coordinates except the αth.

Figure 2 shows a schedule $(\{2,3\},\{2,3\},\{2,3\})$ where all three forbidden rectangles are extended by two rectangles: one in the 'not 2' directions and the other in the 'not 3' directions.

All executions obeying such a schedule are equivalent.

Proposition 3.2. $\vec{T}(X_J)(0,1)$ is either empty or contractible.

Proof. This is Raussen (2010, Proposition 2.8 (2)). The contractibility relies on $J_i \neq \varnothing$. □

Intuitively, too many restriction implies that there are no executions satisfying the schedule, and too few implies that there are 'too many' in a homotopy theoretic sense.

Definition 3.3. Let $X = \vec{I}^n \setminus F$, with F as above. The *index poset* $\mathcal{C}(X)$ is the set of allowed schedules for X. The partial order is $J \leqslant J'$ if $J_i \subset J'_i$ for all i.

A delooping X^M has $l \prod_{i=1}^{n} m_i$ forbidden n-rectangles. Hence, a schedule for X^M is an $l \prod_{i=1}^{n} m_i$-tuple of non-empty subsets of $[1 : n]$. We fix a bijection

$$f : [1 : l \prod_{i=1}^{n} m_i] \to \{(i,K) | i \in [1 : l], K \in [0, M-1]\}.$$

The forbidden rectangles may then be labelled by (i,K) as in Definition 2.9.

Definition 3.4. For a delooping X^M, an element in the index poset is a set

$$\{J_{i,K} | i \in [1 : l], K \in [0, M-1]\}$$

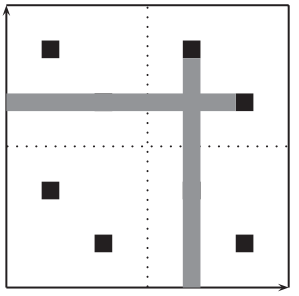
of *non-empty* subsets $J_{i,K} \subset [1 : n]$ for each of the forbidden cubes. The partial order is $J \leqslant J'$ if $J_{i,K} \subset J'_{i,K}$ for all i, K. Hence, $J \leqslant J'$ if $X^M_J \supseteq X^M_{J'}$.

Raussen (2010) gives a pasting scheme to build the trace space $\vec{T}(X)(0,1)$, and thus $\vec{T}(X^M)(0,M)$, as a prod-simplicial complex from the index poset. A schedule J is represented very efficiently as an $l \times \prod_{i=1}^{n} m_i$ binary matrix B_J, where $B_{st} = 1$ if $t \in J_{f(s)}$, where

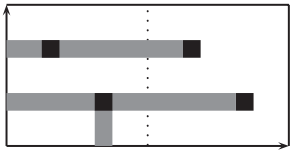
$$f : [1 : \prod_{i=1}^{n} m_i] \to \{(i,K) i \in [1 : l], K \in [0, M-1]\}$$

is the above bijection, and each row corresponds to a 'hole'. Moreover, by Section 4 in Raussen (2010), $\vec{T}(X^M_J)(0,M)$ is empty if and only if the extended rectangles intersect

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A deadlock is a global problem



Schedules $(\{2\},\{1\})$ and $(\{2\},\{2\})$ combined

Fig. 4. Two disallowed schedules for deloopings of Example 2.11

to give a *deadlock* as in Example 1.1 and Figure 1. A point $p \in X_J^M$ is a deadlock if all dipaths starting in p are constant. Hence, p is the infimum of the intersection of n extended rectangles. Since the rectangles producing the deadlock are extended to 0 in all directions except one, all points in the hyperrectangle $[0,p]$ are *unsafe* in the sense that no dipath (in fact, no path) starting there will leave the hyperrectangle and, in particular, they will not reach the final point M, so $\vec{T}(X^M)(0,M) = \varnothing$ – see Figure 4. For more on deadlocks and unsafe areas, see Raussen (2010) and Fajstrup *et al.* (2006).

For large M, the number of rectangles $l\prod_{i=1}^{n} m_i$ makes this unfeasible. In the next section, we will explore the obvious periodicity in a delooping.

4. Periodicity in the trace algorithm

A schedule J for a delooping X^M is a set $J_{(i,K)}$ for each rectangle $R^{(i,K)}$. For fixed K, J is a schedule for R^i, $i \in [1:l]$. Hence, J consists of schedules for the $\prod_{i=1}^{n} m_i$ copies, $X^{M,K}$, of X. For J to be an allowed schedule for X^M, all these corresponding schedules for X must be allowed: in Figure 4, the deadlock produced by the disallowed schedule for X prohibits execution of $X^{(2,2)}$ regardless of how the schedule behaves at the other rectangles. However, this is not a sufficient condition. In Figure 4, two allowed schedules are combined and give a disallowed schedule for $X^{(2,1)}$.

Example 4.1. In Example 2.11, there are three allowed schedules for X, as in Example 1.1. Now, Figure 4 shows a schedule for $X^{(2,2)}$ where the schedule for rectangles $(i,(1,1))$ is the disallowed schedule $(\{1\},\{2\})$ for X. The deadlock prohibits execution of $X^{(2,2)}$ since the rectangles are extended to 0 in the global setting. Hence, an allowed schedule for a delooping is built from allowed schedules for X, but Figure 4 shows that not all such combinations give allowed schedules for deloopings. Combining the schedules $(\{2\},\{1\})$ and $(\{2\},\{2\})$ produces a deadlock since the extensions of rectangles $(2,(1,0))$ and $(2,(0,0))$ intersect and give a deadlock.

Definition 4.2. Given a schedule

$$J = \{J_{i,K} | i \in [1:l], K \in [0, M-1]\}$$

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for a delooping X^M and $K_0 \in [0 : M-1]$, the restriction $J|_{K_0}$ is the schedule

$$(J_{1,K} \ldots, J_{l,K})$$

for X.

Remark 4.3. In $J|_{K_0}$, the rectangle $R^i \subset X$ is extended according to the schedule J_{i,K_0} of R^{i,K_0}, $i = 1, \ldots, l$. This is the schedule for rectangles in

$$X^{M,K_0} = [K_0, K_0 + 1].$$

However,

$$X_{J|_{K_0}} \neq X_J^M \cap X^{M,K_0}$$

since in the latter, the rectangles $R^{i,K}$ with $K > K_0$ may be extended to intersect X^{K_0}.

Proposition 4.4. If

$$\overrightarrow{T}(X_{J1})(0,1) = \varnothing,$$

and J is a schedule for X^M such that $J|_K = J1$ for some $K \in [0 : M-1]$, then

$$\vec{T}(X_J^M)(0, M) = \varnothing.$$

Proof. Since $\vec{T}(X_{J1})(0,1) = \varnothing$, there is a deadlock $p \in X_{J1}$ that is the intersection of (at least) n extended rectangles $R^r_{J1_r}$. Inserting X_{J1} in X^M, the schedule requirement causes the rectangle R^r to be extended to a product of intervals $]a^r_{J1_r}, b^r_{J1_r}[$ and $[0, b^r_i]$ for $i \neq J1_r$, in X^M. Thus, $[0, b^r_i[$ is the extension to the lower ith boundary in X^M. The unsafe region of p will then contain the minimum, that is, 0 of X^M □

A schedule J for X^M is given by schedules J_K for X for all $K \in [1 : M]$, and Proposition 4.4 gives a necessary condition for when J is allowed. However, it is not a sufficient condition – see Examples 4.5 and 4.6.

We will now consider Example 1.2 as the $(3,1,1)$ delooping of $P_aV_a|P_aV_a|P_aV_a$.

Example 4.5. Let $X = \vec{I}^3 \setminus R$ where

$$R =]1/4, 3/4[\times]1/4, 3/4[\times]1/4, 3/4[.$$

Consider the three schedules

$$J = \{1\}$$
$$J = \{2\}$$
$$J = \{3\}.$$

These are all allowed since there are no deadlocks. In the following, we consider combinations of these schedules.

Let $M = (3,1,1)$. Then

$$X^M = [0,3] \times [0,1] \times [0,1] \setminus (R^1 \cup R^2 \cup R^3),$$

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where

$$R^1 = R$$
$$R^2 =]5/4, 7/4[\times]1/4, 3/4[\times]1/4, 3/4[$$
$$R^3 =]9/4, 11/4[\times]1/4, 3/4[\times]1/4, 3/4[.$$

The three schedules may be combined to give 27 schedules for X^M, namely, three tuples of one-element sets $v = (v_1, v_2, v_3)$, which assign v_1 to R^1, v_2 to R^2 and v_3 to R^3. There is no deadlock if $v_i = v_j$ and $j \neq i$, since the corresponding extended rectangles intersect trivially – one is contained in the other.

The 6 remaining schedules are

$$(\{1\}, \{2\}, \{3\})$$
$$(\{1\}, \{3\}, \{2\})$$
$$(\{2\}, \{1\}, \{3\})$$
$$(\{2\}, \{3\}, \{1\})$$
$$(\{3\}, \{1\}, \{2\})$$
$$(\{3\}, \{2\}, \{1\}).$$

For these:

— $(\{1\}, \{2\}, \{3\})$ and $(\{1\}, \{3\}, \{2\})$ give a deadlock at $(1/4, 1/4, 1/4)$. We have

$$R_1^1 =]1/4, 3/4[\times[0, 3/4[\times[0, 3/4[$$
$$R_2^2 = [0, 7/4[\times]1/4, 3/4[\times]0, 3/4[$$
$$R_3^3 = [0, 11/4[\times[0, 3/4[\times]1/4, 3/4[.$$

Hence,

$$(1/4, 1/4, 1/4) \in X^M_{(1,2,3)}$$

and it is a deadlock. Moreover, $(0, 0, 0)$ is unsafe, as can be seen by inspection in this case, or, in general by the deadlock algorithm.

The analysis for $(1, 3, 2)$ is similar.

— $(\{2\}, \{1\}, \{3\})$, $(\{2\}, \{3\}, \{1\})$, $(\{3\}, \{1\}, \{2\})$ and $(\{3\}, \{2\}, \{1\})$ do not produce any deadlocks.

Consider v=($\{2\}$,$\{1\}$,$\{3\}$). Then

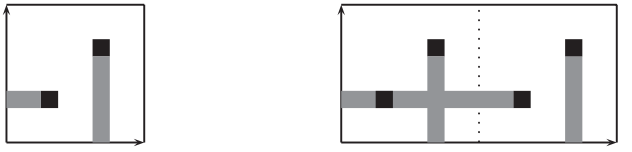
$$R_2^1 = [0, 3/4[\times]1/4, 3/4[\times[0, 3/4[$$
$$R_1^2 =]5/4, 7/4[\times[0, 3/4[\times[0, 3/4[$$
$$R_3^3 = [0, 11/4[\times[0, 3/4[\times]1/4, 3/4[.$$

Since $R_2^1 \cap R_1^2 = \varnothing$, there is no deadlock.

The other cases are similar.

In this example, allowed schedules on X are combined to give a schedule on X^M. Rectangles are extended in the global setting X^M producing deadlocks 'below'. This

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An allowed schedule Repeating the schedule

Fig. 5. Not all schedules can be repeated.

explains why the condition in Proposition 4.4 is not sufficient. Note also the non-commutativity in which $(\{1\},\{2\},\{3\})$ gives a deadlock, but $(\{2\},\{1\},\{3\})$ does not.

An allowed schedule may not even be repeatable.

Example 4.6. Let $X = \vec{I}^2 \setminus F$, where $F = R^1 \cup R^2$ and

$$R^1 =]1/8, 3/8[\times]1/8, 3/8[$$
$$R^2 =]5/8, 7/8[\times]5/8, 7/8[,$$

are two diagonal holes. The schedule $J = (\{2\},\{1\})$ extends the forbidden area to

$$R^1_2 = [0, 3/8[\times]1/8, 3/8[$$
$$R^2_1 =]5/8, 7/8[\times[0, 7/8[.$$

This is an allowed schedule – see Figure 5. However, the concatenated schedule

$$(\{2\},\{1\},\{2\},\{1\})$$

for the 4 forbidden rectangles in $[0,2]\times[0,1]$ has a deadlock at $(5/8, 1/8)$.

The extension of a rectangle $R^{(i,K)} \subset X^{M,K}$ intersects $X^{M,K'}$ for some $K' \leqslant K$, but not all. In Figure 4, the extended rectangles intersect either $X^{(1,0)}$ or $X^{(0,1)}$, but not $X^{(0,0)}$.

A schedule for X^M is allowed if the extended rectangles do not intersect to give a deadlock. That is, if there are no deadlocks in $X^{M,K}$ for all $K \leqslant M$. To make the structure of extensions precise, we will introduce the *downward effect*. It will be sufficient to study the effect on the forbidden area in $X^{M,1} = [0,1]^n$ resulting from extending rectangles in $X^{M,K}$.

Example 4.7. Consider the delooping $X^{(3,2)}$ of Example 2.11. There are 6 copies of X. The schedule $\{2\}$ for $R^{(1,(2,0))}$ extends the rectangle to

$$[0, b^1_1 + 2[\times]a^1_2, b^1_2[.$$

The intersection with $[0,1]^2$ is

$$[0,1]\times]a^1_2, b^1_2[.$$

The effect of the schedule $\{2\}$ for $R^{(1,(1,0))}$ on $[0,1]^2$ is the same. The schedule $\{2\}$ on $R^{(i,(k_1,k_2))}$ where $k_2 \neq 0$ has no effect on $[0,1]^2$.

In general, the effect on $[0,1]^n$ of extending $R^{(i,K)}$ depends not on K, but on $r(K) \in \{0,1\}^n$, with $r(K)_i = min(K_i, 1)$ – this is Lemma 4.13.

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For a set F of n-rectangles as above and a schedule J for F, the resulting forbidden area $G(J,r)$ in a direction $r \in \{0,1\}^n \setminus \{\mathbf{0},\mathbf{1}\}$ is the intersection of the extended rectangles with $[0,1]^n$ when this schedule is applied for rectangles $R^{i,r}$, that is, at X^r.

For a subset Y of I^n, the *downward effect* on Y is

$$D_r(F,J)(S) = S \setminus G(J,r).$$

Example 4.8. For Example 4.1,

$$G((\{2\},\{2\}),(1,0)) = [0,1]\times]a_2^1, b_2^1[\cup[0,1]\times]a_2^2, b_2^2[,$$

but

$$G((\{2\},\{2\}),(0,1)) = \varnothing$$

since rectangles in $X^{(0,1)}$ extended to the left (direction not 2) do not intesect $[0,1]\times[0,1]$.

In Example 4.5,

$$\begin{aligned} G((\{2\}),(1,0,0)) &= [0,1]\times]1/4,3/4[\times[0,3/4[\\ G((\{1\}),(1,0,0) &= \varnothing \\ G((\{2\}),(1,0,1)) &= [0,1]\times]1/4,3/4[\times[0,1]. \end{aligned}$$

Definition 4.9 (notation). For subsets $K, M \in \{1,\ldots,n\}$ and an n-rectangle

$$R = \times_{j=1}^n]a_j, b_j[,$$

let

$$\rho_M^K(R) = \times_{j=1}^n I_j$$

where

$$\begin{aligned} I_j &=]a_j, b_j[&& \text{if } j \notin K \cup M \\ I_j &= [0, b_j[&& \text{if } j \in M \text{ and } j \notin K \\ I_j &=]a_j, 1] && \text{if } j \in K \text{ and } j \notin M \\ I_j &= [0,1] && \text{if } j \in M \cap K \end{aligned}$$

Example 4.10. When $Ji = \{j_i\}$, the ith rectangle R^i is extended to $\rho_{\{j\neq j_i\}}(R^i)$.

Definition 4.11 (downward effect). Let S be the set of subsets of I^n and χ be the set of pairs (F,J), where

$$F = \cup_{i=1}^l R^i$$

with R^i an n-rectangle, and $J = (J1,\ldots,Jl)$ is a schedule. Let $r \in \{0,1\}^n \setminus \{\mathbf{0},\mathbf{1}\}$. The *resulting forbidden area* is

$$G(J,r) = \bigcup_{i=1}^{l} \bigcup_{\{j_i \in J_i | r_i = 0\}} \rho_{\{j|j\neq j_i\}}^{\{k|r_k=1\}}(R^i).$$

The downward effect $D_r : \chi \to map(S,S)$ is defined as follows.

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Let $Y \subset I^n$, $J = (J_1, \ldots, J_l)$ and F be as above. Then

$$D_r(F, J)(Y) = Y \setminus G(J, r).$$

If $X = I^n \setminus F$, with F a union of rectangles and J a schedule, we write $D_r(X_J)$ to denote $D_r(F, J)$.

$D_r(X_J)(Y)$ is the *downward effect from direction r* of X_J on Y.

For a string $X_{J1}X_{J2}\cdots X_{Jk}$, the *combined downward effect* is the composition

$$D_r(X_{J1}X_{J2}\cdots X_{Jk})(Y) = D_r(X_{J1}) \circ D_r(X_{J2}) \circ \cdots \circ D_r(X_{Jk})(Y).$$

The combined down effect is the effect of a union of schedules.

Proposition 4.12. The combined down effect of $X_{J1} \ldots X_{Js}$ is the effect of the union of schedules:

$$D_r(X_{J1}X_{J2} \ldots X_{Js}) = D_r(X_{J1 \vee J2 \ldots \vee Js})$$

where

$$(J_1, \ldots, J_l) \vee (J'_1, \ldots, J'_l) = (J_1 \cup J'_1, \ldots, J_l \cup J'_l).$$

Proof. It suffices to show that

$$F(J, r) \cup F(J', r) = F(J \cup J', r).$$

But this holds because

$$\{j_i \in J_i | r_i = 0\} \cup \{j_i \in J'_i | r_i = 0\} = \{j_i \in J_i \cup J'_i | r_i = 0\},$$

and these determine the extensions of the ith rectangle. □

Lemma 4.13. Let X^M be the M delooping and J be a schedule. Suppose $J_{i,K} = \{\alpha\}$ is a schedule for $R^{i,K}$. Then

$$R^{i,K}_\alpha \cap [0,1]^n = \rho^{\{k | r(K)_j = 1\}}_{\{j | j \neq \alpha\}}(R^i)$$

if $k_\alpha = 0$, and is empty if $k_\alpha \neq 0$. Here

$$r(K)_j = \min\{K_j, 1\}.$$

Proof. The extended rectangle is

$$R^{i,K}_\alpha = \times^n_{j=1} S_j$$

where $S_\alpha =]a^i_\alpha + k_\alpha, b^i_\alpha + k_\alpha[$ and $S_j = [0, b^i_\alpha + k_\alpha[$ otherwise. The intersection is

$$R^{i,K}_\alpha \cap I^n = \times^n_{j=1}(S_j \cap [0,1]).$$

If $k_\alpha \neq 0$, then $S_\alpha \cap [0,1] = \varnothing$, so the intersection is empty.

If $k_\alpha = 0$, we have:

$$\begin{array}{lll} S_\alpha \cap [0,1] = S_\alpha & & \\ S_j \cap [0,1] = [0, b^i_j[& \text{if } k_j = 0 & \text{(that is, if } r(K)_j = 0 \text{ (and } j \neq \alpha) \\ S_j \cap [0,1] = [0,1] & \text{if } k_j \geqslant 1 & \text{(that is, if } r(K)_j = 1). \end{array}$$

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Now

$$\rho^{\{k|r(K)_j=1\}}_{\{j|j\neq\alpha\}}(R^i) = \times^n_{j=1} I_j$$

where:

— $I_j =]a_j, b_j[$ if

$$j \notin \{j|j \neq \alpha\} \cup \{k|r(K)_k = 1\} = \{j|j \neq \alpha\}$$

since $k_\alpha = 0$ implies $r(K)_\alpha = 0$. That is, $j = \alpha$.

— $I_j = [0, b_j[$ if

$$j \in \{j|j \neq \alpha\}$$

and

$$j \notin \{k|r(K)_j = 1\}.$$

— $I_j = [0, 1]$ if

$$j \in \{j|j \neq \alpha\} \cap \{k|r(K)_k = 1\},$$

that is, if $r(K)_j = 1$.

This completes the proof. □

Corollary 4.14. For a schedule J for a delooping X^M, the effect on $[0,1]^n \subset X^M$ of extensions of rectangles in $X^{M,K}$ according to the schedules is

$$I^n \setminus \cup^l_{i=1} \cup_{\alpha\in J_{i,K}} R^{i,K}_\alpha = D_{r(K)}(F, J_{i,K})(I^n).$$

Definition 4.15. For a delooping X^M, $M = (m_1, \ldots, m_n)$ and $K \in [0 : M-1]$, we write X_{JK} to denote X with extensions according to the restriction $J|_K$. The *combined down effect* on X_{JK_0} is the composition

$$D(X^M_J, K_0)(X_{JK_0}) = \circ_{K\in]K_0,M]} D_{r(K-K_0)}(X_{JK})(X_{JK_0}).$$

Here $r(K - K_0)$ is as in Lemma 4.13.

Theorem 4.16. Let X^M be the delooping given by $M = (m_1, \ldots, m_n)$. A schedule J for X^M is allowed if and only if for all $K \in [0 : M-1]$, we have:

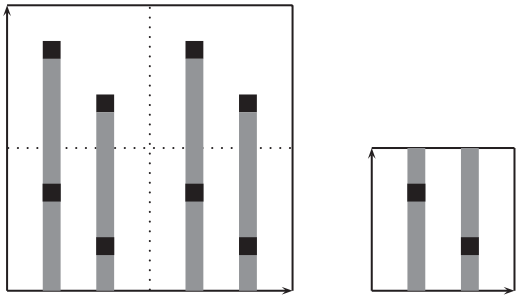
— If p is a deadlock in $D(X^M_J, K)(X_{JK})$, then $p_k = 1$ for some k.

Proof. J is allowed if and only if there are no deadlocks on intersections of extended rectangles. Such deadlocks have coordinates $(a^{i_1}_1 + k_1, \ldots, a^{i_n}_n + k_n)$ where $k_i \in \mathbb{N}$ and $a_j \in]0,1[$. Hence, if p is a deadlock in X^M_J and $p \in X^M_J \cap X^{M,K}$, then it is in the interior of $X^{M,K}$.

By the same reasoning as in Lemma 4.13, extended rectangles $R^{i,K}_j$ will intersect X^{M,K_0} only if $(K - K_0)_j = 0$ and the intersection with X^{M,K_0} is $G(JK, r(K - K_0))$. Therefore, p corresponds to an interior deadlock in $D(X^M_J, K_0)(X_{JK_0})$.

If q is a deadlock in $D(X^M_J, K_0)(X_{JK_0})$, either q is on the intersection of $n-k$ extended rectangles $D_{r(K-K_0)}(R^i_j)(X_{K_0})$ and k upper boundaries, that is, $q_j = 1$ for $j = j_1, \ldots, j_k$. If $q_j = 1$, the corresponding point $\tilde{q} = q + K$ is not a deadlock in X^M, since it has an integer coordinate. □

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Schedule J, $J_{i,K} = \{1\}$ for all i, K The effect on $[0,1]^2$

Fig. 6. An allowed schedule for the $(2,2)$ delooping of Example 2.11

Remark 4.17. In particular, as noted earlier, if X_J is allowed, all X_{JK} are allowed.

Corollary 4.18. A schedule J for X^M is allowed if and only if for all $K \in [0 : M-1]$ there is a k such that

$$\vec{T}(D(X_J^M, K)(X_{JK}))(0, \partial_k^+) \neq \varnothing,$$

where

$$\partial_k^+ = \{x \in D(X_J^M, K)(X_{JK}) | x_k = 1\}.$$

Proof. If there is a dipath to ∂_k^+, then 0 is not in the unsafe area for an interior deadlock, so there are no such deadlocks. If there is an interior deadlock, then 0 is in the associated unsafe area and

$$\vec{T}(D(X_J^M, K)(X_{JK}))(0, \partial_k^+) = \varnothing$$

for all k. □

Example 4.19. Figure 6 shows an allowed schedule J for the $(2,2)$ delooping of Example 2.11. The schedule for each rectangle is $\{1\}$. There is an execution obeying this schedule – for example, serial execution of T_2 twice followed by serial execution of T_1 twice. There is a deadlock in $X^{(2,2)} \cap [0,1]^2$ on the upper boundary. And there is no directed path to the final point $(1,1)$, but there is a directed path to the upper boundary, for example, the restriction of the allowed execution path of $X_J^{(2,2)}$, so the deadlock is not in the interior. This illustrates why Theorem 4.16 only excludes interior deadlocks.

5. An alphabet of building blocks

As we have shown, the allowed schedules for X are the fundamental building blocks for the schedules for deloopings X^M. When $M = (1, 1, \ldots, 1, m, 1, \ldots, 1)$ is delooping in a fixed direction, the allowed schedules for X^M are words in a regular language over the allowed schedules on X.

Example 5.1. Let $r \in \{1, \ldots, n\}$. There is a *concatenation* of X_{J1} and X_{J2} along the rth direction. The subset $X_{J1} \star_r X_{J2} \subset I^v$, where $v \in \mathbb{N}^n$, $v_r = 2$ and $v_j = 1$ otherwise, is

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defined by

$$x = (x_1, \ldots, x_n) \in X_{J1} \star_r X_{J2}$$

if either

$$x_r \geqslant 1 \text{ and } x - e_r \in X_{J2}$$

or

$$x_r \leqslant 1 \text{ and } x \in D_r(X_{J2})(X_{J1}).$$

Iteratively,

$$X_{J1} \star_r X_{J2} \star_r \cdots \star_r X_{Jk} \subset I^M,$$

$M_r = k$ and $M_j = 1$ otherwise. It is defined by

$$x \in X_{J1} \star_r X_{J2} \star_r \cdots \star_r X_{Jk}$$

if

$$x_r \geqslant 1 \text{ and } x - e_r \in X_{J2} \star_r X_{J3} \star_r \cdots \star_r X_{Jk}$$

or

$$x_r \leqslant 1 \text{ and } x \in D_r(X_{J2} \star_r X_{J3} \star_r \cdots \star_r X_{Jk})(X_{J1}).$$

Remark 5.2. The concatenated object $X_{J1} \star_r X_{J2}$ may also be obtained by gluing X to X along the rth face and enumerating the now $2l$ forbidden rectangles consecutively – $R^1, \ldots, R^l$, as before, in the lower copy of X (with $0 \leqslant x_r \leqslant 1$), and R^{l+m}, the copy of R^m, in the upper copy of X (where $1 \leqslant x_r \leqslant 2$). Then $X_{J1} \star_r X_{J2}$ is obtained by extending rectangles according to the schedule

$$J = (J1_1, \ldots, J1_l, J2_1 \ldots, J2_l).$$

Remark 5.3 (notation). In the following, we assume J is a schedule for the rectangles R^i, $i = 1, \ldots, l$ and X_J is the corresponding subset of X. In concatenations $X_{J1} \star_r \ldots \star_r X_{Js}$ and in X^M, we will write 0 to denote the minimal point and f to denote the maximal point.

Definition 5.4. Let

$$\mathcal{A} = \{X_J | \vec{T}(X_j)(0, f) \neq \varnothing\}.$$

Then

$$\mathcal{A}_{r,m} = \{w = a_1 \star_r a_2 \star_r \ldots \star_r a_m | a_i \in \mathcal{A}_1 \text{ and } \vec{T}(w)(0, f) \neq \varnothing\}$$

is the set of allowed r-words of length m.

Let $w = w_1 w_2 \ldots w_m$ be a word on an alphabet $\mathcal{A}$. Then

$$Alph(w) = \{w_i, i = 1, \ldots, m\}$$

are the letters used in w.

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Lemma 5.5 (properties of the operators D_r). Let

$$w \in \mathcal{A}_{r,m}$$
$$z \in \mathcal{A}_{s,m}$$
$$w = w_1 w_2 \dots w_k$$
$$c \in \mathcal{A}.$$

Then:

(1) $D_r(w)(c) = D_r(\vee_{i=1}^{k} w_i)(c)$, where $\vee$ is as in Proposition 4.12.
(2) D_r is commutative in the sense that

$$D_r(w) \circ D_s(z) = D_s(z) \circ D_r(w).$$

(3) D_r is idempotent:

$$D_r(w) \circ D_r(w) = D_r(w).$$

Proof. Parts (1) and (3) follow from Proposition 4.12. The statement in part (2) is just

$$(C \setminus A) \setminus B = (C \setminus B) \setminus A$$

for sets, A, B, C. □

Lemma 5.6. Let $w \in \mathcal{A}_{r,m}$ and $a \in \mathcal{A}$. Then $a \star_r w \in \mathcal{A}_{r,m+1}$ if and only if all deadlocks in $D_r(w)(a)$ are on the upper boundary. Hence:

(1) If $w \in \mathcal{A}_{r,m}$ and $\hat{w}$ is a subsequence of the letters in w, then $\hat{w}$ is allowed.
(2) If $w \star_r w$ is allowed, then all cyclic permutations of w are allowed.
(3) If $w \star_r w$ is allowed, then $w^{r,n} = w \star_r w \star_r w \star_r \dots \star_r w$, the concatenation of n copies of w, is allowed for all n.

Proof. $a \star_r w$ is allowed if $\vec{T}(a \star_r w)(0, f) \neq \varnothing$. The geometric object representing $a \star_r w$ is the subset of I^M, with $M_r = m + 1$ and $M_i = 1$ otherwise, given by

$$x = (x_1, \dots, x_n) \in a \star_r w$$

if $x_r \geqslant 1$ and $x - e_r \in w$, or $x_r \leqslant 1$ and $x \in D_r(w)(a)$.

Hence, since w is allowed and thus has no deadlocks, it follows from Theorem 4.16 that

$$\vec{T}(a \star_r w)(0, f) \neq \varnothing$$

if and only if $D_r(w)(a)$ has no interior deadlocks. Parts (1), (2) and (3) are then consequences of Proposition 4.12 and the Lemma above.

□

Corollary 5.7. $\mathcal{A}_{r+1,m}$ is the set of $a \star_r w$ for which $(a, w) \in \mathcal{A}_1 \times \mathcal{A}_{r,m}$ and

$$D_r(\vee_{b \in alph(w)} b)(a)$$

has no interior deadlocks.

Hence, since $\mathcal{A}$ is finite, the problem is now finite. The following theorem is a consequence of finiteness.

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Theorem 5.8. Let $\mathcal{A}$ be the alphabet of allowed schedules for X. The language $L_r \subset \mathcal{A}^*$, $r \in [1:n]$ consists of words $w = a_1 a_2 \cdots a_m$ such that $a_1 \star_r a_2 \star_r \cdots \star_r a_m$ is allowed. Then L_r is a regular language, which is recognised by an automaton with at most

$$(2^n - 1)^l + 1$$

states.

Proof. The theorem follows from the Myhill–Nerode theorem. Let $L = L_r$ for a fixed r. For $w_1, w_2 \in \mathcal{A}^*$, let $w_1 \sim w_2$ if for all $u \in \mathcal{A}^*$ we have

$$uw_1 \in L \Leftrightarrow uw_2 \in L.$$

The Myhill–Nerode Theorem (Hopcroft and Ullman 1979, Chapter 3)) says that L is a regular language if and only if $|\mathcal{A}^*/\sim|$ is finite. And in that case, there is an automaton with $|\mathcal{A}^*/\sim|$ states that recognises it.

The number of equivalence classes is finite:

— If $w \in \mathcal{A}^* \setminus L$ and $u \in \mathcal{A}^*$, then $uw \in \mathcal{A}^* \setminus L$, so $\mathcal{A}^* \setminus L$ is an equivalence class.
— Let $w \in L$. Then $uw \in L$ if and only if $D_r(\vee_{a \in Alph(w)} a)(u) \in L$. Hence, if

$$\vee\{Alph(w_1)\} = \vee\{Alph(w_2)\},$$

we have $w_1 \sim w_2$.

For all w, $\vee\{Alph(w)\} = (J_1, \ldots, J_l)$, where $\varnothing \neq J_i \subset [1:n]$.

Hence $|\mathcal{A}^*/\sim| \leqslant (2^n - 1)^l + 1$. □

Definition 5.9. An $l \times n$ binary matrix B represents the schedule $J_B = (J1, \ldots, Jl)$ where $s \in Jk \subset [0:n]$ if and only if $B_{ks} = 1$.

Corollary 5.10. The automaton generating L has:

— States: All $l \times n$ binary matrices B for which there is a word $w \in L$ with

$$\vee\{a \in Alph(w)\} = J_B.$$

— Transitions: $a : B \to B \vee a$ if $D_r(B)(a)$ has no interior deadlocks.

Remark 5.11. The automaton may be generated as follows:

— The initial state is $\{\varepsilon\}$.
— Add $a : \{\varepsilon\} \to \{a\}$ the transition a and state $\{a\}$ for all $a \in \mathcal{A}$.
— For all $b \in \mathcal{A}$ and all states B, add a transition (and a state, if $B \vee b$ is not already there) $b : B \to B \vee b$ if $D_r(B)(b)$ has no interior deadlocks.

We now iterate – the iteration will stop since $\mathcal{A}$ is finite.

Proposition 5.12. Let $a_1, a_2 \in \mathcal{A}$, $a_i = J^i$, with $J^i = (J1^i, J2^i, \ldots, Jl^i)$ where $Jk^i \subset [1:n]$ for $k \in [1:l]$. The partial order on $C(X)$ is $a_1 \leqslant a_2$ if $Jk^1 \subset Jk^2$ for all $k \in [1:l]$. Let

$$m = (1, \ldots, 1) + (m-1)e_r.$$

The elements of $C(X^m)$ for the mth delooping are r-words of length m in the regular language L_r over $\mathcal{A}$. The partial order on words is $w_i = a_1^i a_2^i \ldots a_m^i$, $w_1 \leqslant w_2$ if $a_j^1 \leqslant a_j^2$ for all j.

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Proof. The elements of the poset are allowed schedules. Words in the language L_r are the allowed schedules. The partial order is the set inclusion order on restrictions corresponding to individual rectangles. □

Example 5.13. For the 3-cube in Example 4.5, the allowed schedules for X are

$$a = \{1\}$$
$$b = \{2\}$$
$$c = \{3\}$$
$$d = \{1, 2\}$$
$$e = \{1, 3\}$$
$$f = \{2, 3\}.$$

So $\mathcal{A} = \{a, b, c, d, e, f\}$. Consider the deloopings $M = (m, 0, 0)$. The possible states are the 8 binary 1×3 matrices, where (000) is the initial state. They are all allowed states: for example, the word *bca* is allowed, which thus allows the matrix (111). However, not all transitions are allowed. The transition $a : (011) \to (111)$ is not in the automaton, since the words *abc* and *acb* are not allowed.

Example 5.14. For the square with two diagonal holes (see Example 4.6), the allowed schedules are

$$a = (\{2\}\{2\})$$
$$b = (\{2\}\{1\})$$
$$c = (\{1\}\{1\})$$
$$d = (\{1\}\{2\}).$$

So this is the alphabet. Given as 2×2 matrices.

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \; b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; c = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \; d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$Alph(ab) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \; Alph(ac) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$Alph(dc) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \; Alph(bc) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \; Alph(ad) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

All possible states are allowed, but not all transactions: for example, $b : \{b\} \to \{b\}$ is not allowed, as we saw earlier.

Example 5.15. For Example 2.11, there are three allowed schedules:

$$a = (\{2\}, \{2\})$$
$$b = (\{2\}\{1\})$$
$$c = (\{1\}\{1\})$$

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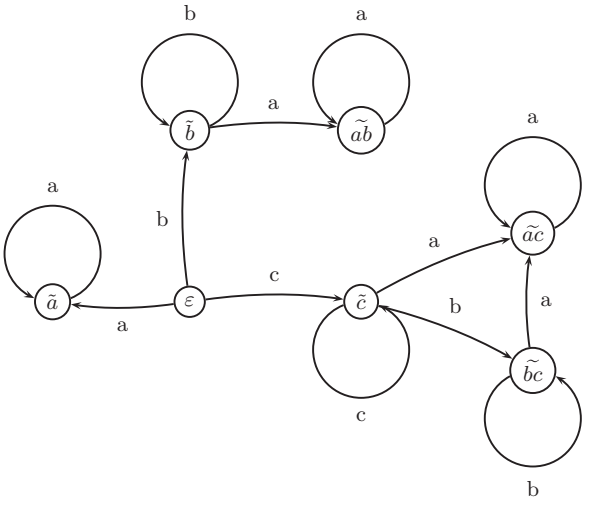


Fig. 7. The automaton for Example 2.11, deloopings $(r, 0)$

(see Figure 1). The 2×2 matrices, are as in Example 5.14. The automaton for deloopings $M = (r, 0)$ has three other states.

$$Alph(ac) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$Alph(ab) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$Alph(bc) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(see Figure 7, where $\widetilde{xy}$ denotes $Alph(xy)$). For any allowed word w, we have aw is allowed, so all states have an outgoing arrow (which may be a loop) labelled a. No word baw or caw is allowed. The alphabet for abc is the same as for ac.

For deloopings in general, $M = (m_1, \ldots, m_n)$, Theorem 4.16 implies that all the n automata generating L_r for $r \in [1 : n]$ should be combined. However, this will not be developed further here.

There are algorithmic gains in the following observations:

— If A, A' are states, $A' \leqslant A$ (as matrices) and $a : A \to A \vee a$ is allowed, then $a : A' \to A' \vee a$ is allowed since $D_r(A)(a) \subset D_r(A')(a)$, so if there is a dipath to an upper face in $D_r(A)(a)$, it is a dipath in $D_r(A')(a)$.
— The possible deadlocks are intersections of extended rectangles. Finding them is a check of inequalities as in Raussen (2010). Some of these inequalities are trivial – for intervals $[0, 1]$.

Remark 5.16. There is a dual of the index poset $\mathcal{C}(X)$ where rectangles are extended upwards. This corresponds to:

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(1) Invert the order on $\vec{I}$ and hence on X to get X^*.
(2) Calculate $\vec{T}(X^*)(1,0)$ as in Raussen (2010).

The resulting automaton will generate words iteratively $w \to wa$, that is, the letter a is a suffix, not a prefix.

6. Conclusions and further work

The periodicity implies that the index poset $\mathcal{C}(X^M)$ is described in terms of deadlocks in X with an enlarged forbidden area of extended rectangles. For a delooping in one direction, the index poset is given as words in a regular language, and the automaton is given.

For general deloopings X^M, the allowed schedules and the corresponding index proset $\mathcal{C}(X^M)$ involves checking for deadlocks in $D_r(X_K)(X_J)$ $r \in \{0,1\}^n$, that is, carry out a check of inequalites, some of which are trivial. This is not developed here. Moreover, for general processes, which are not pure loops, deadlocks in the deloopings appear in a combination of down effects from rectangles after the loops and rectangles within the loop. The down effects from rectangles after the loops give rise to forbidden areas penetrating the whole delooping, and hence there is still a periodicity, which may be used to generate better algorithms. Fajstrup *et al.* (2012) gives an automaton based on some of the ideas here for generating schedules, and hence the trace spaces, for X^M.

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