which can be reduced beautifully to (3) and (2) by the substitutions $t = x^m$ and $u = x^n$ respectively. And in a similar way we obtain

$$\int_0^{+\infty} \frac{\log^{2k-2} x \log(1+x^m)}{x(1+x^n)}\,dx = (-1)^{k-1}\frac{m(2^{2k-1}-1)}{2k(2k-1)}\left(\frac{1}{m^{2k}}+\frac{2k-1}{n^{2k}}\right)\mathrm{B}_{2k}\pi^{2k} \quad (5)$$

(k, m, n positive integers).

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PAUL LEVRIE
Faculty of Applied Engineering, University of Antwerp, Groenenborgerlaan 171, 2020 Antwerpen;
Department of Computer Science, KU Leuven, P.O. Box 2402, 3001 Heverlee, Belgium
e-mail: *paul.levrie@cs.kuleuven.be*
AMRIK SINGH NIMBRAN
B3-304, Palm Grove Heights, Ardee City, Sector 52, Gurugram, Haryana, India 122003
e-mail: *amrikn622@gmail.com*

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106.22 The golden ratio represented by a tangent

It is known that the golden ratio $\Phi = (\sqrt{5}+1)/2$ can be represented by using the trigonometric functions sin and cos as follows:

$$\Phi = 2\sin\frac{3\pi}{10} = 2\cos\frac{\pi}{5},$$

$$\Phi = \left[2\sin\frac{\pi}{10}\right]^{-1} = \left[2\cos\frac{2\pi}{5}\right]^{-1}$$

(see, e.g. [1]). The proofs are based on Ptolemy's theorem and the consideration of two triangles consisting of sides and diagonals of a regular pentagon.

To the best of our knowledge there appears to be no known result for tangent representation of the golden ratio Φ as above. Our purpose is to investigate the relationship between the golden ratio Φ and the trigonometric function tan, and to represent Φ by tan in two ways. We use the notation $\varphi = (\sqrt{5}-1)/2$.

We mention the paper [2] by MacHale which shows that $\sin\theta = \varphi = 1/\Phi$ if $\tan\theta = \cos\theta$. In Glaister [3] the following theorem was established by using the techniques of elementary geometry.

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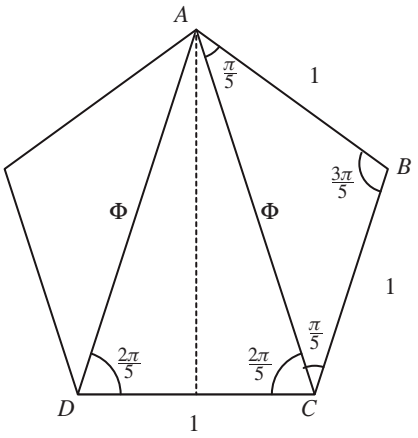


FIGURE 1

Theorem 1

Assume that $0 < \theta < \pi/2$, $0 < \tau < \pi/2$, $\theta + \tau = \pi/2$. Then we obtain

$$\Phi = \tan\tau \text{ if } \tan 2\theta = 2 \qquad (0 < 2\theta < \pi/2). \tag{1}$$

Theorem 2

Under the same assumptions as Theorem 1 we have the following:

$$\Phi = \frac{1}{\tan\theta} \text{ if } \tan 2\theta = 2 \ (0 < 2\theta < \pi/2), \tag{2}$$

$$\Phi = \tan\tau \text{ if } \tan 2\tau = -2 \ (\pi/2 < 2\tau < \pi), \tag{3}$$

$$\Phi = \frac{1}{\tan\theta} \text{ if } \tan 2\tau = -2 \ (\pi/2 < 2\tau < \pi). \tag{4}$$

Proof:

From the hypotheses we see that

$$\tan\theta = \tan\left(\frac{\pi}{2} - \tau\right) = \cot\tau = \frac{1}{\tan\tau}. \tag{5}$$

Hence, (1) implies

$$\Phi = \tan\tau = \frac{1}{\tan\theta}$$

and therefore (2) holds. Letting $\tan 2\tau = -2$ $(\pi/2 < 2\tau < \pi)$, we conclude that

$$\tan 2\tau = \tan 2\left(\frac{\pi}{2} - \theta\right) = \tan(\pi - 2\theta) = -\tan 2\theta = -2,$$

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and hence $\tan 2\theta = 2$. In view of (1) we derive $\Phi = \tan\tau$, and consequently we observe that (3) holds. Combining (3) and (5) yields (4).

Now we state our main theorem.

Theorem 3

We find that

$$\Phi = \left[\tan\left(\frac{1}{2}\,\mathrm{Tan}^{-1}\,2\right)\right]^{-1}, \tag{6}$$

$$\Phi = \tan\left(\frac{\pi}{2} - \frac{1}{2}\,\mathrm{Tan}^{-1}\,2\right), \tag{7}$$

where $\mathrm{Tan}^{-1} = \arctan$ and $\pi/3 < \mathrm{Tan}^{-1}\,2 = 1.1071\ldots < 2\pi/5$.

Proof:

We let $\theta = \frac{1}{2}\,\mathrm{Tan}^{-1}\,2$ to obtain

$$0 < 2\theta = \mathrm{Tan}^{-1}\,2 < \frac{\pi}{2},$$

$$\tan 2\theta = \tan\left(\mathrm{Tan}^{-1}\,2\right) = 2,$$

which implies (6) in view of (2). It is easy to see that

$$\frac{\pi}{2} < \pi - \mathrm{Tan}^{-1}\,2 < \pi,$$

$$\tan\left[2\left(\frac{\pi}{2} - \frac{1}{2}\,\mathrm{Tan}^{-1}\,2\right)\right] = \tan\left(\pi - \mathrm{Tan}^{-1}\,2\right)$$

$$= -\tan\left(\mathrm{Tan}^{-1}\,2\right) = -2.$$

Letting $\tau = \frac{1}{2}\left(\pi - \mathrm{Tan}^{-1}\,2\right)$ in (3) implies

$$\Phi = \tan\left(\frac{\pi}{2} - \frac{1}{2}\,\mathrm{Tan}^{-1}\,2\right)$$

and therefore (7) holds.

Corollary: The following identity holds:

$$\frac{\sqrt{5}-1}{2} = \tan\left(\tfrac{1}{2}\,\mathrm{Tan}^{-1}\,2\right).$$

Proof: Employing (6), we arrive at

$$\frac{\sqrt{5}-1}{2} = \varphi = \frac{1}{\Phi} = \tan\left(\tfrac{1}{2}\,\mathrm{Tan}^{-1}\,2\right).$$

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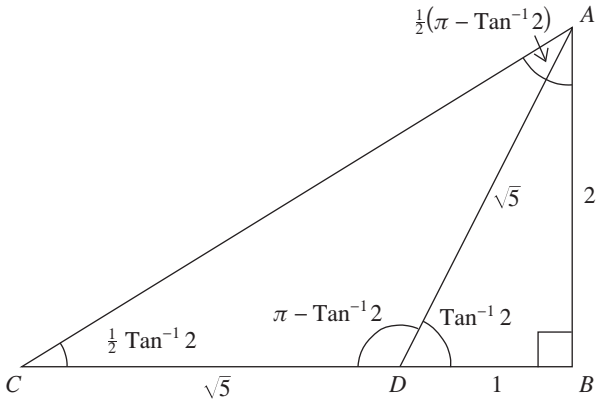


FIGURE 2

We present the following theorem which has the converse statements of Theorems 1 and 2.

Theorem 4

Under the same assumptions as Theorem 1 we have

$$\tan 2\theta = 2 \text{ if } \Phi = \tan\tau \quad \left(\tfrac{\pi}{4} < \tau < \tfrac{\pi}{2}\right), \tag{8}$$

$$\tan 2\theta = 2 \text{ if } \Phi = \frac{1}{\tan\theta} \quad \left(0 < \theta < \tfrac{\pi}{4}\right), \tag{9}$$

$$\tan 2\tau = -2 \text{ if } \Phi = \tan\tau \quad \left(\tfrac{\pi}{4} < \tau < \tfrac{\pi}{2}\right), \tag{10}$$

$$\tan 2\tau = -2 \text{ if } \Phi = \frac{1}{\tan\theta} \quad \left(0 < \theta < \tfrac{\pi}{4}\right), \tag{11}$$

Proof:

First we need the double-angle formula

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha} \quad \left(\alpha \neq \pm\frac{\pi}{4} + \frac{n\pi}{2},\ n = 0, \pm1, \pm2, \ldots\right) \tag{12}$$

and the following relations with respect to φ and Φ:

$$\varphi^2 + \varphi - 1 = 0,\ \Phi^2 - \Phi - 1 = 0,\ \Phi = \frac{1}{\varphi}.$$

Let $\Phi = \tan\tau\left(\frac{\pi}{4} < \tau < \frac{\pi}{2}\right)$. Then we obtain $\varphi = \tan\theta\left(0 < \theta < \frac{\pi}{4}\right)$ in light of (5), and we observe, using (12), that

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{2\varphi}{1 - \varphi^2} = 2,$$

and hence (8) holds. We easily see that (9) follows from (5) and (8). If

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$\Phi = \tan\tau\left(\frac{\pi}{4} < \tau < \frac{\pi}{2}\right)$, then (12) implies that

$$\tan 2\tau = \frac{2\tan\tau}{1-\tan^2\tau} = \frac{2\Phi}{1-\Phi^2} = -2,$$

and therefore (10) follows. Combining (5) with (10) yields (11). This completes the proof.

Remark 1: We find that Theorem 1 follows from (12). In fact, if $\tan 2\theta = 2\left(0 < 2\theta < \frac{\pi}{2}\right)$, then

$$2 = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

by virtue of (12). Hence, we conclude that $x = \tan\theta$ satisfies the quadratic equation $x^2 + x - 1 = 0$. Since $0 < 2\theta < \frac{\pi}{2}$, we get $0 < \theta < \frac{\pi}{4}$, and therefore $0 < \tan\theta < 1$, which implies

$$\tan\theta = \frac{\sqrt{5}-1}{2} = \varphi.$$

Utilising (5) and the relation $\varphi = \frac{1}{\Phi}$, we deduce that $\Phi = \tan\tau$, and thus (1) was proved.

Remark 2: As pointed out in [3], $\pi - \mathrm{Tan}^{-1}2$ denotes the dihedral angle of a dodecahedron.

Remark 3: We see that (7) follows from (6) because

$$\tan\left(\frac{\pi}{2} - \frac{1}{2}\mathrm{Tan}^{-1}2\right) = \cot\left(\frac{1}{2}\mathrm{Tan}^{-1}2\right) = \left[\tan\left(\frac{1}{2}\mathrm{Tan}^{-1}2\right)\right]^{-1} = \Phi.$$

Remark 4: As in Remark 1, we can prove Theorem 2 by applying (12).

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NORIO YOSHIDA
University of Toyama, Toyama, 930-8555 Japan
e-mail: *norio.yoshidajp@gmail.com*

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