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A Weighted Generalization of Gao's $n+D-1$ Theorem

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Let G denote a finite abelian group of order n and Davenport constant D, and put $m = n + D - 1$. Let $x = (x_1, \ldots, x_m) \in G^m$. Gao's theorem states that there is a reordering $(x_{j_1}, \ldots, x_{j_m})$ of x such that

$$\sum_{1 \leqslant i \leqslant n} x_{j_i} = 0.$$

Let $w = (x_1, \ldots, w_m) \in \mathbb{Z}^m$. As a corollary of the main result, we show that there are reorderings $(x_{j_1}, \ldots, x_{j_m})$ of x and $(w_{k_1}, \ldots, w_{k_m})$ of w, such that

$$\sum_{1 \leqslant i \leqslant n} w_{k_i} x_{j_i} = \left(\sum_{1 \leqslant i \leqslant n} w_{k_i} \right) x_{j_1},$$

where x_{j_1} is the most repeated value in x. For $w = (1, \ldots, 1)$, this result reduces to Gao's theorem.

1. Introduction

Let G be an abelian group of order n. The *Davenport constant* of G, denoted by $D(G)$, is the maximal integer k such that there is a sequence of elements of G with length $k-1$ having no non-empty zero-sum subsequence. The investigation of zero-sum subsequences with length n starts with a result of Erdős, Ginzburg and Ziv [2], stating that every sequence of elements of G with length $\geqslant 2n-1$ has a zero-sum subsequence of length n. A result of Gao [4], stating that every sequence of elements of G with length $\geqslant n + D(G) - 1$ has a zero-sum subsequence of length n, unifies these two areas of zero-sum problems. For more details on these questions and some of their applications, the reader may refer to the book of Geroldinger and Halter-Koch [6] and to the survey paper of Caro [1].

Attempts were made to generalize zero-sum problems to the weighted case. Examples of such results may be found in the survey paper of Caro [1], a paper by the author [9] and two more recent papers by Gao and Jin [5] and Grynkiewicz [7].

Let $x = (x_1, \ldots, x_m) \in E^m$, where E is a set. As usual one may consider x as a map from $[1, n]$ into E. We shall put $\rho(x) = \max\{|x^{-1}(a)|; a \in E\}$. Notice that $\rho(x)$ is the maximal repetition of the sequence x.

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Let G be a finite abelian group of order n and Davenport constant D. Let $x = (x_1, \ldots, x_m) \in G^m$ and let $w = (w_1, \ldots, w_m) \in \mathbb{Z}^m$. Let ℓ denote the maximal repetition in x viewed as a sequence.

Our main result states that for $m = n + D - \min(D, \ell) - 1$, there are reorderings $(x_{j_1}, \ldots, x_{j_m})$ of x and $(w_{k_1}, \ldots, w_{k_m})$ of w, and a t, such that

$$n - \min(D, \ell) \leqslant t \leqslant n - 1 \quad \text{and} \quad \sum_{1 \leqslant i \leqslant t} w_{k_i} x_{j_i} = 0.$$

Now take $m = n + D - 1$. Gao's theorem states that there is a reordering $(x_{j_1}, \ldots, x_{j_m})$ of x such that

$$\sum_{1 \leqslant i \leqslant n} x_{j_i} = 0.$$

As a corollary of the main result, we show that there are reorderings $(x_{j_1}, \ldots, x_{j_m})$ of x and $(w_{k_1}, \ldots, w_{k_m})$ of w, such that

$$\sum_{1 \leqslant i \leqslant n} w_{k_i} x_{j_i} = \left(\sum_{1 \leqslant i \leqslant n} w_{k\,i} \right) x_{j_1},$$

where x_{j_1} is the most repeated value in x. For $w = (1, \ldots, 1)$, this result reduces to Gao's theorem.

We shall formulate these results in a less intuitive language which is more efficient for proof formalization.

Theorem 1.1. *Let G denote a finite abelian group of order n and Davenport constant D. Let $x = (x_1, \ldots, x_m) \in G^m$ and let $w = (w_1, \ldots, w_m) \in \mathbb{Z}^m$. Assume moreover that $|x^{-1}(a)| \leqslant \ell$, for all $a \in G$, and that $m = n + D - \min(D, \ell) - 1$.*

Then there are a subset $I \subset [1, m]$ and an injection $f : I \mapsto [1, m]$, such that $n - \min(D, \ell) \leqslant |I| \leqslant n - 1$ and

$$\sum_{i \in I} w_i x_{f(i)} = 0.$$

Corollary 1.2. *Let G denote a finite abelian group of order n and Davenport constant D, and put $m = n + D - 1$. Let $x = (x_1, \ldots, x_m) \in G^m$ be such that $\max_{a \in G} |x^{-1}(a)| = \ell = |x^{-1}(x_m)|$ and put $r = \min(D, \ell)$. Let $w = (w_1, \ldots, w_{m-r}) \in \mathbb{Z}^{m-r}$.*

Then there are an n-subset $I \subset [1, m - r]$ and an injection $f : I \mapsto [1, m]$, such that $m \in f(I)$ and

$$\sum_{i \in I} w_i x_{f(i)} = \left(\sum_{i \in I} w_i \right) x_m.$$

With $w_i = 1$ for all i, Corollary 1.2 reduces to Gao's $n + D - 1$ theorem. We need the following simple and known results.

Theorem A (Scherk [11]). *Let A and B be non-empty finite subsets of an abelian group G. If $|A \cap (-B)| = \{0\}$, then $|A + B| \geqslant |A| + |B| - 1$.*

Notice that Scherck's theorem follows easily from Kneser's theorem (*cf.* [6]).

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Theorem B (folklore). *Let G be a finite abelian group of order n. Let $x = (x_1, \ldots, x_n) \in G^n$ be a sequence of elements of G with a maximal repetition $\leqslant k$. Then x has a non-empty zero-sum subsequence with length $\leqslant k$.*

Theorem B is now a standard tool in zero-sum theory [6]. A proof of Theorem B requires an addition theorem. The oldest such result easily implying Theorem B is Shepherdson's theorem [12]. An almost identical proof follows by Scherck's theorem [11]. When G is cyclic, Theorem B is a special case of Conjecture 4 of Erdős and Heilbronn [3]. In a note added in proof, Erdős and Heilbronn [3] mentioned that Flor proved this conjecture. A generalization of Theorem B to (not necessarily) abelian groups is proved in [8] using a result of Kemperman [10].

In order to make the present work self-complete, we give short proofs of Theorems B and A in the Appendix.

2. Weighted sums

Let G denote a finite abelian group of order n. We need the following weighted generalization of Theorem B.

Lemma 2.1. *Let $x = (x_1, \ldots, x_n) \in G^n$ and let $w = (w_1, \ldots, w_n) \in \mathbb{Z}^n$. Assume that $\max_{a \in G} |x^{-1}(a)| \leqslant \ell$. Then there are a subset $I \subset [1, n]$ with $1 \leqslant |I| \leqslant \ell$, and an injection $f : I \mapsto [1, n]$ such that*

$$\sum_{i \in I} w_i x_{f(i)} = 0.$$

Proof. Put $r = \rho(x)$ and $s = \rho(w)$. Take $a \in G$ such that $|x^{-1}(a)| = r$, and $b \in \mathbb{Z}$ such that $|w^{-1}(b)| = s$.

Assume first that $s \leqslant r$. By Theorem B, applied to w, there is a subset $I \subset [1, n]$ with $|I| \leqslant r$ such that $\sum_{i \in I} w_i \equiv 0 \bmod n$. Take an arbitrary injection $f : I \mapsto x^{-1}(a)$. We have $\sum_{i \in I} w_i x_{f(i)} = \sum_{i \in I} w_i a = \left(\sum_{i \in I} w_i\right) a = 0$.

Now assume that $s > r$. By Theorem B, there is a subset $J \subset [1, n]$ with $|J| \leqslant r$ such that $\sum_{i \in J} x_i = 0$. Take a subset $L \subset w^{-1}(b)$ such that $|L| = |J|$ and take an arbitrary bijection $f : L \mapsto J$. We have $\sum_{i \in L} w_i x_{f(i)} = \sum_{i \in J} b x_i = b\left(\sum_{i \in J} x_i\right) = 0$. □

This lemma suggests the following conjecture.

Conjecture 2.2. *Let $x = (x_1, \ldots, x_n) \in G^n$ and let $w = (w_1, \ldots, w_k) \in \mathbb{Z}^k$. Assume that $\max_{a \in G} |x^{-1}(a)| \leqslant k$. Then there are a non-empty subset $I \subset [1, k]$ and an injection $f : I \mapsto [1, n]$ such that*

$$\sum_{i \in I} w_i x_{f(i)} = 0.$$

Conjecture 2.2 follows easily by Scherck's theorem [11] or by some other additive ingredients if $\gcd(n, w_i) = 1$, for all i. It holds trivially if $D(G) \leqslant k$. If true, this conjecture has very interesting implications.

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Let $x = (x_1, \ldots, x_m) \in G^m$ and let $w = (w_1, \ldots, w_m) \in \mathbb{Z}^m$. Let f be a function on $[1, m]$. We shall write $\langle S \rangle_f = \sum_{i \in S} w_i x_{f(i)}$ for a subset S on which f is defined. Let $I \subset [1, m]$ and let $f : I \mapsto [1, m]$. We shall say that the pair (I, f) is *k-shellable* if there is a partition $I = I_1 \cup \cdots \cup I_t$ such that $1 \leqslant |I_i| \leqslant k$ and $\langle I_i \rangle_f = 0$, for all $1 \leqslant i \leqslant t$. We shall call the partition $I = I_1 \cup \cdots \cup I_t$ a shelling. Notice that $\langle I \rangle_f = 0$ is (I, f) if shellable.

Lemma 2.3. *Let $f : I \mapsto [1, m]$ be such that (I, f) is ℓ-shellable. Then, for every $m_0 \leqslant |I|$, there is a subset $S \subset I$ such that (S, f) is ℓ-shellable (and hence $\langle S \rangle_f = 0$) and $m_0 - \ell + 1 \leqslant |S| \leqslant m_0$.*

Proof. Take a shelling $I = I_1 \cup \cdots \cup I_t$ for (I, f). Take a maximal $s \leqslant t$ such that $|I_1 \cup \cdots \cup I_s| \leqslant m_0$ and put $S = I_1 \cup \cdots \cup I_s$. We must have $|S| \geqslant m_0 - \ell + 1$, since otherwise $s \leqslant t - 1$, and hence $|I_1 \cup \cdots \cup I_{s+1}| \leqslant m_0$, contradicting the maximality of s. □

3. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1.

Claim 1. For any two subsets $A, B \subset [1, m]$ such that $|B| \geqslant |A|$, there exist $R \subset A$ and an injection $g : R \mapsto B$ such that (R, g) is D-shellable and $|R| \geqslant |A| - D + 1$.

Take a maximal subset $R \subset A$ and an injection $g : R \mapsto B$ such that (R, g) is D-shellable (recall that the empty map is injective and hence $(\emptyset, \emptyset)$ is ℓ-shellable). We must have $|R| \geqslant |A| - D + 1$. Suppose that $|R| \leqslant |A| - D$. Take a D-subset of $A' \subset (A \setminus R)$ and D-subset $B' \subset (B \setminus g(R))$. Let h be an arbitrary bijection from R' onto S'. By the definition of the Davenport constant there is a non-empty subset $R' \subset A'$ such that $\langle R' \rangle_h = 0$. We now extend g to $R \cup R'$ by taking $g(x) = h(x)$ for all $x \in R'$. Now $(R \cup R')$ is (D, g)-shellable, contradicting the maximality of R.

Assume first $D \leqslant \ell$. Since $m \geqslant n - 1$, we may take any subset S_0 of $[1, m]$ with cardinality $n - 1$. By Claim 1, there exists $S \subset S_0$ with $n - 1 \geqslant |S| \geqslant n - D + 1$ and an injection $g : S \mapsto [1, m]$ such that $\langle S \rangle_g = 0$. The result holds in this case.

Now assume $\ell < D$. Thus $m = n - \ell + D - 1$.

Claim 2. There is a subset $T \subset [1, m]$ such that $|T| \geqslant D - \ell$ and an injection $g : T \mapsto [1, m]$ such that (T, g) is ℓ-shellable. Take a maximal subset $R \subset [1, m]$ and an injection $g : R \mapsto [1, m]$ such that (R, g) is ℓ-shellable (recall that the empty map is injective and hence $(\emptyset, \emptyset)$ is ℓ-shellable). We must have $|R| \geqslant D - \ell$, since otherwise $m - |R| \geqslant n - \ell + D - 1 - (D - \ell - 1) \geqslant n$. By the lemma there is a non-empty subset $R' \subset [1, m] \setminus R$ with $|R'| \leqslant \ell$ and an injection $h : R' \mapsto [1, m] \setminus g(R)$ such that $\langle R' \rangle_h = 0$. We now extend g to $R \cup R'$ by taking $g(x) = h(x)$ for all $x \in R'$. Now $(R \cup R', g)$ is ℓ-shellable, contradicting the maximality of R.

Put $C = [1, m] \setminus T$. By Claim 1, applied to C, there are a subset $E \subset C$ and injective map $f : E \mapsto [1, m] \setminus g(R)$, such that $n - 1 \geqslant |C| \geqslant |E| \geqslant |C| - D + 1$ and $\langle E \rangle_f = 0$. Now $n - 1 - |E| \leqslant n - 1 - |C| + D - 1 = (m + \ell - D - 1) - |C| + 1 = |T| + \ell - D < |T|$. By Lemma 2.1 there is a subset $F \subset T$ such that $\langle F \rangle_f = 0$ and $n - 1 - |E| \geqslant |F| \geqslant n - 1 - |E| - \ell + 1$. In particular, $n - 1 \geqslant |S| \geqslant n - \ell$, where $S = E \cup F$. We extend f to S, by taking $f(i) = g(i)$ for every $i \in F$. Clearly f is injective on S and $\langle S \rangle_f = \langle E \rangle_f + \langle F \rangle_f = 0$. □

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Proof of Corollary 1.2. Without loss of generality we may assume $x_{m-r+1} = \cdots = x_m$. By Theorem 1.1, applied to $(x_1 - x_m, \ldots, x_{m-r} - x_m)$, there are a subset $I \subset [1, m-r]$ and an injection $f : I \mapsto [1, m-r]$ such that $n - \min(D, \ell) \leqslant |I| \leqslant n-1$ and

$$\sum_{i \in I} w_i (x_{f(i)} - x_m) = 0,$$

and hence

$$\sum_{i \in I} w_i x_{f(i)} = \left(\sum_{i \in I} w_i \right) x_m.$$

Now take an $(n - |I|)$-subset $J \subset [1, m-r] \setminus I$. We may extend f to an injection on $I \cup J$, by putting $f(i) = h(i)$ for every $i \in J$, where h is an arbitrary injection $: J \mapsto [m-r+1, m]$.

We have

$$\begin{aligned} \sum_{i \in I \cup J} w_i x_{f(i)} &= \sum_{i \in I} w_i x_m + \sum_{i \in J} w_i x_m \\ &= \left(\sum_{i \in I \cup J} w_i \right) x_m. \end{aligned}$$

The proof is now complete. □

Appendix: Proofs of Theorem A and Theorem B

Proof of Theorem A. The proof is by induction on $|B|$, the result being obvious for $|B| = 1$. Since $0 \in A \cap B$, we have $|A + B| \geqslant |A \cup B| = |A| + |B| - |A \cap B|$. Therefore we may assume that there exists $b \in A \cap B$, with $b \neq 0$.

Put $B' = B \cap (B - b)$. We have $B - b \neq B$, since otherwise we would have $b \in A \cap -B$, a contradiction. Hence $0 \in B'$ and $|B'| < |B|$.

Put $A' = A \cup A + b$. We have $A' \cap -B' = (A \cap -B) \cap (-B + b) \cup A + b \cap (-B + b) \cap (-B) = \{0\}$. The result follows by induction. □

Proof of Theorem B. One may partition the sequence x as $X_1 \cup \cdots X_k$, where X_i is a set. Suppose that x has no non-empty zero-sum subsequence. We prove by induction that $|X_1 + \cdots + X_j| \geqslant |X_1| + \cdots + |X_j|$, for every $j \leqslant k$. This is true for $j = 1$. Suppose it is true for $j < k$. Put $A = X_1 + \cdots + X_j \cup \{0\}$ and $B = X_{j+1} \cup \{0\}$. Suppose that x has no non-empty zero-sum subsequence: we have $|A| \geqslant |X_1 + \cdots + X_j| + 1$ and $B = |X_{j+1}| + 1$. By Theorem A,

$$|X_1 + \cdots + X_{j+1}| \geqslant |A + B| - 1 \geqslant |A| + |B| - 2 \geqslant |X_1| + \cdots + |X_{j+1}|.$$

Then $|X_1 + \cdots + X_k| \geqslant |X_1| + \cdots + |X_k| = |G|$, and x then has a zero-sum subsequence, a contradiction. □

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