PSEUDO-INVERSES IN SEMIGROUPS*

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Drazin (2) has recently introduced the concept of a pseudo-invertible element of an associative ring or semigroup. In this note we first show that such an element of a semigroup S may be characterized by the fact that some power of it lies in a subgroup of S. The main result (Theorem 2) provides a new criterion for complete simplicity: a simple semigroup is completely simple if and only if all its elements are pseudo-invertible. This is applied to the principal factors of a general semigroup and two previously known results are obtained as corollaries.

1. Let S be a semigroup. Let e be an idempotent element of S ($e^2 = e$) and let G_e denote the set of all elements $x \in S$ such that $ex = x = xe$ and $yx = e = xy$ for some $y \in S$ (dependent on x). Then G_e is a subgroup of S with identity e. Furthermore, if G is any subgroup of S containing e then clearly $G \subseteq G_e$; for this reason G_e is called the *maximal subgroup* of S associated with e.

Lemma 1. *Let S be a semigroup and let x be an element of S such that x^n belongs to a subgroup G of S for some positive integer n. Then, if e is the identity of G,*

(*a*) $ex = xe \in G_e$,

(*b*) $x^m \in G_e$ *for any integer* $m > n$.

Proof. (*a*) Let y be the inverse of x^n in G. Thus we have $ex^n = x^n = x^n e$ and $yx^n = e = x^n y$. Then

$$ex = yx^{n+1} = yx.x^n = yx.x^n e = yx.x^n.x^n y = yx^{2n+1}y$$

and, by symmetry, $yx^{2n+1}y = xe$. Hence $ex = xe$. Then, since $ey = y = ye$, we have

$$xy = xey = exy = yx^n.xy = yx.x^n y = yxe = yex = yx$$

and so $x^{n-1}y = yx^{n-1}$. (We evidently need only consider the case $n > 1$.) Set $z = x^{n-1}y = yx^{n-1}$. Then $z.xe = yx^{n-1}.xe = yx^n.e = e^2 = e$ and similarly $ex.z = e$. Moreover, $e.ex = ex = xe = xe.e$ and so $ex = xe \in G_e$.

(*c*) Let m be any integer greater than n. Select an integer r such that $nr > m$. As above, we have $xy = yx$ where y is the inverse of x^n in G and so $x^{nr-m}y^r = y^r x^{nr-m}$; let this element be denoted by w. Then

$$wx^m = y^r x^{nr-m}.x^m = y^r x^{nr} = (yx^n)^r = e^r = e;$$

in the same way, $x^m w = e$. Furthermore, $ex^m = ex^n.x^{m-n} = x^n.x^{m-n} = x^m$ and similarly $x^m e = x^m$. Thus $x^m \in G_e$.

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Following Drazin, we say that an element x of a semigroup S is *pseudo-invertible* in S if and only if there is an element $\bar{x} \in S$ such that

$$\text{(i)} \quad x\bar{x} = \bar{x}x,$$

$$\text{(ii)} \quad x^n = x^{n+1}\bar{x} \quad \text{for some positive integer } n,$$

and

$$\text{(iii)} \quad \bar{x} = \bar{x}^2 x.$$

If such an element $\bar{x}$ exists, it is uniquely determined by x ((2), Theorem 1) and is called the *pseudo-inverse* of x.

As a special case, we note that if S is a semigroup with a zero 0 and if x is nilpotent (that is, if $x^n = 0$ for some positive integer n), then x is pseudo-invertible and $\bar{x} = 0$.

We now give an alternative characterization of a pseudo-invertible element of S.

THEOREM 1. *An element x of a semigroup S is pseudo-invertible if and only if some power of x lies in a subgroup of S.*

The proof is immediate from Lemma 1 (a) and Theorem 7 of (2). (Lemma 1 (a) shows that the condition $ex = xe \in G_e$ can be deleted from the hypotheses of the second part of Drazin's Theorem 7.)

2. A semigroup S is *simple* if it has no ideals other than S and that consisting of the zero element alone, where such an element exists. (We exclude by convention the semigroup $S = \{x, 0\}$ where 0 is the zero and $x^2 = 0$.) A non-zero idempotent e of a semigroup S is said to be *primitive* if there is no non-zero idempotent of S, other than e itself, satisfying the equations $ex = x = xe$. If a simple semigroup contains a primitive idempotent it is called *completely simple* ((4), (5)): the structure of such semigroups has been fully determined ((4), Theorem 2·93).

We shall say that a semigroup S is pseudo-invertible if every element of S is pseudo-invertible; by Theorem 1 this is equivalent to the condition that some power of each element lies in a subgroup of S.

THEOREM 2. *A simple semigroup is completely simple if and only if it is pseudo-invertible.*

Proof. Let S be a simple semigroup. We may suppose that S has a zero 0; such an element may be adjoined if it is not already present. First, suppose that S is completely simple. Let $x \in S$; then by the structure theorem ((4), Theorem 2·93) we see that either x belongs to a non-zero subgroup of S or $x^2 = 0$. In either case x is pseudo-invertible.

Conversely, suppose that every element of S is pseudo-invertible. Let a be a non-zero element of S. Then since S is simple there exist elements $x, y \in S$ such that $a = xay$ ((4), Lemma 2·11). It follows that $a = x^r a y^r$ for any positive integer r and so x cannot be nilpotent. Since x is pseudo-invertible, $x^s \in G_e$ for some idempotent e and some positive integer s. Hence since $ex^s = x^s \neq 0$ we conclude that $e \neq 0$.

We shall complete the proof by showing that e is primitive. Accordingly, let f be a non-zero idempotent of S such that $ef = f = fe$. Again by Lemma 2·11 of (4) we can find elements $h', k' \in S$ such that $e = h'fk'$. Set $h = eh'f$ and $k = fk'e$. Then

$$eh = h = hf = hfe = he \quad \text{and} \quad ke = k = fk = efk = ek.$$

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Moreover, $hk = eh'f^2k'e = e^3 = e$. From these relations it follows that

$$e = hk = hek = h.hk.k = h^2k^2 = \ldots = h^rk^r$$

for any positive integer r.

Since h is pseudo-invertible, $h^n \in G_g$ for some idempotent $g \in S$ and some positive integer n. Write $u = h^n$, $v = k^n$ and let w be the inverse of u in G_g. Then

$$eu = u = ue, \quad ev = v = ve, \quad e = uv = u^2v^2,$$

$$gu = u = ug \quad \text{and} \quad wu = g = uw.$$

Now $gv^2u^2 = w^2u^2v^2u^2 = w^2eu^2 = w^2u^2 = g$. Hence

$$e = uv = ugv = ugv^2u^2v = (ugv)(vu)(uv) = e(vu)e = vu.$$

But $fv = fk^n = k^n = v$, since $fk = k$. Thus $f = fe = fvu = vu = e$, as required.

3. Two elements of a semigroup S are said to belong to the same $\mathfrak{f}$-class of S if they generate the same principal ideal of S (Green (3)). The $\mathfrak{f}$-class containing the element x of S will be denoted by F_x. If F is any $\mathfrak{f}$-class, we shall denote by $P(F)$ the ideal of S generated by an element of F. There is a natural partial ordering of the $\mathfrak{f}$-classes: we write $F \leqslant F'$ if and only if $P(F) \subseteq P(F')$. If every set of $\mathfrak{f}$-classes of S contains a minimal member with respect to this partial ordering we shall say that S satisfies the condition $\mathscr{M}_{\mathfrak{f}}$; this may be described as 'the minimal condition on principal ideals of S'.

Let F be any $\mathfrak{f}$-class of S, let $P = P(F)$ and let N be the set of non-generators of P. If $N \neq \emptyset$, it is an ideal of S and so we can form the quotient semigroup (difference semigroup) P/N ((4), p. 389); if $N = \emptyset$ we take P/N to mean P itself. The semigroup P/N is called the *principal factor* of S corresponding to F; we denote it by $Q = Q(F)$. As remarked by Green (3), either Q is simple or $Q^2 = \{0\}$, where 0 is the zero of Q. (In the case where $N = \emptyset$, P is simple without zero.)

Theorem 2 may then be extended as follows.

THEOREM 3. *Let S be a pseudo-invertible semigroup. Then every simple principal factor of S is completely simple.*

Conversely, let S be a semigroup satisfying the condition $\mathscr{M}_{\mathfrak{f}}$ and let every simple principal factor of S be completely simple. Then S is pseudo-invertible.

Proof. First, let S be pseudo-invertible. Let Q be a simple principal factor of S and let a be any non-zero element of Q. Then a may be regarded as an element of S and $Q = Q(F_a)$. By hypothesis, $a^n \in G_e$ for some idempotent $e \in S$ and some positive integer n. Then $G_e \subseteq P(F_a)$. If $G_e \subseteq F_a$, then G_e may be regarded as a non-zero subgroup of Q and so a, *qua* element of Q, is pseudo-invertible. On the other hand, if G_e contains a non-generator of $P(F_a)$ then $G_e \cap F_a = \emptyset$; in particular, a^n is a non-generator of $P(F_a)$ and so a, *qua* element of Q, is nilpotent. In either case a is pseudo-invertible in Q. Hence by Theorem 2, Q is completely simple.

Conversely, let S be a semigroup satisfying $\mathscr{M}_{\mathfrak{f}}$ and such that every simple principal factor is completely simple. Let $a \in S$ and let $\mathfrak{F}_a$ denote the set of all $\mathfrak{f}$-classes of S containing powers of a. Then $\mathfrak{F}_a$ contains a minimal member F_{a^n} for some integer n.

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Now $F_{a^{2n}} \in \mathfrak{F}_a$ and $F_{a^{2n}} \leqslant F_{a^n}$, from which we see that $F_{a^{2n}} = F_{a^n}$. Let $Q = Q(F_{a^n})$. Since a^n and a^{2n} may both be regarded as non-zero elements of Q, it follows that $Q^2 \neq \{0\}$ and so Q is simple; hence Q is completely simple, by hypothesis. Then, from the structure theorem for completely simple semigroups, we deduce that a^n must belong to a non-zero subgroup of Q; that is, a^n, *qua* element of S, belongs to a subgroup of S. Thus a is pseudo-invertible in S.

A semigroup is *semisimple* if every principal factor is simple and *completely semisimple* if every principal factor is completely simple. Theorem 3 shows, in particular, that a pseudo-invertible semisimple semigroup is completely semisimple and, conversely, that a completely semisimple semigroup satisfying $\mathscr{M}_f$ is pseudo-invertible. It is an open question whether or not the second statement remains true if we remove the condition $\mathscr{M}_f$.

If every element of a semigroup S has finite order ((4), p. 388) then S is said to be *periodic*.

COROLLARY 1. *Every simple principal factor of a periodic semigroup is completely simple.*

For, if an element a of a semigroup S has finite order, then there is a positive integer m such that the elements a^r $(r \geqslant m)$ constitute a cyclic group: thus a is pseudo-invertible.

Corollary 1, for the case of a simple semigroup, was previously obtained by Rees ((4), Lemma 2·4).

We conclude by deducing a result due to Clifford ((1), Theorem 2 (in part)).

COROLLARY 2. *A semigroup which consists of a union of groups is completely semisimple.*

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