Ergod. Th. & Dynam. Sys. (2016), **36**, 2351–2383 © *Cambridge University Press, 2015*
doi:10.1017/etds.2015.32

Conjugation between circle maps with several break points

ABDELHAMID ADOUANI

University of Carthage, Faculty of Science of Bizerte, Department of Mathematics, Jarzouna 7021, Tunisia
(e-mail: arbi.abdelhamid@gmail.com)

(*Received* 14 *February* 2013 *and accepted in revised form* 16 *March* 2015)

Abstract. Let f and g be two class P-homeomorphisms of the circle S^1 with break point singularities, which are differentiable maps except at some singular points where the derivative has a jump. Assume that f and g have irrational rotation numbers and the derivatives Df and Dg are absolutely continuous on every continuity interval of Df and Dg, respectively. We prove that if the product of the f-jumps along all break points of f is distinct from that of g then the homeomorphism h conjugating f and g is a singular function, i.e. it is continuous on S^1, but $\mathrm{Dh}(x) = 0$ almost everywhere with respect to the Lebesgue measure. This result generalizes previous results for one and two break points obtained by Dzhalilov, Akin and Temir, and Akhadkulov, Dzhalilov and Mayer. As a consequence, we get in particular Dzhalilov–Mayer–Safarov's theorem: if the product of the f-jumps along all break points of f is distinct from 1, then the invariant measure μ_f is singular with respect to the Lebesgue measure.

1. *Introduction*

Denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle and $p : \mathbb{R} \longrightarrow S^1$ the canonical projection. Let f be an orientation preserving homeomorphism of S^1. The homeomorphism f admits a lift $\widehat{f} : \mathbb{R} \longrightarrow \mathbb{R}$ that is an increasing homeomorphism of $\mathbb{R}$ such that $p \circ \widehat{f} = f \circ p$. Conversely, the projection of such a homeomorphism of $\mathbb{R}$ is an orientation preserving homeomorphism of S^1. Let $x \in S^1$. We call the *orbit* of x by f the subset $O_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$.

Historically, the study of the dynamics of circle homeomorphisms was initiated by Poincaré in 1885 [**25**], who introduced the rotation number of a homeomorphism f of S^1 as $\rho(f) = \lim_{n\to+\infty} ((\widehat{f}^n(\widehat{x}) - \widehat{x})/n) \pmod 1$, where $\widehat{x} \in \mathbb{R}$ is such that $p(\widehat{x}) = x$. Poincaré showed that this limit exists and does not depend on either x or the lift $\widehat{f}$ of f.

We say that f is semi-conjugate to the rotation $R_{\rho(f)}$ if there exists an orientation preserving surjective continuous map $h : S^1 \longrightarrow S^1$ of degree one such that $h \circ f = R_{\rho(f)} \circ h$.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

POINCARÉ'S THEOREM. *Let f be a homeomorphism of S^1 with irrational rotation number $\rho(f)$. Then f is semi-conjugate to the rotation $R_{\rho(f)}$.*

A natural question is whether the semi-conjugation h could be improved to be a conjugation, that is, h to be a homeomorphism. In this case, we say that f is topologically conjugate to the rotation $R_{\rho(f)}$. In this direction, Denjoy [**7**] proved the following result.

DENJOY'S THEOREM. [**7**] *Every C^2-diffeomorphism f of S^1 with irrational rotation number $\rho(f)$ is topologically conjugate to the rotation $R_{\rho(f)}$.*

Denjoy asked whether or not C^2-diffeomorphisms f of S^1 are ergodic with respect to the Lebesgue measure m (f is said to be ergodic with respect to m if any f-invariant measurable set A has measure $m(A)$ equal to 0 or 1). Simultaneously, Herman and Katok gave a positive answer to this question.

HERMAN–KATOK'S THEOREM. [**18**, **14**] *Every C^2-diffeomorphism f with irrational rotation number is ergodic with respect to the Lebesgue measure m.*

It is well known that a homeomorphism f of S^1 with irrational rotation number preserves a unique normalized measure on S^1, denoted by μ_f. By taking $h(0) = 0$, the conjugating homeomorphism h is unique and related to μ_f by $h(x) = p(\mu_f([0, x])) \in S^1$ for $x \in S^1$. Uniqueness of μ_f implies that μ_f is either singular or absolutely continuous with respect to m; in the second case, h is an absolutely continuous function. Recall that μ_f is said to be *singular* with respect to the Lebesgue measure m on S^1 if there exists a measurable subset E of S^1 such that $\mu_f(E) = 1$ and $m(E) = 0$. In fact, if μ_f is absolutely continuous with respect to the Lebesgue measure m and f is a C^2-diffeomorphism, then μ_f is necessarily equivalent to m (i.e. m is absolutely continuous with respect to μ_f and conversely), as a consequence of Herman–Katok's theorem above.

In what follows, we denote $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$ and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.

Definition. A real number $\alpha \in]0, 1[$ is called *Diophantine with exponent* $\delta \geq 0$ if there is a constant $c(\alpha) > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c(\alpha)}{q^{2+\delta}} \quad \text{for any } \frac{p}{q} \in \mathbb{Q}. \tag{1.1}$$

A number that is neither rational nor Diophantine is called a *Liouville number*.

Every real number $\alpha \in]0, 1[$ has a continued fraction expansion represented by

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} := [a_1, a_2, \ldots, a_n, \ldots],$$

where $a_m \in \mathbb{N}^*$, $m \in \mathbb{N}^*$, are called *partial quotients* of α. When $(a_m)_{m\in\mathbb{N}^*}$ is bounded, α is said to be of *bounded type*. This is equivalent to the fact that (1.1) holds with $\delta = 0$.

The problem of smoothness of the conjugacy h of smooth diffeomorphisms to rotations is now very well understood (see for instance [**16–21**, **22**, **28**]). We recall the last two important results in this area.

Katznelson and Ornstein (see also [**21**, **27**]) proved the following.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

KATZNELSON–ORNSTEIN'S THEOREM. [**18**] *Let f be a C^1-diffeomorphism of S^1 with irrational rotation number $\rho(f)$. If* Df *is absolutely continuous, $D \log \mathrm{Df} \in L^p(S^1)$ for some $p > 1$ and $\rho(f)$ is of bounded type, then μ_f is equivalent to the Lebesgue measure m.*

This implies that the map conjugating f and the rotation $R_{\rho(f)}$ is absolutely continuous.

KHANIN–TEPLINSKY'S THEOREM. [**21**] *Let f be a $C^{2+\alpha}$-diffeomorphism of S^1 for some $0 < \alpha < 1$ and let the rotation number $\rho = \rho(f)$ be a Diophantine number with exponent $\delta \in (0, \alpha)$. Then the map conjugating f and the rotation R_ρ is a $C^{1+\alpha-\delta}$-diffeomorphism.*

Notice that the condition $f \in C^{2+\alpha}$ is sharp, because there is a set of full Lebesgue measure in $[0, 1]$ such that for any rotation number in this set there are C^2-diffeomorphisms for which the conjugating map is singular [**15**].

We refer the reader to the books [**24**, **6**] for a thorough account of circle homeomorphisms.

The situation is more complicated for circle homeomorphisms with break points or, shortly, class P-homeomorphisms (see the definition below). This class is known to satisfy the conclusion of Denjoy's theorem (Corollary 2.5) (see also [**18**] and [**16**, Ch. VI]) and, with additional regularity, of Herman–Katok's theorem (see [**5**]). However, Katznelson–Ornstein's theorem [**18**] cannot be extended in general to class P. The study of the regularity of the invariant measures of class P-homeomorphisms then arises naturally.

1.1. *Class P-homeomorphisms.* The following definition is due to Herman.

Definition 1.1. [**16**, p. 74] An orientation preserving homeomorphism f of S^1 is called a *class P*-homeomorphism if it is differentiable except at finitely many points, the so-called *break points* of f, at which left and right derivatives (denoted, respectively, by Df_- and Df_+) exist and such that the derivative $\mathrm{Df} : S^1 \longrightarrow \mathbb{R}_+^*$ has the following properties:

- there exist two constants $0 < a < b < +\infty$ such that $a < \mathrm{Df}(x) < b$ for every x, where Df exists;
- $a < \mathrm{Df}_+(c) < b$ and $a < \mathrm{Df}_-(c) < b$ at the break points c;
- log Df has bounded variation on S^1.

We point out that the third condition implies the first two.

The ratio $\sigma_f(c) := (\mathrm{Df}_-(c))/(\mathrm{Df}_+(c))$ is called the *f-jump* in c.

Denote:

- $C(f) = \{c_0, c_1, c_2, \ldots, c_p\}$, the set of break points of f in S^1;
- $c_{p+1} := c_0$;
- $\pi_{s,O_f(c)}(f) := \prod_{x \in C(f) \cap O_f(c)} \sigma_f(x)$ for every $c \in S^1$;
- $\pi_s(f)$, the product of f-jumps in the break points of f:

$$\pi_s(f) = \prod_{c \in C(f)} \sigma_f(c).$$

We denote the total variation of log Df by $V = \mathrm{Var}(\log \mathrm{Df})$. We have

$$V := \sum_{j=0}^{p} \mathrm{Var}_{[c_j, c_{j+1}]} \log \mathrm{Df} + |\log(\sigma_f(c_j))| < +\infty.$$

In this case, V is the total variation of log Df, $\log \mathrm{Df}_-$, $\log \mathrm{Df}_+$.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

We notice the following properties:

- if f is a class P-homeomorphism of S^1 which is C^1 on S^1, then f is a C^1-diffeomorphism;
- if f is a class P-homeomorphism of S^1, then $D^2 f \in L^1(S^1)$: indeed, this follows from [**17**, Theorem 18.14, p. 284] since Df has bounded variation on S^1;
- if f is a class P-homeomorphism of S^1 and Df is absolutely continuous on every connected interval of $S^1 \backslash C(f)$, then $f \in C^1(S^1 \backslash C(f))$.

Among the simplest examples of class P-homeomorphisms, there are:

- C^2-diffeomorphisms;
- piecewise linear PL-homeomorphisms.

An orientation preserving homeomorphism f of S^1 is called a *PL-homeomorphism* if f is differentiable except at finitely many break points $(c_i)_{1 \le i \le p}$ of S^1 such that the derivative Df is constant on each $]c_i, c_{i+1}[$.

Denote:

- $\mathcal{P}(S^1)$, the set of class P-homeomorphisms of S^1;
- $\mathrm{PL}(S^1)$, the set of piecewise linear PL-homeomorphisms of S^1.

We notice that $\mathcal{P}(S^1)$ is a group which contains $\mathrm{PL}(S^1)$ (cf. [**1**]).

Definition 1.2. [**2**] Let $f \in \mathcal{P}(S^1)$. We say that f has the *(D)-property* if the product of f-jumps at the break points of each orbit is trivial, that is, $\pi_{s, O_f(c)}(f)=1$ for every $c \in C(f)$.

In particular, if f has the (D)-property, then $\pi_s(f) = 1$. Conversely, if all break points belong to the same orbit and $\pi_s(f) = 1$, then f has the (D)-property.

- Invariant measures of PL-homeomorphisms with two break points were first studied by Herman in [**16**] and by Liousse in [**23**] for 'generic' PL-homeomorphisms with *several break points* and with irrational rotation numbers of bounded type.
- Invariant measures of general (non-PL) class P-homeomorphisms with *one break point* have been studied by Dzhalilov and Khanin in [**10**]. The situation of their properties is the opposite of the $C^{2+\varepsilon}$ diffeomorphism case, $\varepsilon > 0$. They proved the following result: let f be a circle homeomorphism with a single break point a. If the rotation number $\rho(f)$ is irrational and $f \in C^{2+\varepsilon}(S^1 \backslash \{a\})$ for some $\varepsilon > 0$, then the measure μ_f is singular with respect to the Lebesgue measure m.

In a next step, invariant measures of class P-homeomorphisms with *two break points* were recently studied by Dzhalilov *et al* in [**11**]. Their main result is the following.

DZHALILOV–LIOUSSE–MAYER'S THEOREM. [**11**, Theorems 1.4 and 1.5] *Let $f \in \mathcal{P}(S^1)$ have irrational rotation number α. Suppose that f has two break points x_1 and x_2 and* Df *is absolutely continuous on every connected interval of $S^1 \backslash \{x_1, x_2\}$ and $D^2 f \in L^1(S^1)$. Then the measure μ_f is singular with respect to the Lebesgue measure provided one of the following condition holds:*

(i) $\sigma_f(x_1)\sigma_f(x_2) \neq 1$*; or*

(ii) α *is of bounded type and x_1 and x_2 are on distinct orbits with $\sigma_f(x_1)\sigma_f(x_2) = 1$.*

Invariant measures of class P-homeomorphisms with *several break points* have been studied by Dzhalilov [**8**], by Akhadkulov [**5**] in the case where the break points *lie on a same orbit* and by the authors [**1**] for class P-homeomorphisms *having the (D)-property*.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

AKHADKULOV'S THEOREM. [**5**, Theorems 2 and 3] *Let* $f \in \mathcal{P}(S^1)$ *have irrational rotation number* α. *Suppose that* Df *is absolutely continuous on every connected interval of* $S^1 \backslash C(f)$ *and* $D^2 f \in L^p(S^1)$ *for some* $p > 1$. *If the break points of* f *belong to the same orbit and* $\pi_s(f) \neq 1$, *then the measure* μ_f *is singular with respect to the Lebesgue measure* m.

The above theorems show that the conjugation map between f and $R_{\rho(f)}$ is singular.

- Recently, there has been significant progress in the problem of the regularity of the conjugating map between two class P-homeomorphisms with one or two break points and the same irrational rotation numbers. The case of one break point with the same jump, the so-called rigidity problem, was studied by Khanin and Khmelev in [**20**] and by Khanin and Teplinsky in [**19**]. Let

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} := [a_1, a_2, \ldots, a_n, \ldots]$$

be the continued fraction expansion of the irrational rotation number α and define $M_o = \{\alpha : \exists C > 0, \forall n \in \mathbb{N}^*, a_{2n-1} \leq C\}$, $M_e = \{\alpha : \exists C > 0, \forall n \in \mathbb{N}^*, a_{2n} \leq C\}$. A circle homeomorphism f is called a diffeomorphism of smoothness $C^{2+\alpha}$, $\alpha \in]0, 1[$, with a break at a point b and size $c > 0$, $c \neq 1$, if the following conditions hold:

(1) $f \in C^{2+\alpha}(]b, b+1[)$ and $\inf_{x \neq b} \mathrm{Df}(x) > 0$;

(2) there exist non-equal one-sided derivatives $\mathrm{Df}_-(b)$ and $\mathrm{Df}_+(b)$ with $(\mathrm{Df}_-(b))/(\mathrm{Df}_+(b)) = c$.

Then Khanin–Teplinsky's result is the following.

THEOREM. [**19**] *Let* f *and* g *be two circle diffeomorphisms of smoothness* $C^{2+\alpha}$, $\alpha \in]0, 1[$, *with breaks of the same size* c *and the same rotation number* $\rho \in M_e$ *in the case of* $c > 1$, *or* $\rho \in M_o$ *in the case of* $0 < c < 1$. *There exists an orientation preserving* C^1*-smooth circle diffeomorphism* ϕ *such that* $\phi \circ f \circ \phi^{-1} = g$.

- The case of *one* break point with *distinct* jumps was studied by Dzhalilov *et al* in [**9**] and for *two* break points by Akhadkulov *et al* in [**4**] (see Corollaries 1.3 and 1.4 below).

In the present paper, we consider the general case of class P-homeomorphisms having *several break points* with coinciding irrational rotation number. The main purpose is to prove the following.

MAIN THEOREM. *Let* f *and* g *be two class* P*-homeomorphisms of the circle* S^1 *and the same irrational rotation number. Assume that the derivatives* Df *and* Dg *are absolutely continuous on every continuity interval of* Df *and* Dg, *respectively. If* $\pi_s(f) \neq \pi_s(g)$, *then the homeomorphism map* h *conjugating* f *and* g *is a singular function, i.e. it is continuous on* S^1 *and* $\mathrm{Dh}(x) = 0$ *almost everywhere with respect to the Lebesgue measure.*

The condition of Df: 'Df is absolutely continuous on every continuity interval of Df' is called the *Katznelson–Ornstein (KO) condition*. As a consequence, we have the following.

COROLLARY 1.3. [**9**] *Let* $f_i \in C^{2+\varepsilon}(S^1 \backslash \{x_i\})$ *for some* $\varepsilon > 0$, *with one break point* x_i, $i = 1, 2$, *and* $\mathrm{Df}_i(x) > 0$ *for all* $x \in S^1 \backslash \{x_i\}$. *Assume that:*

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

(i) *the rotation numbers* $\rho(f_i)$ *of* f_i, $i = 1, 2$, *are irrational and coincide, i.e.* $\rho(f_1) = \rho(f_2) = \rho$;
(ii) *the jumps* $\sigma_{f_1}(x_1)$ *and* $\sigma_{f_2}(x_2)$ *are positive but do not coincide, i.e.* $\sigma_{f_1}(x_1) \neq \sigma_{f_2}(x_2)$.

Then the map h conjugating f_1 *and* f_2 *is a singular function.*

COROLLARY 1.4. [**4**] *Let* $f_i \in \mathcal{P}(S^1)$ *be circle homeomorphisms with two break points* a_1, b_1 *respectively* a_2, b_2 *and such that the derivatives* $\mathrm{Df_i}$ *are absolutely continuous on every connected interval of* $S^1 \backslash \{a_i, b_i\}$. *Assume that:*
(1) *the rotation numbers* $\rho(f_i)$ *of* f_i, $i = 1, 2$, *are irrational and coincide, i.e.* $\rho(f_1) = \rho(f_2) = \rho$, $\rho \in \mathbb{R}\backslash\mathbb{Q}$;
(2) *the products of the jumps of* f_1 *and* f_2 *are different, i.e.* $\sigma_{f_1}(a_1)\sigma_{f_1}(b_1) \neq \sigma_{f_2}(a_2)\sigma_{f_2}(b_2)$;
(3) $\mu_1([a_1, b_1]) = \mu_2([a_2, b_2])$, *where* μ_i *is an invariant probability measure of* f_i, $i = 1, 2$.

Then the map h conjugating f_1 *and* f_2 *is singular.*

COROLLARY 1.5. [**12**, Theorem 1.5] *Let* $f \in \mathcal{P}(S^1)$ *of* S^1 *have rotation number irrational such that* f *satisfies the Katznelson–Ornstein (KO) condition. If* $\pi_s(f) \neq 1$, *then the invariant measure* μ_f *is singular with respect to the Lebesgue measure.*

Corollary 1.5, due to Akhadkulov *et al* [**4**], generalizes Theorem A of Adouani and Marzougui [**3**] obtained for a weaker condition on the product of f-jumps.

2. *Notation and preliminary results*

2.1. *Dynamical partitions.* Let f be a homeomorphism of S^1 with irrational rotation number $\alpha = \rho(f)$. We identify α to its lift $\widehat{\alpha}$ in $]0, 1[$. Let $(a_n)_{n\in\mathbb{N}^*}$ be the partial quotients of α in the continued fraction expansion. For $n \in \mathbb{N}^*$, the fractions $[a_1, a_2, \ldots, a_n]$ are written in the form of irreducible fractions p_n/q_n. The sequence p_n/q_n converges to α and we say that p_n/q_n are *rational approximations* of α. Their denominators q_n satisfy the following recursion relation:

$$q_n = a_n q_{n-1} + q_{n-2}, \quad n \geq 2, \quad q_0 = 1, \quad q_1 = a_1.$$

Let $x_0 \in S^1$ be fixed. Denote

$$\Delta_0^{(n)}(x_0) = \begin{cases} [x_0,\ f^{q_n}(x_0)] & \text{if } n \text{ is even,} \\ [f^{q_n}(x_0),\ x_0] & \text{if } n \text{ is odd,} \end{cases}$$
$$\Delta_i^{(n)}(x_0) := f^i(\Delta_0^{(n)}(x_0)), \quad i \in \mathbb{Z}.$$

In all that follows, we deal with the case n odd (the case n even is obtained by reversing the orientation of S^1).

We then have the following result.

LEMMA 2.1. (See [**26**]) *The segments* $\Delta_i^{(n-1)}(x_0) = f^i(\Delta_0^{(n-1)}(x_0))$, $0 \leq i < q_n$, *and* $\Delta_j^{(n)}(x_0) = f^j(\Delta_0^{(n)}(x_0))$, $0 \leq j < q_{n-1}$, *cover* S^1 *and their interiors are mutually disjoint.*

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

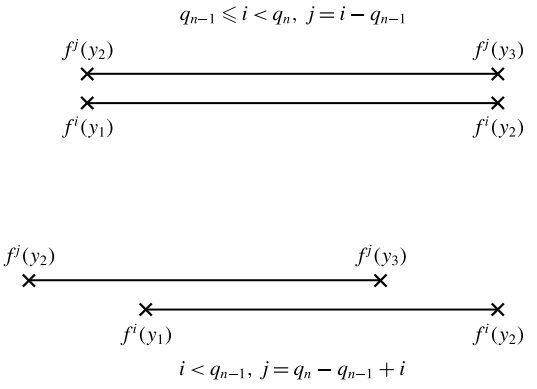


FIGURE 1.

The partition denoted by $\xi_n(x_0) := \{\Delta_i^{(n-1)}(x_0);\ \Delta_j^{(n)}(x_0),\ 0 \le i < q_n,\ 0 \le j < q_{n-1}\}$, is called the *nth dynamical partition* of the point x_0. It is defined by the order of points $x_0, \ldots, f^{q_{n-1}+q_n-1}(x_0)$. The process of passing from $\xi_n(x_0)$ to $\xi_{n+1}(x_0)$ is described by keeping intact the elements $\Delta_j^{(n)}(x_0)$, while each element $\Delta_i^{(n-1)}(x_0)$ $(0 \le i < q_n)$ is partitioned into $a_{n+1}+1$ subsegments; since $q_{n+1} = a_{n+1}q_n + q_{n-1}$,

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{a_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

For every $n \ge 1$, $\xi_{n+1}(x_0)$ is finer than $\xi_n(x_0)$ in the following sense: every element $\xi_n(x_0)$ is a union of elements $\xi_{n+1}(x_0)$ (see Figure 1).

The following lemmas are easy to check.

LEMMA 2.2. *Let $n \ge 1$ be an integer and $y_1 \in S^1$. Set $y_2 = f^{q_{n-1}}(y_1)$, $y_3 = f^{q_{n-1}}(y_2)$, $\Delta_i^{(n-1)}(y_1) = f^i([y_1, y_2])$ and $\Delta_j^{(n-1)}(y_2) = f^j([y_2, y_3])$, $0 \le i, j < q_n$. Then $\Delta_i^{(n-1)}(y_1) \cap \Delta_j^{(n-1)}(y_2)$ has non-empty interior if and only if one of the following conditions holds:*

(1) *$j = i - q_{n-1}$ and $q_{n-1} \le i < q_n$; in this case we have*

$$\Delta_i^{(n-1)}(y_1) = \Delta_j^{(n-1)}(y_2);$$

(2) *$j = q_n - q_{n-1} + i$ and $0 \le i < q_{n-1}$; in this case we have*

$$\Delta_j^{(n-1)}(y_2) = f^{q_n}(\Delta_i^{(n-1)}(y_1)).$$

LEMMA 2.3. *Let $n \ge 1$ be an integer and $y_1 \in S^1$. Set $y_2 = f^{q_{n-1}}(y_1)$ and $y_3 = f^{q_{n-1}}(y_2)$. Then, for every $x \in S^1 \setminus O_f(y_1)$, there exists a unique $0 \le i = i_n(x) < q_n$ such that $x \in f^i(]y_1, y_2[) \cup f^{\varphi_n(i)}(]y_2, y_3[)$, where $\varphi_n : [0, q_n[\longrightarrow [0, q_n[$ is a bijective map given by*

$$\varphi_n(i) = \begin{cases} q_n - q_{n-1} + i & \text{if } 0 \le i < q_{n-1}, \\ i - q_{n-1} & \text{if } q_{n-1} \le i < q_n. \end{cases}$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

2.2. *Useful inequalities.* The proofs of the following results are based on the dynamical partition of S^1.

FINZI DISTORTION LEMMA. [**13**] *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number. For all $z \in S^1$, $x, y \in [z, f^{q_{n-1}}(z)]$ and for all integers $0 \le k < q_n$, we have*

$$e^{-V} \le \frac{\mathrm{Df}^k(x)}{\mathrm{Df}^k(y)} \le e^V.$$

Notice that the Finzi distortion lemma is used for the proof of Herman–Katok's theorem.

The following lemma plays a key role for studying metrical properties of the homeomorphism f.

DENJOY'S INEQUALITY. *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number. For all $x \in S^1$ and all $n \in \mathbb{N}$, we have*

$$e^{-V} \le \mathrm{Df}^{q_n}(x) \le e^V,$$

where $V = \mathrm{Var}(\log \mathrm{Df})$.

The proof of this inequality results from the Finzi distortion lemma. It can also be obtained by applying Denjoy–Koksma's inequality to the function log Df (cf. [**16**]).

Here and further on, by $m([a, b])$ we mean the length of the smaller one of the two arcs $[a, b]$ and $[b, a]$ on S^1.

Using Denjoy's inequality, one has the following result.

COROLLARY 2.4. (Geometric inequality) *There exists a constant $C > 0$ such that, for any $x_0 \in S^1$, $n \ge 1$ and any element $\Delta^{(n)}$ of the dynamical partition $\xi_n(x_0)$, we have $m(\Delta^{(n)}) \le C\lambda^n$, where $\lambda = (1 + e^{-V})^{-1/2} < 1$.*

From Corollary 2.4, it follows that every orbit of every $x \in S^1$ is dense in S^1 and this implies the following generalization of the classical Denjoy theorem.

COROLLARY 2.5. (Denjoy's theorem: the class P) *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number $\alpha = \rho(f)$. Then f is topologically conjugate to the rotation R_α.*

In the following lemma, we have to compare different iterates of intervals.

LEMMA 2.6. *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number $\alpha = \rho(f)$. Let $n \in \mathbb{N}^*$ and $z_1 \in S^1$. Set $z_2 = f^{q_{n-1}}(z_1)$ and $z_3 = f^{q_{n-1}}(z_2)$. Then, for any segments $K_1, K_2 \subset [z_1, z_3]$, one has*

$$e^{-2V} \frac{m(K_1)}{m(K_2)} \le \frac{m(f^j(K_1))}{m(f^j(K_2))} \le e^{2V} \frac{m(K_1)}{m(K_2)}$$

for all $j = 0, \ldots, q_n$.

Proof. If $j = q_n$, Lemma 2.6 is a consequence of Denjoy's inequality. We suppose that $0 \le j < q_n$. Let $\mathcal{L}$ be the set of segments in $[z_1, z_2]$ or in $[z_2, z_3]$. If $K \in \mathcal{L}$, then $K \subset [z, f^{q_{n-1}}(z)]$ with $z = z_1$ or $z = z_2$.

Since

$$\frac{m(f^j(K))}{\mathrm{Df}^j(z_2)} = \int_K \frac{\mathrm{Df}^j(t)}{\mathrm{Df}^j(z_2)}\, dm(t),$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

by the Finzi distortion lemma applied to f on $[z,\ f^{q_{n-1}}(z)]$, we get

$$e^{-V} \le \frac{\mathrm{Df}^j(t)}{\mathrm{Df}^j(z_2)} \le e^V \quad \text{for every } t \in K.$$

By integrating this last inequality, we obtain

$$e^{-V} m(K) \le \frac{m(f^j(K))}{\mathrm{Df}^j(z_2)} \le e^V m(K). \tag{2.1}$$

Let $K_1, K_2 \subset [z_1, z_3]$ be two segments. We distinguish three cases.

(a) $K_1 \in \mathcal{L}$ and $K_2 \in \mathcal{L}$: in this case, the inequality (2.1) implies that

$$e^{-V} m(K_1) \le \frac{m(f^j(K_1))}{\mathrm{Df}^j(z_2)} \le e^V m(K_1)$$

and

$$e^{-V} m(K_2) \le \frac{m(f^j(K_2))}{\mathrm{Df}^j(z_2)} \le e^V m(K_2);$$

therefore,

$$e^{-2V} \frac{m(K_1)}{m(K_2)} \le \frac{m(f^j(K_1))}{m(f^j(K_2))} \le e^{2V} \frac{m(K_1)}{m(K_2)}.$$

(b) $K_1 \notin \mathcal{L}$ and $K_2 \in \mathcal{L}$: in this case, take $K_1 = K_1' \cup K_1''$, where $K_1' \subset [z_1, z_2]$ and $K_1'' \subset [z_2, z_3]$ are two segments; then $K_1' \in \mathcal{L}$ and $K_1'' \in \mathcal{L}$. By (a), we see that

$$e^{-2V} \frac{m(K_1')}{m(K_2)} \le \frac{m(f^j(K_1'))}{m(f^j(K_2))} \le e^{2V} \frac{m(K_1')}{m(K_2)}$$

and

$$e^{-2V} \frac{m(K_1'')}{m(K_2)} \le \frac{m(f^j(K_1''))}{m(f^j(K_2))} \le e^{2V} \frac{m(K_1'')}{m(K_2)}.$$

Adding these inequalities, we get

$$e^{-2V} \frac{m(K_1)}{m(K_2)} \le \frac{m(f^j(K_1))}{m(f^j(K_2))} \le e^{2V} \frac{m(K_1)}{m(K_2)}.$$

(c) $K_1 \notin \mathcal{L}$ and $K_2 \notin \mathcal{L}$: in this case, take $K_2 = K_2' \cup K_2''$, where $K_2' \subset [z_1, z_2]$ and $K_2'' \subset [z_2, z_3]$ are two segments. By applying (b) to K_1 and K_2' (respectively K_2''), we obtain the inequality of the lemma. □

2.3. *Some ratio tools.* Let us introduce the notion of ratio distortion with respect to a homeomorphism.

Definition 2.7. Let x_1, x_2, x_3 be real numbers such that $x_1 < x_2 < x_3$.

(i) The *ratio* of the triple (x_1, x_2, x_3) is the real number defined as

$$\mathrm{r}(x_1, x_2, x_3) = \frac{x_2 - x_1}{x_3 - x_2}.$$

(ii) The *ratio distortion* of the triple (x_1, x_2, x_3) with respect to a strictly increasing function $F : \mathbb{R} \longrightarrow \mathbb{R}$ is the real number defined as

$$\mathrm{Dr}_F(x_1, x_2, x_3) = \frac{\mathrm{r}(F(x_1), F(x_2), F(x_3))}{r(x_1, x_2, x_3)} = \frac{F(x_2) - F(x_1)}{x_2 - x_1} \frac{x_3 - x_2}{F(x_3) - F(x_2)}.$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

Notice that:

- the ratio r is translation invariant:
$$\mathrm{r}(x_1 + a, x_2 + a, x_3 + a) = \mathrm{r}(x_1, x_2, x_3), \quad a \in \mathbb{R};$$
- the ratio distortion Dr is multiplicative with respect to composition: for two functions F and G on $\mathbb{R}$, we have
$$\mathrm{Dr}_{G\circ F}(x_1, x_2, x_3) = \mathrm{Dr}_G(F(x_1), F(x_2), F(x_3))\,\mathrm{Dr}_F(x_1, x_2, x_3).$$

Let $z_1, z_2, z_3 \in S^1$ with lifts $\widehat{z_1}, \widehat{z_2}, \widehat{z_3}$ and $z_1 \prec z_2 \prec z_3 \prec z_1$ ordered on S^1. Define
$$\widetilde{z_i} := \begin{cases} \widehat{z_i} & \text{if } \widehat{z_1} < \widehat{z_i} < 1, \\ 1 + \widehat{z_i} & \text{if } 0 \le \widehat{z_i} < \widehat{z_1}, \end{cases}$$
where $i = 2, 3$. Obviously, we have $\widetilde{z_1} < \widetilde{z_2} < \widetilde{z_3}$. The vector $(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3}) \in \mathbb{R}^3$ is called the *lifted vector* of $(z_1, z_2, z_3) \in (S^1)^3$.

Consider now a homeomorphism f of S^1 with lift $\widehat{f}$. We define the *ratio distortion* of the triple (z_1, z_2, z_3), $z_1, z_2, z_3 \in S^1$, with respect to f by
$$\mathrm{Dr}_f(z_1, z_2, z_3) := \mathrm{Dr}_{\widehat{f}}(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3}),$$
where $(\widetilde{z_1}, \widetilde{z_2}, \widetilde{z_3})$ is the lifted vector of (z_1, z_2, z_3). This definition is independent of the lift of f, since the ratio is invariant by translation.

Definition 2.8. Let $R > 1$ be a real number and $x_0 \in S^1$. We say that a triple (z_1, z_2, z_3) of S^1 $(z_1 \prec z_2 \prec z_3)$ satisfies the conditions (a) and (b) for the point x_0 and the constant R if:

(a) $R^{-1} \le (m([z_2, z_3]))/(m([z_1, z_2])) \le R$;

(b) $\max_{1\le i\le 3} m([x_0, z_i]) \le R\, m([z_1, z_2])$.

We call two intervals in S^1 *R-comparable* if the ratio of their lengths is in $[R^{-1}, R]$.

2.4. *Reduction.* In this subsection, we will reduce any homeomorphism $f \in \mathcal{P}(S^1)$ with several break (or non-break) points to one with break points lying on different orbits.

Definition 2.9. Let $c \in C(f)$. A *maximal f-connection* of c is a segment $[f^{-p}(c), \ldots, f^q(c)] := \{f^s(c) : -p \le s \le q\}$ of the orbit $O_f(c)$ which contains all the break points of f contained on $O_f(c)$ and such that $f^{-p}(c)$ (respectively $f^q(c)$) is the first (respectively last) break point of f on $O_f(c)$.

We have the following properties:

- two break points of f are on the same maximal f-connection if and only if they are on the same orbit;
- two distinct maximal f-connections are disjoint.

Denote:

- $M_i(f) = [c_i, \ldots, f^{N_i}(c_i)](N_i \in \mathbb{N})$, the maximal f-connections of $c_i \in C(f)$ $(0 \le i \le p)$;
- $M(f) = \coprod_{i=0}^{p} M_i(f)$.

So, we have the decomposition $C(f) = \coprod_{i=0}^{p} C_i(f)$, where $C_i(f) = C(f) \cap M_i(f)$, $0 \le i \le p$. We also have
$$\prod_{d\in C_i(f)} \sigma_f(d) = \prod_{d\in M_i(f)} \sigma_f(d).$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

PROPOSITION 2.10. (See [**2**, Proposition 3.1]) *Let* $f \in \mathcal{P}(S^1)$ *with irrational rotation number and let* $(k_0, \ldots, k_p) \in \mathbb{Z}^{p+1}$. *Then there exists a piecewise quadratic homeomorphism* $K \in \mathcal{P}(S^1)$ *such that* $F := K \circ f \circ K^{-1} \in \mathcal{P}(S^1)$ *with* $C(F) \subset \{K(f^{k_i}(c_i)) = F^{k_i}(K(c_i)), i = 0, 1, \ldots, p\}$, *and such that* $\sigma_F(F^{k_i}(K(c_i))) = \pi_{s, O_f(c_i)}(f)$, $i = 0, 1, \ldots, p$.

COROLLARY 2.11. *Let* $f \in \mathcal{P}(S^1)$ *with irrational rotation number. Suppose that* f *satisfies the (KO) condition. Then there exists a piecewise quadratic homeomorphism* $K \in \mathcal{P}(S^1)$ *such that* $F = K \circ f \circ K^{-1} \in \mathcal{P}(S^1)$ *with* $C(F) \subset \{K(c_0), \ldots, K(c_p)\}$, *where* $c_0, \ldots, c_p \in C(f)$ *are on pairwise distinct orbits.*

Proof. Take $k_i = 0$ for all i in Proposition 2.10. So, we get $F = K \circ f \circ K^{-1} \in \mathcal{P}(S^1)$ with $C(F) \subset \{K(c_0), \ldots, K(c_p)\}$, where $c_0, \ldots, c_p \in C(f)$ are on pairwise distinct orbits. □

In particular, we have the following result.

COROLLARY 2.12. *Let* f, K *and* F *be as in Corollary 2.11.*

(i) *If* f *does not satisfy the (D)-property, then there exists* $0 \le i \le p$ *such that* $K(c_i)$ *is the unique break point of* F *in its orbit.*

(ii) *If* f *satisfies the (D)-property, then* F *is a* C^1*-diffeomorphism with* DF *absolutely continuous on* S^1.

PROPOSITION 2.13. *Let* $f, g \in \mathcal{P}(S^1)$ *have the same irrational rotation number such that the break points of* f *(respectively* g*) belong to pairwise distinct* f*-orbits (respectively* g*-orbits) and let* h *be the conjugating homeomorphism from* f *to* g. *Then there exist an integer* $q \ge p$, $c_0, c_1, \ldots, c_q$ *and* $d_0, d_1, \ldots, d_q \in S^1$ *belonging to pairwise distinct* f*-orbits and* g*-orbits respectively such that* $C(f) \subset \{c_0, c_1, \ldots, c_q\}$ *and* $C(g) \subset \{d_0, d_1, \ldots, d_q\}$ *and a homeomorphism* u *of* S^1 *such that* $G := u \circ f \circ u^{-1} \in \mathcal{P}(S^1)$ *has break points belonging to pairwise distinct* G*-orbits and satisfying:*

(i) $C(G) \subset \{u(c_i), i = 0, 1, \ldots, q\}$;

(ii) $\sigma_G(u(c_i)) = \sigma_g(d_i)$, $i = 0, 1, \ldots, q$;

(iii) $\pi_s(G) = \pi_s(g)$;

(iv) h *is singular if and only if so is* u.

Proof. Add to $C(f)$ all points of $h^{-1}(C(g))$ that are not in some singular f-orbit and add to $C(g)$ all points of $h(C(f))$ that are not in some singular g-orbit, that is, set

$$\{c_0, \ldots, c_q\} := C(f) \cup \left(h^{-1}(C(g)) \setminus \bigcup_{c \in C(f)} O_f(c) \right)$$

and

$$\{d_0, \ldots, d_q\} := C(g) \cup \left(h(C(f)) \setminus \bigcup_{d \in C(g)} O_g(d) \right).$$

Then:

- $C(f) \subset \{c_0, \ldots, c_q\}$ and $C(g) \subset \{d_0, \ldots, d_q\}$;
- the points $c_0, \ldots, c_q$ belong to pairwise distinct f-orbits;

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

- the points $d_0, \ldots, d_q$ belong to pairwise distinct g-orbits;
- for every $i = 0, \ldots, q$, there exists a unique integer $k_i \in \mathbb{Z}$ such that $h(c_i) = g^{k_i}(d_i)$.

By Proposition 2.10, there exists a piecewise quadratic homeomorphism $K \in \mathcal{P}(S^1)$ such that $G := K \circ g \circ K^{-1} \in \mathcal{P}(S^1)$ with $C(G) \subset \{K(g^{k_i}(d_i)), i = 0, 1, \ldots, q\}$ such that $\sigma_G(K(g^{k_i}(d_i))) = \sigma_g(d_i)$, $i = 0, 1, \ldots, q$. Moreover, the break points of G belong to pairwise distinct G-orbits. One has

$$\pi_s(G) = \prod_{i=0}^{q} \sigma_G(K(g^{k_i}(d_i))) = \prod_{i=0}^{q} \sigma_g(d_i) = \pi_s(g).$$

Set $u = K \circ h$; thus, $u \circ f = G \circ u$; moreover, $\sigma_G(u(c_i)) = \sigma_g(d_i)$, $i = 0, 1, \ldots, q$, and $u(c_i) = K(h(c_i)) = K(g^{k_i}(d_i))$. Hence, $C(G) \subset \{u(c_0), \ldots, u(c_q)\}$. As $K \in \mathcal{P}(S^1)$ and $u = K \circ h$, h is singular if and only if so is u. □

From now on, we denote

$$B := \{c_0, \ldots, c_q\} \quad \text{and} \quad c_0 = c.$$

In what follows, we may assume that $f, g \in \mathcal{P}(S^1)$ have the same irrational rotation number α and satisfy the following conditions:

- the points of B belong to pairwise distinct f-orbits;
- the break points of g belong to pairwise distinct g-orbits;
- the maps f, g satisfy the Katznelson–Ornstein (KO) condition;
- $C(f) \subset B$;
- the conjugation h is such that $C(g) \subset h(B)$.

3. *Primary cells*

3.1. *Primary cell.* Let $x_0 \in S^1$, $\delta > 0$ be fixed and $U_\delta(x_0)$ be a δ-neighbourhood of x_0. Let $f \in \mathcal{P}(S^1)$ with irrational rotation number $\alpha = \rho(f)$. Let $c \in B$ and n be an odd integer. By Lemma 2.1, either $c \in \Delta_{i_n(c)}^{(n-1)}(x_0)$ for some $0 \le i_n(c) < q_n$ or $c \in \Delta_{i_n(c)}^{(n)}(x_0)$ for some $0 \le i_n(c) < q_{n-1}$. Set

$$y_2 = f^{-i_n(c)}(c), \quad y_1 = f^{-q_{n-1}}(y_2), \quad y_3 = f^{q_{n-1}}(y_2).$$

Notice that:

- y_1, y_2 and y_3 are defined with respect to f, x_0 and c;
- the number $i_n(c)$ depends on x_0.

PROPOSITION 3.1. *Under the notation above, there exists $N = N(x_0, \delta) \in \mathbb{N}$ such that, for all $n \ge N$, there is a triple $(y_1, y_2, y_3) = (y_1(n), y_2(n), y_3(n))_{n \ge N}$ with the following properties:*

(c-0) (y_1, y_2, y_3) *(respectively* $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3)))_{n \ge N} \subset U_\delta(x_0)$*;*

(c-1) $y_2 \in \Delta_0^{(n-1)}(x_0)$ *or* $y_2 \in \Delta_0^{(n)}(x_0)$*;*

(c-2) $m(f^j([y_1, y_3])) \le K\lambda^n$ *for every* $0 \le j < q_n$*, where* $\lambda = (1 + e^{-V})^{-1/2}$ *and* K *is a constant independent of* n*;*

(c-3) $\sum_{j=0}^{q_n - 1} m(f^j([y_1, y_3])) \le 2$*;*

(c-4) *the triples* (y_1, y_2, y_3) *and* $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))$ *satisfy conditions (a) and (b) of Definition 2.8 for* x_0 *and the constant* $R = e^{3V} + e^V + 1$*;*

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

(c-5)
- $f^i([y_1, y_3])$ *does not contain* c *in its interior for every* $0 \le i \ne i_n(c) < q_n$;
- $f^{i_n(c)}([y_1, y_3])$ *does not contain any break point of* f *other than* c.

Proof. The property (c-1) is obvious by construction. The property (c-2) holds by Corollary 2.4. Since $[y_1, y_2] = \Delta_0^{(n-1)}(y_1)$ and $[y_2, y_3] = \Delta_0^{(n-1)}(y_2)$, one has that $f^i([y_1, y_2])$ and $f^i([y_2, y_3])$ are elements of the partitions $\xi_n(y_1)$ and $\xi_n(y_2)$, respectively; hence, the first part of (c-5) is satisfied. For the second part of (c-5): choose an integer $n_0 \ge (\log(K^{-1}L_0))/(\log \lambda)$, where $L_0 = \min_{x,y\in B, x\ne y} m([x, y])$. Then, for every $n \in \mathbb{N}_0$, $n > n_0$, if there is $d \in B\backslash\{c\}$ such that $d \in f^{i_n(c)}([y_1, y_3])$, then, by (c-2), one has $m(f^{i_n(c)}([y_1, y_3])) \le K\lambda^n < K\lambda^{n_0} \le L_0 \le m([c, d])$, which is a contradiction. The property (c-3) is satisfied since

$$\sum_{j=0}^{q_n-1} m(f^j([y_1, y_3])) = \sum_{j=0}^{q_n-1} m(f^j([y_1, y_2])) + \sum_{j=0}^{q_n-1} m(f^j([y_2, y_3])) \le 2.$$

Let us show the property (c-4): by Denjoy's inequality, one has $e^{-V} \le (m([y_2, y_3]))/(m([y_1, y_2])) \le e^V$. On the other hand,

$$\max_{1\le i\le 3} m([x_0, y_i]) \le m([y_1, y_3]) \le (1 + e^V)m([y_1, y_2]).$$

Hence, (y_1, y_2, y_3) satisfies conditions (a) and (b) for $R_1 = e^V + 1$. For the triple $f^{q_n}(y_1, y_2, y_3)$, Lemma 2.6 yields

$$e^{-2V}\frac{m([y_2, y_3])}{m([y_1, y_2])} \le \frac{m(f^{q_n}([y_2, y_3]))}{m(f^{q_n}([y_1, y_2]))} \le e^{2V}\frac{m([y_2, y_3])}{m([y_1, y_2])}.$$

Thus,

$$e^{-3V} \le \frac{m(f^{q_n}([y_2, y_3]))}{m(f^{q_n}([y_1, y_2]))} \le e^{3V}.$$

Hence, $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))$ satisfies condition (a) for $R_2 = e^{3V}$. We will now check condition (b). We consider two cases.

- If $y_2 \in \Delta_0^{(n-1)}(x_0)$, then $x_0 \in f^{q_n}([y_1, y_3])$ since $f^{q_n}(y_3)$ lies to the right of y_2. Therefore,
$$\max_{1\le i\le 3} m([x_0, f^{q_n} y_i]) \le m(f^{q_n}([y_1, y_3])) \le (1 + e^V)m(f^{q_n}([y_1, y_2])).$$
- If $y_2 \in \Delta_0^{(n)}(x_0)$, then either $x_0 \in f^{q_n}([y_1, y_3]))$, in which case one has as above
$$\max_{1\le i\le 3} m([x_0, f^{q_n} y_i]) \le (1 + e^V)m(f^{q_n}([y_1, y_2])),$$
or x_0 is outside $f^{q_n}([y_1, y_3])$; then $f^{q_n}(y_3)$ lies to the left of x_0. Thus,
$$\max_{1\le i\le 3} m([x_0, f^{q_n} y_i]) \le m(f^{q_n}([y_1, y_3])) + m([y_2, y_3]).$$

By Denjoy's inequality, one has $e^{-V} \le (m(f^{q_n}([y_1, y_2])))/(m([y_1, y_2])) \le e^V$, so

$$\begin{aligned}\max_{1\le i\le 3} m([x_0, f^{q_n} y_i]) &\le (1 + e^V)m(f^{q_n}([y_1, y_2])) + e^V m([y_1, y_2]) \\ &\le (1 + e^V + e^{2V})m(f^{q_n}([y_1, y_2])).\end{aligned}$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

Therefore, condition (b) with the constant $R = 1 + e^V + e^{2V}$ holds for $f^{q_n}(y_1, y_2, y_3)$. In conclusion, conditions (a) and (b) with the constant $R = 1 + e^V + e^{3V}$ hold for (y_1, y_2, y_3) and $f^{q_n}(y_1, y_2, y_3)$. It remains to show the property (c-0): since $x_0 \in [y_1, y_3]$, it follows by (c-2) that there exists $N = N(x_0, \delta) \in \mathbb{N}$, $N > n_0$, such that $(y_1, y_2, y_3) = (y_1(n), y_2(n), y_3(n))_{n\geq N} \subset U_\delta(x_0)$ and $(f^{q_n}(y_1), f^{q_n}(y_2), f^{q_n}(y_3))_{n\geq N} \subset U_\delta(x_0)$ since, by (c-4), we have

$$m([x_0, f^{q_n} y_i]) \leq Rm(f^{q_n}([y_1, y_2])) \leq Re^V m([y_1, y_2]).$$

The proof is complete. □

Definition. (Primary cell) The triple (y_1, y_2, y_3) given in Proposition 3.1 is called a *primary cell* associated to (f, x_0, c, δ, N) for the constant $R = e^{3V} + e^V + 1$.

In what follows, set $\mathbb{N}_0 = \{n \in \mathbb{N} : n \geq N\}$. Let $c \in B$, $d \notin O_f(c)$ and $(y_1, y_2, y_3) := (y_1(n), y_2(n), y_3(n))_{n\in\mathbb{N}_0}$ be the primary cell associated to (f, x_0, c, δ, N).

For every $n \in \mathbb{N}_0$ and, by Lemma 2.3, denote by:

- $i_n(c, d)$ the unique integer $0 \leq i_n(c, d) < q_n$ such that either $d \in f^{i_n(c,d)}([y_1, y_2])$ or $d \in f^{j_n(c,d)}([y_2, y_3])$, where $j_n(c, d) = \varphi_n(i_n(c, d))$.

Set

$$t_n(c, d) = \frac{m([f^{-i_n(c,d)}(d), y_2])}{m([f^{j_n(c,d)-i_n(c,d)}(y_2), y_2])}.$$

Write for short $t_n(c, d) := t_n(d)$, $i_n(c, d) := i_n(d)$ and $j_n(c, d) := j_n(d)$.

One always has $f^{-i_n(d)}(d) \in [f^{q_n} y_1, y_2]$; indeed, in the first case, $f^{-i_n(d)}(d) \in [y_1, y_2]$ and, in the second case, $f^{-i_n(d)}(d) \in [f^{q_n} y_1, f^{q_n} y_2] \subset [f^{q_n} y_1, y_2]$. One can write

$$t_n(d) = \begin{cases} \dfrac{m([f^{-i_n(d)}(d), y_2])}{m([y_1, y_2])} & \text{if } q_{n-1} \leq i_n(d) < q_n, \\ \dfrac{m([f^{-i_n(d)}(d), y_2])}{m([f^{q_n} y_1, y_2])} & \text{if } 0 \leq i_n(d) < q_{n-1}. \end{cases}$$

Let $d \in B\backslash\{c\}$ and A be an infinite subset of $\mathbb{N}_0$. Denote by:

- $T(d, A)$ the set of limit points of $(t_n(d))_{n\in A}$. Therefore, $T(d, A) \subset [0, 1]$.

We say that d satisfies:

- the $P_1(d, A)$-property if $T(d, A) \subset]0, 1[$ and, for every $n \in A$, $0 \leq i_n(d) < q_{n-1}$;
- the $P_2(d, A)$-property if $T(d, A) \subset]0, 1[$ and, for every $n \in A$, $q_{n-1} \leq i_n(d) < q_n$.

Set

$$Q_{11}(A) = \{d \in B\backslash\{c\} : d \text{ satisfies the } P_1(d, A)\text{-property}\},$$
$$Q_{12}(A) = \{d \in B\backslash\{c\} : d \text{ satisfies the } P_2(d, A)\text{-property}\},$$
$$Q_2(A) = \{d \in B\backslash\{c\} : T(d, A) = \{0\}\},$$
$$Q_3(A) = \{d \in B\backslash\{c\} : T(d, A) = \{1\}\}.$$

PROPOSITION 3.2. *There exists an infinite subset* $\mathbb{N}_1$ *of* $\mathbb{N}_0$ *such that* $Q_{11}(\mathbb{N}_1)$, $Q_{12}(\mathbb{N}_1)$, $Q_2(\mathbb{N}_1)$ *and* $Q_3(\mathbb{N}_1)$ *form a partition of* $B\backslash\{c\}$.

Proof. For A an infinite subset of $\mathbb{N}_0$, write $Q(A) := Q_{11}(A) \sqcup Q_{12}(A) \sqcup Q_2(A) \sqcup Q_3(A)$.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

- First we shall prove that there exists an infinite subset M_1 of $\mathbb{N}_0$ such that $c_1 \in Q(M_1)$: if $0 \in T(c_1, \mathbb{N}_0)$, there exists an infinite subset M_1 of $\mathbb{N}_0$ such that the sequence $(t_n(c_1))_{n\in M_1}$ converges to 0; that is, $c_1 \in Q_2(M_1)$.
  If $1 \in T(c_1, \mathbb{N}_0)$, there exists an infinite subset M_1 of $\mathbb{N}_0$ such that the sequence $(t_n(c_1))_{n\in M_1}$ converges to 1; that is, $c_1 \in Q_3(M_1)$.
  If $T(c_1, \mathbb{N}_0) \cap \{0, 1\} = \emptyset$, then $T(c_1, \mathbb{N}_0) \subset]0, 1[$. Since the subsets $M_{11} := \{n \in \mathbb{N}_0 : 0 \le i_n(c_1) < q_{n-1}\}$ and $M_{12} := \{n \in \mathbb{N}_0 : q_{n-1} \le i_n(c_1) < q_n\}$ form a partition of the infinite set $\mathbb{N}_0$, one of them is infinite. We conclude that $c_1 \in Q_{11}(M_1) \sqcup Q_{12}(M_1)$, where
  $$M_1 := \begin{cases} M_{11} & \text{if } M_{11} \text{ is infinite,} \\ M_{12} & \text{if } M_{11} \text{ is finite.} \end{cases}$$
- In the same way, there exists an infinite subset M_2 of M_1 such that $c_2 \in Q(M_2)$. As $Q(M_1) \subset Q(M_2)$, $\{c_1, c_2\} \subset Q(M_2)$. We continue in this fashion, obtaining a decreasing family $M_1, M_2, \ldots, M_p$ of p infinite subsets of $\mathbb{N}_0$ such that $\{c_1, c_2, \ldots, c_k\} \subset Q(M_k)$ for every $1 \le k \le p$. In particular, $B\backslash\{c\} := \{c_1, c_2, \ldots, c_p\} \subset Q(M_p)$ and so $B\backslash\{c\} = Q(M_p)$. Set $\mathbb{N}_1 := M_p$; we obtain $B\backslash\{c\} = Q_{11}(\mathbb{N}_1) \sqcup Q_{12}(\mathbb{N}_1) \sqcup Q_2(\mathbb{N}_1) \sqcup Q_3(\mathbb{N}_1)$. The proof is complete. □

In what follows, in view of Proposition 3.2, we write for simplicity $\mathbb{N}_0$ instead of $\mathbb{N}_1$. So, one has
$$B\backslash\{c\} = Q_{11}(\mathbb{N}_0) \sqcup Q_{12}(\mathbb{N}_0) \sqcup Q_2(\mathbb{N}_0) \sqcup Q_3(\mathbb{N}_0).$$

3.2. *The distinguished points of f.* For $a \in S^1$, $n \in \mathbb{N}$ and $\gamma > 0$, set:

- $l_n := m(\Delta^{(n)}(y_1))$;
- $V_{n,\gamma}^{(f)}(a) := [a - \gamma l_{n-1}, a + \gamma l_{n-1}]$;
- $k_n(d) = \begin{cases} i_n(d) & \text{if } d \in \{c\} \cup Q_2(\mathbb{N}_0), \\ j_n(d) & \text{if } d \in Q_3(\mathbb{N}_0). \end{cases}$

PROPOSITION 3.3. *Under the notation above, there exist an infinite subset M_2 of $\mathbb{N}_0$ and a positive constant $\alpha_0 > 0$ such that:*

(1) *for every $d \in \overline{E}(c) := \{c\} \cup Q_2(M_2) \cup Q_3(M_2)$ and every $0 < \gamma < e^{-2V}$, there exists $n_\gamma \in M_2$ such that for any $n \in M_2$, $n \ge n_\gamma$, there exists a unique integer $0 \le k_n(d) < q_n$ such that*
$$f^{-k_n(d)}(d) \in V_{n,\gamma}^{(f)}(y_2) \subset]y_1, y_3[;$$

(2) *for every $n \in M_2$, $\bigcup_{i=0}^{q_n-1} f^{-i}(B\backslash\overline{E}(c)) \subset S^1\backslash V_{n,\alpha_0}^{(f)}(y_2)$;*

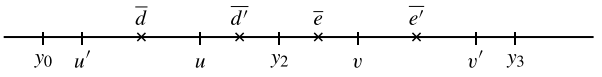
(3) *for every $d \in Q_{11}(M_2)$ and $n \in M_2$, we have $f^{-i_n(d)}(d) \in]f^{q_n}(y_1) + \alpha_0 l_{n-1}, y_2 - \alpha_0 l_{n-1}[$;*

(4) *for every $d \in Q_{12}(M_2)$ and $n \in M_2$, we have $f^{-j_n(d)}(d) \in]y_2 + \alpha_0 l_{n-1}, y_3 - \alpha_0 l_{n-1}[$.*

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

$$\overline{d} := f^{-i_n(d)}(d),\ \overline{d'} := f^{-i_n(d')}(d'),\ \overline{e} := f^{-j_n(e)},\ \overline{e'} := f^{-j_n(e')}(e')$$

$$d \in Q_{11}(M_0),\ d' \in Q_2(M_0),\ e \in Q_3(M_0),\ e' \in Q_{12}(M_0)$$



$$y_0 := f^{q_n}(y_1),\ u := y_2 - \alpha_0 l_{n-1},\ v := y_2 + \alpha_0 l_{n-1}$$

$$u' := y_0 + \alpha_0 l_{n-1},\ v' := y_3 - \alpha_0 l_{n-1}$$

FIGURE 2.

Proof. (See Figure 2.) Let $d \in B\backslash\{c\}$.

Assertion (1):

- The intervals $f^{i_n(d)}([y_1, y_2])$ and $f^{j_n(d)}([y_2, y_3])$ are comparable: by Denjoy's inequality, for every $n \in \mathbb{N}$,

$$e^{-V} \le \frac{m(f^{i_n(d)}([y_1, y_2]))}{m(f^{j_n(d)}([y_2, y_3]))} \le e^{V}.$$

  Those intervals are comparable to $[f^{j_n(d)}(y_2), f^{i_n(d)}(y_2)]$, which implies, by Lemma 2.6 and Denjoy's inequality, that the four intervals

$$[y_1, y_2], \quad [y_2, y_3], \quad [f^{j_n(d)-i_n(d)}(y_2), y_2], \quad [y_2, f^{i_n(d)-j_n(d)}(y_2)]$$

  are comparable. In particular, the ratios

$$\frac{m([f^{-i_n(d)}(d), y_2])}{m([y_1, y_2])} \quad \text{and} \quad \frac{m([f^{-i_n(d)}(d), y_2])}{m([f^{j_n(d)-i_n(d)}(y_2), y_2])} = t_n(d)$$

  are comparable.

  So, for $d \in Q_2(\mathbb{N}_0)$, $\lim_{n\to+\infty} t_n(d) = 0$, which is equivalent to the assertion

$$\text{for all } \gamma > 0 \quad \text{there exists } n_\gamma \in \mathbb{N}_0 \text{ such that, for every } n \in \mathbb{N}_0,$$
$$n \ge n_\gamma,\ f^{-i_n(d)}(d) \in V^{(f)}_{n,\gamma}(y_2).$$

- Similarly, the ratios $(m([y_2, f^{-j_n(d)}(d)]))/(m([y_1, y_2]))$ and $(m([y_2, f^{-j_n(d)}(d)]))/(m([y_2, f^{i_n(d)-j_n(d)}(y_2)]))$ are comparable, which, by Lemma 2.6, is comparable to the ratio

$$\frac{m([f^{j_n(d)-i_n(d)}(y_2), f^{-i_n(d)}(d)])}{m([f^{j_n(d)-i_n(d)}(y_2), y_2])} = 1 - t_n(d).$$

  So, for $d \in Q_3(\mathbb{N}_0)$, $\lim_{n\to+\infty} t_n(d) = 1$, which is equivalent to the assertion

$$\text{for all } \gamma > 0 \quad \text{there exists } n_\gamma \in \mathbb{N}_0 \text{ such that, for every } n \in \mathbb{N}_0,\ n \ge n_\gamma,$$
$$f^{-j_n(d)}(d) \in V^{(f)}_{n,\gamma}(y_2).$$

Assertion (2): Let $d \in Q_1(\mathbb{N}_0)$. Then there exist an infinite subset $M_2(d)$ of $\mathbb{N}_0$ and $0 < \alpha(d) < 1$ such that, for every $n \in M_2(d)$, $t_n(d) \ge \alpha(d)$ and $1 - t_n(d) \ge \alpha(d)$. Since $Q_1(\mathbb{N}_0)$ is finite, there exist an infinite subset M_2 of $\mathbb{N}_0$ and $0 < \alpha_0 < 1$ such that the points $f^{-i_n(d)}(d)$ and $f^{-j_n(d)}(d)$ are contained in the set $S^1\backslash V^{(f)}_{n,\alpha_0}(y_2)$ for all $d \in Q_1(M_2)$.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

By Lemma 2.3, for every $0 \le i < q_n$, $i \ne i_n(d)$, $j = \varphi_n(i)$, we have $d \notin f^i([y_1, y_2]) \cup f^j([y_2, y_3])$. Hence, for every $0 \le i < q_n$, $i \ne i_n(d)$, $f^{-i}(d) \notin [y_1, y_2]$ and for every $0 \le j < q_n$, $j \ne j_n(d) = \varphi_n(i_n(d))$, $f^{-j}(d) \notin [y_2, y_3]$. In particular, for every $0 \le i < q_n$, $i \ne i_n(d)$, $i \ne j_n(d)$, $f^{-i}(d) \notin [y_1, y_3]$. So, $f^{-i}(d) \notin V^{(f)}_{n,\alpha_0}(y_2)$ since $V^{(f)}_{n,\alpha_0}(y_2) \subset [y_1, y_3]$.

Then

$$\bigcup_{0 \le i < q_n} f^{-i}(B \backslash \overline{E}(c)) \subset S^1 \backslash V^{(f)}_{n,\alpha_0}(y_2).$$

Assertion (3): Let $d \in Q_{11}(\mathbb{N}_0)$. Then there exists an infinite subset $M_2(d)$ of $\mathbb{N}_0$ such that, for every $n \in M_2(d)$, $0 \le i_n(d) < q_{n-1}$,

$$f^{-i_n(d)}(d) \in [y_1, y_2] \cup f^{j_n(d)-i_n(d)}([y_2, y_3]) = [y_1, y_2] \cup [f^{q_n}(y_1), f^{q_n}(y_2)] = [f^{q_n}(y_1), y_2]$$

and

$$\alpha_0 \le t_n(d) = \frac{m([f^{-i_n(d)}(d), y_2])}{m([f^{q_n}(y_1), y_2])} \le 1 - \alpha_0.$$

Since the intervals $[f^{q_n}(y_1), y_2]$ and $[y_1, y_2]$ are comparable, there exists $0 < \alpha_1 < \alpha_0$ such that, for all $d \in Q_{11}(M_2(d))$ and $n \in M_2(d)$,

$$f^{-i_n(d)}(d) \in [f^{q_n}(y_1) + \alpha_1 l_{n-1}, y_2 - \alpha_1 l_{n-1}].$$

Since $Q_{11}(\mathbb{N}_0)$ is finite, there exists an infinite subset M_2 of $\mathbb{N}_0$ such that, for all $d \in Q_{11}(M_2)$ and $n \in M_2$,

$$f^{-i_n(d)}(d) \in [f^{q_n}(y_1) + \alpha_1 l_{n-1}, y_2 - \alpha_1 l_{n-1}].$$

Assertion (4): Let $d \in Q_{12}(\mathbb{N}_0)$. Then there exists an infinite subset $M_2(d)$ of $\mathbb{N}_0$ such that, for every $n \in M_2(d)$, $q_{n-1} \le i_n(d) < q_n$, $f^{-j_n(d)}(d) \in [y_2, y_3]$ and

$$\alpha_0 \le t_n(d) = \frac{m([f^{-i_n(d)}(d), y_2])}{m([y_1, y_2])} \le 1 - \alpha_0.$$

Since the ratios

$$\frac{m([f^{-j_n(d)}(d), y_3])}{m([y_2, y_3])} \quad \text{and} \quad \frac{m([f^{-i_n(d)}(d), y_2])}{m([y_1, y_2])}$$

are comparable, there exists $0 < \alpha_2 < \alpha_0$ such that $f^{-j_n(d)}(d) \in [y_2 + \alpha_2 l_{n-1}, y_3 - \alpha_2 l_{n-1}]$ for all $d \in Q_{12}(M_2(d))$. Since $Q_{12}(\mathbb{N}_0)$ is finite, there exists an infinite subset M_2 of $\mathbb{N}_0$ such that $f^{-j_n(d)}(d) \in [y_2 + \alpha_2 l_{n-1}, y_3 - \alpha_2 l_{n-1}]$ for all $d \in Q_{12}(M_2)$ and $n \in M_2$. □

PROPOSITION 3.4. *There exist an infinite subset M_3 of M_2, k points $c_1, c_2, \ldots, c_k \in Q_{11}(M_3)$, a partition $(\overline{E}(c_j))_{1 \le j \le k}$ of $Q_{11}(M_3)$ with $c_j \in \overline{E}(c_j)(1 \le j \le k)$ and a positive constant $0 < \beta_0 < \alpha_0$ such that, for every $1 \le j \le k$, we have:*

(1) *for every $d \in \overline{E}(c_j)$ and $\gamma > 0$, there exists $n_\gamma \in M_3$ such that for any $n \in M_3$, $n \ge n_\gamma$, there exists a unique integer $0 \le k_n(d) < q_{n-1}$ such that*

$$f^{-k_n(d)}(d) \in V^{(f)}_{n,\gamma}(f^{-i_n(c_j)}(c_j)) \subset]f^{q_n}(y_1) + \alpha_0\, l_{n-1}, y_2 - \alpha_0\, l_{n-1}[,$$

where $k_n(d) = i_n(d)$;

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

(2) *for every* $n \in M_3$, $\bigcup_{i=0}^{q_{n-1}-1} f^{-i}(B \setminus \overline{E}(c_j)) \subset S^1 \setminus V_{n,\beta_0}^{(f)}(f^{-i_n(c_j)}(c_j))$.

Proof. Assertion (1): Set $Q_{11}(M_2) = \{d_1, d_2, \ldots, d_m\}$. By Proposition 3.3(3), there exists $\alpha_0 > 0$ such that, for every $d \in Q_{11}(M_2)$ and $n \in M_2$, $f^{-i_n(d)}(d) \in]f^{q_n}(y_1) + \alpha_0\, l_{n-1},\ y_2 - \alpha_0\, l_{n-1}[$ (since $V_{n,\alpha_0}^{(f)}(y_2)$ does not contain $f^{-i_n(d)}(d)$ and $f^{-j_n(d)}(d)$).

For $d \in Q_{11}(M_2)$ and $n \in M_2$, set

$$t_n'(d_1, d) := \frac{m([f^{-i_n(d)}(d),\ f^{-i_n(d_1)}(d_1)])}{l_{n-1}}.$$

There exist an infinite subset $\mathbb{N}_2$ of M_2 and a positive constant γ_1 such that

$$Q_{11}(M_2) = \overline{E}(d_1) \amalg F_1,$$

where

$$\overline{E}(d_1) := \{d \in Q_{11}(M_2) : (t_n'(d_1, d))_{n\in\mathbb{N}_2} \text{ converges to } 0\}$$

and

$$F_1 := \{d \in Q_{11}(M_2) : t_n'(d_1, d) \geq \gamma_1,\ \forall n \in \mathbb{N}_2\}.$$

Write $\overline{E}(d_1) = \{d_1, \ldots, d_{m_1-1}\}$ and $F_1 = \{d_{m_1}, \ldots, d_m\}$ for some integer $2 \leq m_1 \leq m+1$.

For $d \in F_1$, set

$$t_n'(d_{m_1}, d) := \frac{m([f^{-i_n(d)}(d),\ f^{-i_n(d_{m_1})}(d_{m_1})])}{l_{n-1}},\ n \in \mathbb{N}_2.$$

There exist an infinite subset $\mathbb{N}_3$ of $\mathbb{N}_2$ and a positive constant $\gamma_2 < \gamma_1$ such that

$$F_1 = \overline{E}(d_{m_1}) \amalg F_2,$$

where

$$\overline{E}(d_{m_1}) := \{d \in F_1 : (t_n'(d_{m_1}, d))_{n\in\mathbb{N}_3} \text{ converges to } 0\}$$

and

$$F_2 := \{d \in F_1 : t_n'(m_1, d) \geq \gamma_2,\ \forall n \in \mathbb{N}_3\}.$$

Write $\overline{E}(d_{m_1}) = \{d_{m_1}, \ldots, d_{m_2-1}\}$ and $F_2 = \{d_{m_2}, \ldots, d_m\}$ for some integer $m_1 + 1 \leq m_2 \leq m$. Repeating this process enables us to conclude that there exist an infinite subset $\mathbb{N}_k$ of M_2, a positive constant $\gamma_k > 0$, k points $d_1, d_{m_1}, \ldots, d_{m_{k-1}}$ and a partition $(\overline{E}(d_{m_j}))_{j=0,1,\ldots,k-1}$ of $Q_{11}(M_2)$ (where $d_{m_0} = d_1$) such that, for every $j = 0, 1, \ldots, k-1$,

$$\overline{E}(d_{m_j}) = \{d \in Q_{11}(M_2) : (t_n'(d_{m_j}, d))_{n\in\mathbb{N}_k} \text{ converges to } 0\}$$

and

$$Q_{11}(M_2) \setminus \overline{E}(d_{m_j}) = \{d \in Q_{11}(M_2) : t_n'(d_{m_j}, d) \geq \gamma_k,\ \forall n \in \mathbb{N}_k\},$$

where

$$t_n'(d_{m_j}, d) := \frac{m([f^{-i_n(d)}(d),\ f^{-i_n(d_{m_j})}(d_{m_j})])}{l_{n-1}},\ n \in \mathbb{N}_k.$$

Here $c_i = d_{m_{i-1}}$, $i = 1, 2, \ldots, k$, and $M_3 = \mathbb{N}_k$. Set $\beta := \gamma_k$.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

Assertion (2): Let $d \in B\backslash\overline{E}(c_j)$, where $1 \le j \le k$. Let $0 \le i < q_{n-1}$ and $n \in M_3$.

• If $i \ne i_n(d)$, then $d \notin f^i([y_1, y_2]) \cup f^{\varphi_n(i)}([y_2, y_3])$, where $\varphi_n(i) = q_n - q_{n-1} + i$. Hence,

$$f^{-i}(d) \notin [y_1, y_2] \cup f^{q_n}([y_2, y_3]) = [f^{q_n}(y_1), y_2].$$

Since $V^{(f)}_{n,\beta}(f^{-i_n(c_j)}(c_j)) \subset [f^{q_n}(y_1), y_2]$, so $f^{-i}(d) \notin V^{(f)}_{n,\beta}(f^{-i_n(c_j)}(c_j))$.

• If $i = i_n(d)$, since $0 \le i_n(d) < q_{n-1}$, $d \in Q_{11}(M_3)$ or $d \in E(c)$. In the first case $t'_n(c_j, d) \ge \beta$, that is, $f^{-i}(d) \notin V^{(f)}_{n,\beta}(f^{-i_n(c_j)}(c_j))$. In the second case, for every $d \in E(c)$, by Assertion (1)(a) there exists $n(d) \in M_3$ such that, for every $n \ge n(d)$, $f^{-i_n(d)}(d) \in V^{(f)}_{n,\beta/4}(y_2)$. Since $E(c)$ is finite, there exists $n_0 \in M_3$ such that, for every $d \in E(c)$ and $n \in M_3$, $n \ge n_0$, we have $f^{-i_n(d)}(d) \in V^{(f)}_{n,\beta/4}(y_2)$. Since $f^{-i_n(d)}(d) \in V^{(f)}_{n,\beta/4}(y_2) \cap V^{(f)}_{n,\beta/4}(f^{-i_n(c_j)}(c_j)) = \emptyset$, $f^{-i_n(d)}(d) \notin V^{(f)}_{n,\beta/4}(f^{-i_n(c_j)}(c_j))$. We conclude that for every $1 \le j \le k$ and every $n \in \{n \in M_3, n \ge n_0\}$, we have

$$\bigcup_{i=0}^{q_{n-1}-1} f^{-i}(B\backslash E(c_j)) \subset S^1\backslash V^{(f)}_{n,\beta/4}(f^{-i_n(c_j)}(c_j)).$$

Set $\beta_0 := \min(\alpha_0, \beta/4)$. □

PROPOSITION 3.5. *There exist an infinite subset $\mathbb{M}_0$ of M_3, $c_{k+1}, c_{k+2}, \ldots, c_l \in Q_{12}(\mathbb{M}_0)$, a partition $(\overline{E}(c_i))_{k+1\le i\le l}$ of $Q_{12}(\mathbb{M}_0)$ with $c_i \in \overline{E}(c_i)(k+1 \le i \le l)$ and a positive constant $\gamma_0 < \beta_0$ such that, for every $k+1 \le i \le l$, we have:*

(1) *for every $d \in \overline{E}(c_i)$ and $\gamma > 0$, there exists $n_\gamma \in \mathbb{M}_0$ such that for any $n \in \mathbb{M}_0$, $n \ge n_\gamma$, there exists a unique integer $0 \le k_n(d) < q_n - q_{n-1}$ such that*

$$f^{-k_n(d)}(d) \in V^{(f)}_{n,\gamma}(f^{-j_n(c_i)}(c_i)) \subset]y_2 + \alpha_0\, l_{n-1},\ y_3 - \alpha_0\, l_{n-1}[,$$

where $k_n(d) = j_n(d)$;

(2) *for every $n \in \mathbb{M}_0$, $\bigcup_{j=0}^{q_n-q_{n-1}-1} f^{-j}(B\backslash\overline{E}(c_i)) \subset S^1\backslash V^{(f)}_{n,\gamma_0}(f^{-j_n(c_i)}(c_i))$.*

Proof. By Proposition 3.3(4), there exists $\gamma_0 < \beta_0$ such that, for every $d \in Q_{12}(M_3)$ and $n \in M_3$, $f^{-j_n(d)}(d) \in]y_2 + \gamma_0\, l_{n-1},\ y_3 - \gamma_0\, l_{n-1}[$. The remaining proof is the same as in Proposition 3.4. □

Definition. (Distinguished points) In view of Propositions 3.3–3.5, the points c_j for $j = 0, \ldots, l$ $(0 \le l \le p)$ are called *distinguished points* of f.

We have the following partition:

$$B = \coprod_{0\le i\le l} \overline{E}(c_i).$$

Let $(y_1, y_2, y_3) \subset U_\delta(x_0)$ be the primary cell for the constant $R = e^{3V} + e^V + 1$ associated to $(f, x_0, c, \mathbb{M}_0, \delta)$ and let $\gamma_0 > 0$ be as in Proposition 3.5. Let b be a distinguished point of f and $n \in \mathbb{M}_0$. Set

$$y_1(b) = f^{-q_{n-1}}(y_2(b)), \quad y_3(b) = f^{q_{n-1}}(y_2(b)),$$

where

$$y_2(b) = \begin{cases} f^{-i_n(b)}(b) & \text{if } b \in \{c, c_1, \ldots, c_k\}, \\ f^{-j_n(b)}(b) & \text{if } b \in \{c_{k+1}, \ldots, c_l\}, \end{cases}$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

where $j_n(b) = \varphi_n(i_n(b))$.

The triple $(y_1(b), y_2(b), y_3(b)) \subset U_\delta(x_0)$ is called *the primary cell associated* to $(f, x_0, b, c, \mathbb{M}_0, \delta)$.

COROLLARY 3.6. *For every distinguished point b of f and every $d \in \overline{E}(b)$, we have*

$$\lim_{n\to+\infty, n\in\mathbb{M}_0} \frac{m([f^{-k_n(d)}(d), y_2(b)])}{m([y_1(b), y_2(b)])} = 0,$$

where

$$k_n(d) = \begin{cases} i_n(d) & \text{if } d \in \{c\} \cup Q_2(\mathbb{M}_0), \\ j_n(d) & \text{if } d \in Q_3(\mathbb{M}_0). \end{cases}$$

Proof. It is a consequence of Assertion (1) in Propositions 3.3–3.5. □

4. *Secondary cells*

We introduce in this section a new triple (z_1, z_2, z_3) in $U_\delta(x_0)$ depending on two real parameters β and γ. Let $\beta, \gamma \in]0, \gamma_0[$ $(\beta < \gamma \le R)$. For $n \in \mathbb{M}_0$ $(n \ge n_\beta)$ and b a distinguished point of f (i.e. $b \in \{c_0, \ldots, c_l\}$), we define the points $z_1, z_2, z_3 \in S^1$ $(z_1 \prec z_2 \prec z_3)$ as follows: $z_2 = y_2(b)$ and $z_1, z_3 \in V^{(f)}_{n,\gamma}(z_2) \setminus V^{(f)}_{n,\beta}(z_2)$.

Definition. (Secondary cell) The triple (z_1, z_2, z_3) is called a (β, γ)-secondary cell associated to $(f, x_0, b, \mathbb{M}_0, \delta)$.

Set

$$q_n(b) = \begin{cases} q_n & \text{if } b = c, \\ q_{n-1} & \text{if } b \in \{c_1, \ldots, c_k\}, \\ q_n - q_{n-1} & \text{if } b \in \{c_{k+1}, \ldots, c_l\}. \end{cases}$$

PROPOSITION 4.1. *Under the notation above, for every $n \in \mathbb{M}_0$, $n \ge n_\beta$, we have:*

(b-0) $[z_1, z_3] \subset V^{(f)}_{n,\gamma}(y_2(b)) \subset [y_1(b), y_3(b)]$;

(b-1) *for any $d \in \overline{E}(b)$, there exists a unique $0 \le k_n(d) < q_n(b)$ such that $f^{-k_n(d)}(d) \in V^{(f)}_{n,\beta}(y_2(b))$;*

(b-2) $m(f^j([z_1, z_3])) \le K\lambda^n$ *for every $0 \le j < q_n(b)$, where $\lambda = (1 + e^{-V})^{-1/2}$ and K is a constant independent of n, $V = \mathrm{Var}(\log(\mathrm{Df}))$;*

(b-3) $\sum_{j=0}^{q_n(b)-1} m(f^j([z_1, z_3])) \le 2$;

(b-4) *the triples (z_1, z_2, z_3) and $(f^{q_n(b)}(z_1), f^{q_n(b)}(z_2), f^{q_n(b)}(z_3))$ satisfy conditions (a) and (b) of Definition 2.8 for x_0 and the constant $R(\beta) = (Re^{2V})/\beta$, where $R = e^{3V} + e^V + 1$;*

(b-5) *$f^i([z_1, z_3])$ does not contain any point in B for every $0 \le i < q_n(b)$ and $i \ne k_n(d)$, $d \in \overline{E}(b)$;*

(b-6) *for every $d \in \overline{E}(b)$, $d \in f^{k_n(d)}(]z_1, z_3[)$.*

Proof.

(b-0): This results from the definition of (z_1, z_2, z_3) for γ small.

(b-1): This follows from Assertion (1) in Propositions 3.3–3.5.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

(b-2) and (b-3) hold since $[z_1, z_3] \subset [y_1(b), y_3(b)]$ and by applying the same proof to $(y_1(b), y_2(b), y_3(b))$ instead of (y_1, y_2, y_3) of the properties (c-2) and (c-3) of Proposition 3.1.

(b-4): We have

$$\frac{\beta}{\gamma} = \frac{\beta m([y_1(b), y_2(b)])}{\gamma m([y_1(b), y_2(b)])} \le \frac{m([z_2, z_3])}{m([z_1, z_2])} \le \frac{\gamma m([y_1(b), y_2(b)])}{\beta m([y_1(b), y_2(b)])} = \frac{\gamma}{\beta};$$

then, by Lemma 2.6, we get

$$\begin{aligned} e^{-2V}\frac{\beta}{\gamma} &= e^{-2V}\frac{\beta m([y_1(b), y_2(b)])}{\gamma m([y_1(b), y_2(b)])} \\ &\le e^{-2V}\frac{m([z_2, z_3])}{m([z_1, z_2])} \le \frac{m(f^{q_n}([z_2, z_3]))}{m(f^{q_n}([z_1, z_2]))} \\ &\le e^{2V}\frac{m([z_2, z_3])}{m([z_1, z_2])} \\ &\le e^{2V}\frac{\gamma m([y_1(b), y_2(b)])}{\beta m([y_1(b), y_2(b)])} = e^{2V}\frac{\gamma}{\beta}. \end{aligned}$$

Hence, (z_1, z_2, z_3) satisfies conditions (a) and (b) of Definition 2.8 for the constant $R(\beta, \gamma) = R/\beta e^{2V}$. On the other hand, since $[z_1, z_3] \subset [f^{q_n}(y_1(b)), y_3(b)]$, $x_0 \in [f^{q_n}(y_1(b)), y_3(b)]$ and $\max(m([f^{q_n}(y_1(b)), x_0]), m([x_0, y_3(b)])) \le Rm([y_1(b), y_2(b)])$, it follows that

$$\begin{aligned} \max_{1\le i\le 3} m([x_0, z_i]) &\le \max(m([f^{q_n}(y_1(b)), x_0]); m([x_0, y_3(b)])) \\ &\le Rm([y_1(b), y_2(b)]) \le \frac{R}{\beta}m([z_1, z_2]). \end{aligned}$$

Furthermore, as

$$[z_1, z_3] \subset \begin{cases}]y_1(b), y_3(b)[& \text{if } b = c, \\ ]f^{q_n}(y_1(b)), y_2(b)[& \text{if } b \in \{c_1, c_2, \dots, c_k\}, \\ ]y_2(b), y_3(b)[& \text{if } b \in \{c_{k+1}, \dots, c_l\}, \end{cases}$$

$$f^{q_n(b)}([z_1, z_3]) \subset \begin{cases} f^{q_n}(]y_1(b), y_3(b)[) & \text{if } b = c, \\ f^{q_n-1}(]f^{q_n}(y_1(b)), y_2(b)[) & \text{if } b \in \{c_1, c_2, \dots, c_k\}, \\ f^{q_n-q_{n-1}}(]y_2(b), y_3(b)[) & \text{if } b \in \{c_{k+1}, \dots, c_l\}. \end{cases}$$

Since $x_0 \in [f^{q_n}(y_1(b)), y_3(b)]$ and

$$\begin{aligned} &f^{q_n}(]y_1(b), y_3(b)[) \cup f^{q_{n-1}}(]f^{q_n}(y_1(b)), y_2(b)[) \\ &\quad \cup f^{q_n-q_{n-1}}(]y_2(b), y_3(b)[) \subset [f^{q_n}(y_1(b)), y_3(b)], \end{aligned}$$

we get

$$\begin{aligned} \max_{1\le i\le 3} m([f^{q_n(b)}(z_i), x_0]) &\le \max(m([f^{q_n}(y_1(b)), x_0]); m([x_0, y_3(b)])) \\ &\le \frac{R}{\beta}m([z_1, z_2]) \\ &\le \frac{R}{\beta}e^{2V}m(f^{q_n(b)}([z_1, z_2])) \quad \text{(by Denjoy's inequality).} \end{aligned}$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

Therefore, $(f^{q_n(b)}(z_1), f^{q_n(b)}(z_2), f^{q_n(b)}(z_3))$ satisfies conditions (a) and (b) of Definition 2.8 for the constant $R(\beta, \gamma) = (R/\beta)e^{2V}$.

(b-5): If $d \in B$ and $0 \le i < q_n(b)$ is an integer such that $d \in f^i([z_1, z_3])$, then $f^{-i}(d) \in [z_1, z_3] \subset V^{(f)}_{n,\gamma}(y_2(b)) \subset V^{(f)}_{n,\gamma_0}(y_2(b))$. According to Assertion (2) of Propositions 3.3–3.5, we obtain $f^{-i}(d) \notin \bigcup_{j=0}^{q_n(b)-1} f^{-j}(B\backslash\overline{E}(b))$, $0 \le i < q_n(b)$. In particular, $f^{-i}(d) \notin f^{-i}(B\backslash\overline{E}(b))$. Hence, $d \in \overline{E}(b)$ is such that $0 \le i < q_n(b)$ and $f^{-i}(d) \in V^{(f)}_{n,\gamma}(y_2(b))$. By Assertion (b-1), $i = k_n(d)$.

(b-6): Let $n \in \mathbb{M}_0$, $n \ge n_\beta$. By (b-1), $f^{-k_n(d)}(d) \in V^{(f)}_{n,\beta}(y_2(b)) = [z_2 - \beta l_{n-1}, z_2 + \beta l_{n-1}]$. Since $z_1 \prec z_2 - \beta l_{n-1} \prec z_2 \prec z_2 + \beta l_{n-1} \prec z_3$ and $y_2(b) = z_2$, we have $V^{(f)}_{n,\beta}(z_2) \subset]z_1, z_3[$. It follows that $f^{-k_n(d)}(d) \in]z_1, z_3[$. □

PROPOSITION 4.2. *Let $(y_1(b), y_2(b), y_3(b)) \subset U_\delta(x_0)$ be the primary cell and (z_1, z_2, z_3) an (β, γ)-secondary cell associated to $(f, x_0, b, \mathbb{M}_0, \delta)$. Let h be the homeomorphism conjugating f to g. Assume that it admits in x_0 a positive derivative* $\mathrm{Dh}(x_0) > 0$. *Then:*

(1) *$(h(y_1(b)), h(y_2(b)), h(y_3(b)))$ is a primary cell associated to $(g, h(x_0), h(b), \mathbb{M}_0)$;*

(2) *there is an integer $n_\beta \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \ge n_\beta$, $(h(z_1), h(z_2), h(z_3))$ is a $(\beta/2, 2\gamma)$-secondary cell associated to $(g, h(x_0), h(b), \mathbb{M}_0)$;*

(3) *there is an integer $n_\gamma \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \ge n_\gamma$,*

$$h(V^{(f)}_{n,\gamma}(y_2(b))) \subset V^{(g)}_{n,2\gamma}(h(y_2(b))) \subset h(V^{(f)}_{n,4\gamma}(y_2(b))).$$

Proof.

(1) is trivial.

(2) Write $\omega_0 = \mathrm{Dh}(x_0) > 0$. One has

$$\max_{i=1,2,3} m([x_0, y_i(b)]) \le Rm([y_1(b), y_2(b)]) \le RK\lambda^n$$

and

$$\max_{i=1,2,3} m([x_0, z_i]) \le Rm([y_1(b), y_2(b)]) \le RK\lambda^n;$$

then

$$\lim_{n\to+\infty} \frac{m([h(y_i(b)), h(x_0)])}{m([y_i(b), x_0])} = \lim_{n\to+\infty} \frac{m([h(z_i), h(x_0)])}{m([z_i, x_0])} = \omega_0 \quad (i = 1, 2, 3).$$

If $x_0 \in [y_1(b), y_2(b)]$,

$$\begin{aligned}
m([h(y_1(b)), h(y_2(b))]) &= m([h(y_1(b)), h(x_0)]) + m([h(x_0), h(y_2(b))]) \\
&= \omega_0 m([y_1(b), x_0]) + o(m([y_1(b), x_0])) \\
&\quad + \omega_0 m([x_0, y_2(b)]) + o(m([x_0, y_2(b)])) \\
&= \omega_0(m([y_1(b), x_0]) + m([x_0, y_2(b)])) + o(m([y_1(b), x_0])) \\
&\quad + o(m([x_0, y_2(b)])) \\
&= \omega_0 m([y_1(b), y_2(b)]) + o(m([y_1(b), x_0])) \\
&\quad + o(m([x_0, y_2(b)])) \\
&= \omega_0 m([y_1(b), y_2(b)]) + o(m([y_1(b), y_2(b)])) \quad \text{(by } (b\text{-}4)\text{)}.
\end{aligned}$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

If $x_0 \notin [y_1(b), y_2(b)]$, there exists $s \in \{-1, 1\}$ such that

$$\begin{aligned} m([h(y_1(b)), h(y_2(b))]) &= s(m([h(y_1(b)), h(x_0)]) - m([h(x_0), h(y_2(b))])) \\ &= s(\omega_0 m([y_1(b), x_0]) + o(m([y_1(b), x_0])) - \omega_0 m([x_0, y_2(b)]) \\ &\quad - o(m([x_0, y_2(b)]))) \\ &= s(\omega_0 m([y_1(b), x_0]) - \omega_0 m([x_0, y_2(b)])) + o(m([y_1(b), x_0])) \\ &\quad + o(m([x_0, y_2(b)])) \\ &= \omega_0 m([y_1(b), y_2(b)]) + o(m([y_1(b), x_0])) \\ &\quad + o(m([x_0, y_2(b)])) \\ &= \omega_0 m([y_1(b), y_2(b)]) + o(m([y_1(b), y_2(b)])) \quad \text{(by } (b\text{-4})\text{)}. \end{aligned}$$

Similarly, $m([h(z_1), h(z_2)]) = \omega_0 m([z_1, z_2]) + o(m([y_1(b), y_2(b)]))$. As a consequence, we get

$$\frac{m([h(z_1), h(z_2)])}{m([h(y_1(b)), h(y_2(b))])} = \frac{m([z_1, z_2])}{m([y_1(b), y_2(b)])} + \varepsilon_n \quad \text{where } \lim_{n\to+\infty} \varepsilon_n = 0.$$

Hence,

$$\lim_{n\to+\infty} \left(\frac{m([h(z_1), h(z_2)])}{m([h(y_1(b)), h(y_2(b))])} - \frac{m([z_1, z_2])}{m([y_1(b), y_2(b)])} \right) = 0. \tag{4.1}$$

Similarly,

$$\lim_{n\to+\infty} \left(\frac{m([h(z_2), h(z_3)])}{m([h(y_1(b)), h(y_2(b))])} - \frac{m([z_2, z_3])}{m([y_1(b), y_2(b)])} \right) = 0. \tag{4.2}$$

Since (z_1, z_2, z_3) is a (β, γ)-secondary cell associated to $(f, x_0, b, \mathbb{M}_0, \delta)$, there exists an integer $n'_\beta \in \mathbb{M}_0$ such that, for every $n \in \mathbb{M}_0$ $(n \geq n'_\beta)$, we have

$$z_2 = y_2(b), \quad \beta \leq \frac{m([z_1, z_2])}{m([y_1(b), y_2(b)])} \leq \gamma, \quad \beta \leq \frac{m([z_2, z_3])}{m([y_1(b), y_2(b)])} \leq \gamma.$$

By (4.1) and (4.2), there exists an integer $n_\beta \in \mathbb{M}_0$ $(n_\beta \geq n'_\beta)$ such that, for every $n \in \mathbb{M}_0$ $(n \geq n'_\beta)$,

$$h(z_2) = h(y_2(b)), \quad \frac{\beta}{2} \leq \frac{m([h(z_1), h(z_2)])}{m([h(y_1(b)), h(y_2(b))])} \leq 2\gamma, \quad \frac{\beta}{2} \leq \frac{m([h(z_2), h(z_3)])}{m([h(y_1(b)), h(y_2(b))])} \leq 2\gamma.$$

(3) Let $n \in \mathbb{M}_0$, $t_1 := y_2(b) - \gamma l_{n-1}$, $t_2 := y_2(b)$ and $t_3 := y_2(b) + \gamma l_{n-1}$, where $l_{n-1} = m(\Delta^{(n-1)}(y_1(b)))$. Then $(t_1, t_2, t_3)_{n\in\mathbb{M}_0}$ is a (γ, γ)-secondary cell associated to $(f, x_0, b, \mathbb{M}_0, \delta)$. By (2), $(h(t_1), h(t_2), h(t_3))_{n\in\mathbb{M}_0}$ is a $(\gamma/2, \gamma)$-secondary cell associated to $(g, h(x_0), h(b), \mathbb{M}_0)$. Since h is an orientation preserving continuous function on S^1, so is $h(V^{(f)}_{n,\gamma}(y_2(b))) = [h(t_1), h(t_3)]$. As the interval $V^{(g)}_{n,2\gamma}(h(y_2(b)))$ contain the points $h(t_1)$ and $h(t_3)$, $h(V^{(f)}_{n,\gamma}(y_2(b))) \subset V^{(g)}_{n,2\gamma}(h(y_2(b)))$. Since h^{-1} is the homeomorphism conjugating g to f and admits in $h(x_0)$ a positive derivative $Dh^{-1}(h(x_0)) = 1/(Dh(x_0)) > 0$, we can apply the last result to the primary cell $(h(y_1(b)), h(y_2(b)), h(y_3(b)))$ associated to $(g, h(x_0), h(b), \mathbb{M}_0)$ to obtain $V^{(g)}_{n,2\gamma}(h(y_2(b))) \subset h(V^{(f)}_{n,4\gamma}(y_2(b)))$. □

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

PROPOSITION 4.3. *Under the notation of Proposition 4.2, for every $n \in \mathbb{M}_0$, $n \geq n_\beta$, we have:*

(h(b)-0) $[h(z_1), h(z_3)] \subset V^{(g)}_{n,2\gamma}(h(y_2(b))) \subset [h(y_1(b)), h(y_3(b))]$;

(h(b)-1) *for any $d \in \overline{E}(b)$, $k_n(d)$ is the unique integer in $[0, q_n(b)[$ such that $g^{-k_n(d)}(h(d)) \in V^{(g)}_{n,\beta/2}(h(y_2(b)))$;*

(h(b)-2) $m(g^j([h(z_1), h(z_3)])) \leq K'(\lambda')^n$ *for every $0 \leq j < q_n(b)$, where $\lambda' = (1 + e^{-V'})^{-1/2}$ and K' is a constant independent of n, $V' = \mathrm{Var}(\log(Dg))$;*

(h(b)-3) $\sum_{j=0}^{q_n(b)-1} m(g^j([h(z_1), h(z_3)])) \leq 2$;

(h(b)-4) $g^i([h(z_1), h(z_3)])$ *does not contain any point of $h(B)$ for every $0 \leq i < q_n(b)$ and $i \neq k_n(d)$, $d \in \overline{E}(b)$;*

(h(b)-5) *for every distinguished point b of f and every $d \in \overline{E}(b)$,*

$$\lim_{n \to +\infty} \frac{m([g^{-k_n(d)}(h(d)), h(y_2(b))])}{m([h(y_1(b)), h(y_2(b))])} = 0;$$

(h(b)-6) *for every $d \in \overline{E}(b)$, $h(d) \in g^{k_n(d)}(]h(z_1), h(z_3)[)$.*

Proof. Assertion (h(b)-0) is a consequence of Proposition 4.2(3).

(h(b)-1): This follows from Assertion (1) in Propositions 3.3–3.5. Let $d \in \overline{E}(b)$; there exists $n_\beta \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \geq n_\beta$, there exists $0 \leq k_n(d) < q_n(b)$ such that $f^{-k_n(d)}(d) \in V^{(f)}_{n,\beta/4}(z_2)$; then $g^{-k_n(d)}(h(d)) = h(f^{-k_n(d)}(d)) \in h(V^{(f)}_{n,\beta/4}(z_2)) \subset V^{(g)}_{n,\beta/2}(h(z_2))$. Uniqueness: if $0 \leq i, j < q_n(b)$ satisfy $g^{-i}(h(d)), g^{-j}(h(d)) \in V^{(g)}_{n,\beta/2}(h(z_2))$, then $f^{-i}(d), f^{-j}(d) \in h^{-1}(V^{(g)}_{n,\beta/2}(h(z_2))) \subset V^{(f)}_{n,\beta}(z_2)$. Therefore, by Proposition 4.1(b-1), $i = j$.

(h(b)-2) and (h(b)-3): By Proposition 4.2(2), $(h(z_1), h(z_2), h(z_3))$ is a $(\beta/2, 2\gamma)$-secondary cell associated to $(g, h(x_0), h(b), \mathbb{M}_0)$. Therefore, the proof is similar to Proposition 4.1(b-2) and (b-3) by replacing f by g.

(h(b)-4): Let $0 \leq i < q_n(b)$ and assume that $g^i([h(z_1), h(z_3)])$ contains a point $d' = h(d)$ of g for a point d of B. Then $f^i([z_1, z_3])$ contains the point d of B. By Proposition 4.1(b-1) and (b-5), it follows that $d \in \overline{E}(b)$ and $i = k_n(d)$.

(h(b)-5): Let b be a distinguished point of f, $d \in \overline{E}(b)$ and $\gamma > 0$. By Proposition 3.1(b-6), there is an integer $n'_\gamma \in \mathbb{M}_0$ such that, for every $n \in \mathbb{M}_0$, $n \geq n'_\gamma$, $f^{-k_n(d)}(d) \in V^{(f)}(y_2(b))$. On another hand, by Proposition 4.2(3), there is an integer $n_\gamma \in \mathbb{M}_0$, $n_\gamma \geq n'_\gamma$ such that, for every $n \in \mathbb{M}_0$, $n \geq n_\gamma$: $h(V^{(f)}_{n,\gamma/2}(y_2(b))) \subset V^{(g)}_{n,\gamma}(h(y_2(b)))$. As $g^{-k_n(d)}(h(d)) = h(f^{-k_n(d)}(d))$, it follows that $g^{-k_n(d)}(h(d)) \in V^{(g)}_{n,\gamma}(h(y_2(b)))$ for every $n \geq n_\gamma$, $n \in \mathbb{M}_0$. Since $\gamma > 0$ is arbitrary, we get

$$\lim_{n \to +\infty} \frac{m([g^{-k_n(d)}(h(d)), h(y_2(b))])}{m([h(y_1(b)), h(y_2(b))])} = 0.$$

(h(b)-6): Let $n \in \mathbb{M}_0$, $n \geq n_\beta$.

By (h(b)-1), $g^{-k_n(d)}(h(d)) \in V^{(g)}_{n,\beta/2}(h(y_2(b))) = [h(z_2) - (\beta/2)l'_{n-1}, h(z_2) + (\beta/2)l'_{n-1}]$, where $l'_n = m(\Delta^{(n)})(h(y_1))$. Since $h(z_1) \prec h(z_2) - (\beta/1)l'_{n-1} \prec h(z_2) \prec h(z_2) + (\beta/2)l'_{n-1} \prec h(z_3)$ and $h(y_2(b)) = h(z_2)$, we have $V^{(g)}_{n,\beta/2}(h(z_2)) \subset]h(z_1), h(z_3)[$. It follows that $g^{-k_n(d)}(h(d)) \in]h(z_1), h(z_3)[$. □

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

5. *Control of distortions*

PROPOSITION 5.1. *Let $\mathbb{M}_0$ and γ_0 be as in Proposition 4.2. Let b be a distinguished point of f and β, $\gamma \in]0, \gamma_0[(\beta < \gamma)$. Assume that the conjugation homeomorphism h from f to g admits at x_0 a positive derivative $\mathrm{Dh}(x_0) = \omega_0 > 0$. Then there exists an integer $n_\beta \in \mathbb{M}_0$ such that, for every $n \in \mathbb{M}_0$, $n \geq n_\beta$, we have*

$$\left| \frac{\mathrm{Dr}_{g^{q_n(b)}}(h(z_1), h(z_2), h(z_3))}{\mathrm{Dr}_{f^{q_n(b)}}(z_1, z_2, z_3)} - 1 \right| \leq \frac{2}{1-\gamma_0}\beta.$$

We need the following lemma.

LEMMA 5.2. [**10**, Lemma 3] *Let h be a circle homeomorphism, $x_0 \in S^1$ and $R > 1$ a constant. If h admits at x_0 a positive derivative $\mathrm{Dh}(x_0) = \omega_0 > 0$, then there exists a constant $C > 0$ (dependent only on R and ω_0) such that, for any $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that, for any triple $(z_1, z_2, z_3) \subset U_\delta(x_0) := [x_0 - \delta, x_0 + \delta]$ $(z_1 \prec z_2 \prec z_3)$ satisfying the conditions (a) and (b) for the point x_0 and the constant R, one has*

$$|\mathrm{Dr}_h(z_1, z_2, z_3) - 1| \leq C\varepsilon.$$

Remark 2. One can show precisely that $C = (8R^2/\omega_0)$.

Proof of Proposition 5.1. Applying Lemma 5.2 to x_0 and $R(\beta) = R\beta^{-1}e^{2V} > 1$, there exists a constant $C_\beta = 8\omega_0^{-1}R^2(\beta)$ (see Remark 2) such that for $\varepsilon = \varepsilon(\beta) := (\omega_0/(8R^2e^{4V}))\beta^3$ there exists $\delta(\beta) > 0$ (dependent on β and x_0) such that for any triple $(t_1, t_2, t_3) \subset U_{\delta(\beta)}(x_0)^3$ $(t_1 \prec t_2 \prec t_3)$ satisfying conditions (a) and (b) for the point x_0 and the constant $R(\beta)$, we have

$$|\mathrm{Dr}_h(t_1, t_2, t_3) - 1| \leq C_\beta\varepsilon(\beta) = \beta.$$

Furthermore, by Proposition 4.1(b-4), the triples (z_1, z_2, z_3) and $(f^{q_n(b)}(z_1), f^{q_n(b)}(z_2), f^{q_n(b)}(z_3))$ satisfy conditions (a) and (b) for the point x_0 and the constant $R(\beta)$. For every $n \in \mathbb{M}_0$, we have

$$\max_{i=1,2,3} m([x_0, z_i]) \leq R(\beta)m([y_1, y_2]) \leq R(\beta)K\lambda^n$$

and

$$\max_{i=1,2,3} m([x_0, f^{q_n(b)}(z_i)]) \leq R(\beta)m([y_1, y_2]) \leq R(\beta)K\lambda^n.$$

As $\lim_{n\to+\infty} \lambda^n = 0$, there exists $n_\beta \in \mathbb{M}_0$ (dependent on β) such that, for every $n \in \mathbb{M}_0$, $n \geq n_\beta$, $U_{\delta(\beta)}(x_0)$ contains the intervals $[z_1, z_3]$ and $f^{q_n(b)}([z_1, z_3])$. In particular, for every $n \in \mathbb{M}_0$, $n \geq n_\beta$,

$$|\mathrm{Dr}_h(z_1, z_2, z_3) - 1| < \beta \quad \text{and} \quad |\mathrm{Dr}_h(f^{q_n(b)}(z_1), f^{q_n(b)}(z_2), f^{q_n(b)}(z_3)) - 1| < \beta.$$

Since $h \circ f^{q_n(b)} = g^{q_n(b)} \circ h$, we get

$$\begin{aligned} &\mathrm{r}(g^{q_n(b)}(h(z_1)), g^{q_n(b)}(h(z_2)), g^{q_n(b)}(h(z_3))) \\ &\quad = \mathrm{r}(h(f^{q_n(b)}(z_1)), h(f^{q_n(b)}(z_2)), h(f^{q_n(b)}(z_3))). \end{aligned}$$

Set

$$D_n := \frac{\mathrm{Dr}_{g^{q_n(b)}}(h(z_1), h(z_2), h(z_3))}{\mathrm{Dr}_{f^{q_n(b)}}(z_1, z_2, z_3)}.$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

One has

$$\begin{aligned} D_n &= \frac{\mathrm{r}(g^{q_n(b)}(h(z_1)),\, g^{q_n(b)}(h(z_2)),\, g^{q_n(b)}(h(z_3)))}{\mathrm{r}(h(z_1),\, h(z_2),\, h(z_3))} \\ &\quad \times \frac{\mathrm{r}(z_1,\, z_2,\, z_3)}{\mathrm{r}(f^{q_n(b)}(z_1),\, f^{q_n(b)}(z_2),\, f^{q_n(b)}(z_3))} \\ &= \frac{\mathrm{r}(h(f^{q_n(b)}(z_1)),\, h(f^{q_n(b)}(z_2)),\, h(f^{q_n(b)}(z_3)))}{\mathrm{r}(f^{q_n(b)}z_1,\, f^{q_n(b)}z_2,\, f^{q_n(b)}z_3)} \frac{\mathrm{r}(z_1,\, z_2,\, z_3)}{\mathrm{r}(h(z_1),\, h(z_2),\, h(z_3))} \\ &= \frac{\mathrm{Dr}_h(f^{q_n(b)}(z_1),\, f^{q_n(b)}(z_2),\, f^{q_n(b)}(z_3))}{\mathrm{Dr}_h(z_1,\, z_2,\, z_3)}. \end{aligned}$$

Therefore, $D_n \in [(1-\beta)/(1+\beta),\, (1+\beta)/(1-\beta)] \subset [1-(2\beta)/(1-\gamma_0),\, 1+(2\beta)/(1-\gamma_0)]$. This implies that for every $n \in \mathbb{M}_0$, $n \geq n_\beta$, $|D_n - 1| \leq 2/(1-\gamma_0)\beta$. □

The next proposition is an another distortion control which is opposite to the one of Proposition 5.1; this allows us to prove that the conjugation from f to g is singular with respect to the Lebesgue measure.

PROPOSITION 5.3. *Let $\mathbb{M}_0$ and γ_0 be as in Proposition 4.2. Let b be a distinguished point of f. Then, for any $\varepsilon > 0$, there exists $0 < \gamma < \gamma_0$ such that for any $\beta \in]0, \gamma[$ there exists $n_\beta \in \mathbb{M}_0$ such that for all $n > n_\beta$, $n \in \mathbb{M}_0$, the (β, γ)-secondary cell (z_1, z_2, z_3) associated to $(f, x_0, b, \mathbb{M}_0, \delta)$ satisfies the following inequality:*

$$\left| \frac{\mathrm{Dr}_{g^{q_n(b)}}(h(z_1), h(z_2), h(z_3))}{\mathrm{Dr}_{f^{q_n(b)}}(z_1, z_2, z_3)} - \Pi(b) \right| \leq A\varepsilon$$

for some constant $A > 0$, where

$$\Pi(b) = \prod_{d \in \overline{E}(b)} \frac{\sigma_g(h(d))}{\sigma_f(d)}.$$

We need two lemmas.

LEMMA 5.4. *Under the hypothesis of Proposition 5.3, for any $\varepsilon > 0$, there exists $0 < \gamma < \gamma_0$ such that for any $\beta \in]0, \gamma[$ there exists $n_\beta \in \mathbb{M}_0$ such that for any $n \in \mathbb{M}_0$, $n \geq n_\beta$, the (β, γ)-secondary cell (z_1, z_2, z_3) associated to $(f, x_0, b, \mathbb{M}_0, \delta)$ satisfies*

$$|\mathrm{Dr}_{f^{q_n(b)}}(z_1, z_2, z_3) - \nu(b)| \leq C_1\varepsilon,$$

where $\nu(b) = \prod_{d \in \overline{E}(b)} \sigma_f(d)$ and $C_1 > 0$ (depending on $V(f)$, $\|\mathrm{Df}\|_\infty$ and $\|\mathrm{Dg}\|_\infty$).

Proof. Let $\varepsilon > 0$. Since Df is absolutely continuous on every interval $[c_i, c_{i+1}] (0 \leq i \leq p)$, there exists a real $0 < \eta_0 < e^{2V}\gamma_0$ such that for every disjoint continuity interval $([a_j, b_j])_{j \in J}$ of Df (these latter are the $[c_i, c_{i+1}]$ $(0 \leq i \leq p)$) such that if $\sum_{j \in J} |a_j - b_j| < \eta_0$ then $\sum_{j \in J} |\mathrm{Df}(a_j) - \mathrm{Df}(b_j)| < \varepsilon$. Let $0 < \beta < \gamma < \frac{1}{5}e^{-2V}\eta_0$ and (z_1, z_2, z_3) be the (β, γ)-secondary cell associated to $(f, x_0, b, \mathbb{M}_0, \delta)$.

For $n \in \mathbb{M}_0$, denote the following:

- $I := \{0, 1, \ldots, q_n(b) - 1\} \backslash \{k_n(d) : d \in \overline{E}(b)\}$;
- $D_i(f) := \mathrm{Dr}_f(f^i(z_1), f^i(z_2), f^i(z_3))$, $0 \leq i < q_n(b)$;

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

- $P := \prod_{i\in I} D_i(f)$; $P_1 := \prod_{d\in\overline{E}(b)} D_{k_n(d)}(f)$.
  One has
$$\mathrm{Dr}_{f^{qn(b)}}(z_1, z_2, z_3) = PP_1.$$

Step 1. We consider $D_i(f)$, $i \in I$. By Proposition 4.1(b-5), Df is continuous on the interval $[f^i(z_1), f^i(z_3)]$. Then, by the mean value theorem, there exist $a_i \in]f^i(z_1), f^i(z_2)[$ and $b_i \in]f^i(z_2), f^i(z_3)[$ such that $D_i(f) = (\mathrm{Df}(a_i))/(\mathrm{Df}(b_i))$. Since $([a_i, f^i(z_2)])_{i\in I}$ are disjoint intervals of continuity of Df, by Lemma 2.6, they satisfy the inequalities
$$\begin{aligned}\sum_{i\in I} m([a_i, f^i(z_2)]) &\le \sum_{i\in I} m([f^i(z_1), f^i(z_2)])\\ &\le \sum_{i\in I} e^{2V}\frac{m([z_1, z_2])}{m([y_1(b), y_2(b)])} m([f^i(y_1(b)), f^i(y_2(b))])\\ &\le e^{2V}\gamma \sum_{i\in I} m([f^i(y_1(b)), f^i(y_2(b))])\\ &\le e^{2V}\gamma\\ &\le \eta_0.\end{aligned}$$
Consequently,
$$\sum_{i\in I} |\mathrm{Df}(f^i(z_2)) - \mathrm{Df}(a_i)| \le \varepsilon. \tag{5.1}$$

Also, $([f^i(z_2), b_i])_{i\in I}$ are disjoint intervals of continuity of Df, so, by Lemma 2.6, they satisfy
$$\begin{aligned}\sum_{i\in I} m([f^i(z_2), b_i]) &\le \sum_{i\in I} m([f^i(z_2), f^i(z_3)])\\ &\le \sum_{i\in I} e^{2V}\frac{m([z_2, z_3])}{m([y_1(b), y_2(b)])} m([f^i(y_1(b)), f^i(y_2(b))])\\ &\le e^{2V}\gamma \sum_{i\in I} m([f^i(y_1(b)), f^i(y_2(b))]) \le \eta_0.\end{aligned}$$
Consequently,
$$\sum_{i\in I} |\mathrm{Df}(f^i(z_2)) - \mathrm{Df}(b_i)| \le \varepsilon. \tag{5.2}$$

Combining (5.1) and (5.2), we get
$$\sum_{i\in I} |\mathrm{Df}(b_i) - \mathrm{Df}(a_i)| \le 2\varepsilon. \tag{5.3}$$

Set $D_i^*(f) := \max(D_i(f), (D_i(f))^{-1})$ and $P^* := \prod_{i\in I} D_i^*(f)$.
Since Df takes its values in $[m_1, M_1]$ ($m_1 > 0$), the following properties hold:
$$\begin{gathered}0 \le D_i^*(f) - 1 \le \frac{1}{m_1}|\mathrm{Df}(b_i) - \mathrm{Df}(a_i)|,\\ \sum_{i\in I} |D_i^*(f) - 1| \le \frac{2}{m_1}\varepsilon,\\ |\log D_i(f)| = \log D_i^*(f) \le D_i^*(f) - 1,\\ \frac{1}{P^*} \le P \le P^*.\end{gathered}$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

It follows that

$$e^{-C_0\varepsilon} \le P \le e^{C_0\varepsilon} \quad \text{where } C_0 = \frac{2}{m_1}. \tag{5.4}$$

Step 2. We consider $D_{k_n(d)}(f)$, $d \in \overline{E}(b)$. We have $d \in]f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_3)[$.

Case 2a. $d \in]f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)[$. By Proposition 4.1(b-5), Df is continuous on the intervals $[f^{k_n(d)}(z_1),\ d]$, $[d,\ f^{k_n(d)}(z_2)]$ and $[f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)]$; by the mean value theorem, there exist $t_1 \in]f^{k_n(d)}(z_1),\ d[$, $t_2 \in]d,\ f^{k_n(d)}(z_2)[$ and $t_3 \in]f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)[$ such that

$$D_{k_n(d)}(f) = (1 - r_n(d))\frac{\mathrm{Df}(t_1)}{\mathrm{Df}(t_3)} + r_n(d)\frac{\mathrm{Df}(t_2)}{\mathrm{Df}(t_3)},$$

where

$$r_n(d) = \frac{m([d,\ f^{k_n(d)}(z_2)])}{m([f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)])}.$$

We have

$$|D_{k_n(d)}(f) - \sigma_f(d)| \le \left|\frac{\mathrm{Df}(t_1)}{\mathrm{Df}(t_3)} - \frac{D_-f(d)}{D_+f(d)}\right| + r_n(d)\frac{|\mathrm{Df}(t_1) - \mathrm{Df}(t_2)|}{\mathrm{Df}(t_3)}.$$

Since

$$\left|\frac{\mathrm{Df}(t_1)}{\mathrm{Df}(t_3)} - \frac{D_-f(d)}{D_+f(d)}\right| \le \frac{M_1}{m_1^2}(|\mathrm{Df}(t_1) - D_-f(d)| + |\mathrm{Df}(t_3) - D_+f(d)|)$$

and, by Lemma 2.6,

$$\begin{aligned} r_n(d) &= \frac{m(f^{k_n(d)}([f^{-k_n(d)}(d),\, z_2]))}{m(f^{k_n(d)}([z_1,\, z_2]))} \\ &\le e^{2V}\frac{m([f^{-k_n(d)}(d),\, z_2])}{m([z_1,\, z_2])} \\ &\le e^{2V}\frac{m([y_1(b),\, y_2(b)])}{l([z_1,\, z_2])}\frac{m([f^{-k_n(d)}(d),\, z_2])}{m([y_1(b),\, y_2(b)])} \\ &\le \frac{e^{2V}}{\beta}t_n(b,\, d) \end{aligned}$$

with

$$t_n(b,\, d) := \frac{m([f^{-k_n(d)}(d),\, z_2])}{l([y_1(b),\, y_2(b)])}.$$

It follows that

$$|D_{k_n(d)}(f) - \sigma_f(d)| \le K_0\left(\frac{t_n(b,\, d)}{\beta} + |\mathrm{Df}(t_1) - D_-f(d)| + |\mathrm{Df}(t_3) - D_+f(d)|\right),$$

where

$$K_0 = \max\left(\frac{M_1}{m_1}e^{2V},\ \frac{M_1}{m_1^2}\right).$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

Since $\lim_{n\to+\infty} t_n(b, d) = 0$ (Corollary 3.6), there exists $n_{d,\beta} \in \mathbb{M}_0$ such that for any $n \in \mathbb{M}_0$, $n \geq n_{d,\beta}$, one has $t_n(b, d) < \beta\varepsilon$. The intervals $(]t_1, d[;]d, t_3[)_{d\in\overline{E}(b)}$ are disjoint intervals of continuity of Df; they satisfy, by Lemma 2.6,

$$\begin{aligned} m([t_1, d]) + m([d, t_3]) &\leq 2m([f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)]) + m([f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)]) \\ &\leq 3e^{2V}\gamma m([f^{k_n(d)}(y_1(b)),\ f^{k_n(d)}(y_2(b))]) \\ &\leq 3e^{2V}\gamma < \eta_0. \end{aligned}$$

So,

$$|\mathrm{Df}(t_1) - D_-f(d)| + |\mathrm{Df}(t_3) - D_+f(d)| < \varepsilon.$$

Hence, there exists $n_{d,\beta} \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \geq n_{d,\beta}$,

$$|D_{k_n(d)}(f) - \sigma_f(d)| \leq 2K_0\varepsilon. \tag{5.5}$$

Case 2b. $d \in]f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)[$. By Proposition 4.1(b-5), Df is continuous on the intervals $[f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)]$, $[f^{k_n(d)}(z_2), d]$ and $[d,\ f^{k_n(d)}(z_3)]$. By the mean value theorem, there exist $s_1 \in]d,\ f^{k_n(d)}(z_3)[$, $s_2 \in]f^{k_n(d)}(z_2), d[$ and $s_3 \in]f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)[$ such that

$$D^{-1}_{k_n(d)}(f) = (1 - r'_n(d))\frac{\mathrm{Df}(s_1)}{\mathrm{Df}(s_3)} + r'_n(d)\frac{\mathrm{Df}(s_2)}{\mathrm{Df}(s_3)},$$

where

$$r'_n(d) = \frac{m([f^{k_n(d)}(z_2), d])}{m([f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)])}.$$

As in Case 2a, we have

$$|D_{k_n(d)}(f)^{-1} - \sigma_f(d)^{-1}| \leq \left|\frac{\mathrm{Df}(s_1)}{\mathrm{Df}(s_3)} - \frac{D_+f(d)}{D_-f(d)}\right| + r'_n(d)\frac{|\mathrm{Df}(s_1) - \mathrm{Df}(s_2)|}{\mathrm{Df}(s_3)}.$$

Since

$$\left|\frac{\mathrm{Df}(s_1)}{\mathrm{Df}(s_3)} - \frac{D_+f(d)}{D_-f(d)}\right| \leq \frac{M_1}{m_1^2}(|\mathrm{Df}(s_1) - D_+f(d)| + |\mathrm{Df}(s_3) - D_-f(d)|)$$

and, by Lemma 2.6,

$$\begin{aligned} r'_n(d) &= \frac{m(f^{k_n(d)}([z_2,\ f^{-k_n(d)}(d)]))}{m(f^{k_n(d)}([z_2, z_3]))} \\ &\leq e^{2V}\frac{m([z_2,\ f^{-k_n(d)}(d)])}{m([z_2, z_3])} \\ &\leq e^{2V}\frac{m([y_1(b),\ y_2(b)])}{m([z_2, z_3])}\frac{m([z_2,\ f^{-k_n(d)}(d)])}{m([y_1(b),\ y_2(b)])} \\ &\leq \frac{e^{2V}}{\beta}t_n(b, d), \end{aligned}$$

where

$$t_n(b, d) := \frac{m([z_2,\ f^{-k_n(d)}(d)])}{m([y_1(b),\ y_2(b)])}.$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

It follows that

$$|D_{k_n(d)}(f)^{-1} - \sigma_f(d)^{-1}| \le K_0\left(\frac{t_n(b,d)}{\beta} + |\mathrm{Df}(s_1) - D_+f(d)| + |\mathrm{Df}(s_3) - D_-f(d)|\right),$$

where

$$K_0 = \max\left(\frac{M_1}{m_1}e^{2V}, \frac{M_1}{m_1^2}\right).$$

By Corollary 3.6, $\lim_{n\to+\infty} t_n(b,d) = 0$. Hence, there exists $n_{d,\beta} \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \ge n_{d,\beta}$, $t_n(b,d) < \beta\varepsilon$. The intervals $(]d, s_1[;]s_3, d[)_{d\in\overline{E}(b)}$ are disjoint intervals of continuity of Df; they satisfy, by Lemma 2.6,

$$\begin{aligned} m([d,s_1]) + m([s_3,d]) &\le m([f^{k_n(d)}(z_1),\ f^{k_n(d)}(z_2)]) + 2m([f^{k_n(d)}(z_2),\ f^{k_n(d)}(z_3)]) \\ &\le 3e^{2V}\gamma m([f^{k_n(d)}(y_1(b)),\ f^{k_n(d)}(y_2(b))]) \\ &\le 3e^{2V}\gamma < \eta_0. \end{aligned}$$

So,

$$|\mathrm{Df}(s_1) - D_+f(d)| + |\mathrm{Df}(s_3) - D_-f(d)| < \varepsilon.$$

Hence, there exists $n_{d,\beta} \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \ge n_{d,\beta}$,

$$|D_{k_n(d)}(f)^{-1} - \sigma_f(d)^{-1}| \le 2K_0\varepsilon. \tag{5.6}$$

Therefore, from the Cases 2a and 2b, we conclude that there exists $n_{d,\beta} \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \ge n_{d,\beta}$,

$$|D_{k_n(d)}(f) - \sigma_f(d)| \le K_0\varepsilon. \tag{5.7}$$

Since $\overline{E}(b)$ is finite, there exists $n_{b,\beta} \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \ge n_{b,\beta}$,

$$\left|\prod_{d\in\overline{E}(b)} D_{k_n(d)}(f) - \prod_{d\in\overline{E}(b)} \sigma_f(d)\right| \le \varepsilon. \tag{5.8}$$

Hence, (5.4) and (5.8) imply that there exists $n_{b,\beta} \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \ge n_{b,\beta}$,

$$|\mathrm{Dr}_{f^{q_n(b)}}(z_1,z_2,z_3) - \nu(b)| \le C_1\varepsilon, \tag{5.9}$$

where $\nu(b) = \prod_{d\in\overline{E}(b)} \sigma_f(d)$ and C_1 is a positive constant. □

LEMMA 5.5. *Under the hypothesis of Proposition 5.3, for any $\varepsilon > 0$, there exists $0 < \gamma < \gamma_0$ such that, for any $\beta \in]0,\gamma[$, there exists $n_\beta \in \mathbb{M}_0$ such that for any $n \in \mathbb{M}_0$, $n \ge n_\beta$, the (β,γ)-secondary cell (z_1,z_2,z_3) associated to $(f, x_0, b, \mathbb{M}_0, \delta)$ satisfies*

$$|\mathrm{Dr}_{g^{q_n(b)}}(h(z_1),h(z_2),h(z_3)) - \nu(h(b))| \le C_2\varepsilon,$$

where $\nu(h(b)) = \prod_{d\in\overline{E}(b)} \sigma_g(h(d))$ and C_2 is a positive constant.

Proof. The proof is a consequence of Lemma 5.4 and Proposition 4.3(h(b)-5) applied to the $(\beta/2,\gamma)$-secondary cell $(h(z_1),h(z_2),h(z_3))$ associated to $(g, h(x_0), h(b), \mathbb{M}_0)$ instead of the (β,γ)-secondary cell (z_1,z_2,z_3) associated to $(x_0, b, \mathbb{M}_0, \delta)$. □

Proof of Proposition 5.3. The proof results from Lemmas 5.4 and 5.5. □

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

6. *Proof of main theorem*

Let $f, g \in \mathcal{P}(S^1)$ with the same irrational rotation number α. Suppose that f and g satisfy the (KO) condition. By Corollary 2.11, there exist two piecewise quadratic homeomorphisms $K, L \in \mathcal{P}(S^1)$ such that $F = L \circ f \circ L^{-1}$ and $G = K \circ g \circ K^{-1}$ have the following properties:

(1) $F, G \in \mathcal{P}(S^1)$ and have the same irrational rotation number α;
(2) the break points of F (respectively G) are on *pairwise disjoint* F-orbits (respectively G-orbits);
(3) F and G satisfy the (KO) condition;
(4) let h be the conjugating map between f and g, i.e. $h \circ f = g \circ h$, and set $v = K \circ h \circ L^{-1}$. Then:
   - v is an absolutely continuous (respectively a singular) function if and only if so is h;
   - $v \circ F = G \circ v$.

Therefore, we may assume that all break points of f (respectively g) are on *pairwise disjoint* f-orbits (respectively g-orbits). Now, by Proposition 2.13, there exists a homeomorphism u of S^1 such that $G = u \circ f \circ u^{-1} \in \mathcal{P}(S^1)$ with the following properties:

- $C(f) \subset B := \{c_i : i = 0, 1, \ldots, q\}$ $(q \geq p)$;
- $C(G) \subset u(B) := \{u(c_i) : i = 0, 1, \ldots, q\}$;
- $\sigma_G(u(c_i)) = \sigma_g(d_i)$, $i = 0, 1, \ldots, q$;
- $\pi_s(G) = \pi_s(g)$;
- h is singular if and only if so is u.

So, we may assume that $u = h$ and $G = g$. Hence, we get

$$\frac{\pi_s(g)}{\pi_s(f)} = \prod_{i=0}^{q} \frac{\sigma_g(h(c_i))}{\sigma_f(c_i)}.$$

To show that the conjugation homeomorphism h from f to g is singular with respect to the Lebesgue measure m, it suffices to prove that its derivative Dh is zero on a set of Lebesgue total measure.

On the contrary: h admits at a point x_0 a positive derivative: $\mathrm{Dh}(x_0) = \omega_0 > 0$. Let $b \in \{c_0, c_1, \ldots, c_l\}$ be a distinguished point of f and $\varepsilon > 0$. For $0 < \beta < \gamma < \gamma_0$, write

$$D_n(b, \beta, \gamma) := \frac{\mathrm{Dr}_{g^{q_n(b)}}(h(z_1), h(z_2), h(z_3))}{\mathrm{Dr}_{f^{q_n(b)}}(z_1, z_2, z_3)}.$$

By Proposition 5.1, there exists $n_{b,\beta} \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \geq n_{b,\beta}$,

$$|D_n(b, \beta, \gamma) - 1| \leq \frac{2}{1 - \gamma_0}\beta. \tag{6.1}$$

By Proposition 5.3, there exists $\gamma_\varepsilon < \gamma_0$ such that for every $0 < \beta < \gamma_\varepsilon$ there exists $n_{b,\beta}(\varepsilon) \in \mathbb{M}_0$ such that for every $n \in \mathbb{M}_0$, $n \geq n_{b,\beta}(\varepsilon)$, the (β, γ)-secondary cell (z_1, z_2, z_3) $(\gamma < \gamma_\varepsilon)$ associated to $(f, x_0, b, \mathbb{M}_0, \delta)$ satisfies

$$|D_n(b, \beta, \gamma) - \Pi(b)| \leq A\varepsilon, \tag{6.2}$$

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

where

$$\Pi(b) := \prod_{d\in\overline{E}(b)} \frac{\sigma_g(h(d))}{\sigma_f(d)}$$

and A is a positive constant.

Let $\beta < \gamma < \gamma_\varepsilon$ and $n \in \mathbb{M}_0,\ n \geq n_{b,\beta}(\varepsilon)$. Then (6.1) and (6.2) imply that

$$|\Pi(b) - 1| \leq |D_n(b, \beta, \gamma) - 1| + |D_n(b, \beta, \gamma) - \Pi(b)| \leq \frac{2\beta}{1-\gamma_0} + A\varepsilon.$$

Since ε and β are arbitrary, it follows that

$$\Pi(b) = 1 \quad \text{for every } b \in \{c_0, c_1, \ldots, c_l\}.$$

Hence,

$$\prod_{d\in\overline{E}(b)} \frac{\sigma_g(h(d))}{\sigma_f(d)} = 1 \quad \text{for every } b \in \{c_0, c_1, \ldots, c_l\}. \tag{6.3}$$

As $B = \coprod_{0\leq i\leq l} \overline{E}(c_i)$ and $h(B) = \coprod_{0\leq i\leq l} h(\overline{E}(c_i))$ (by Proposition 2.13),

$$\frac{\pi_s(g)}{\pi_s(f)} = \prod_{i=0}^{q} \frac{\sigma_g(h(c_i))}{\sigma_f(c_i)} = \prod_{i=0}^{l} \prod_{d\in\overline{E}(c_i)} \frac{\sigma_g(h(d))}{\sigma_f(d)} = 1.$$

Thus, $\pi_s(g) = \pi_s(f)$, which is a contradiction. □

Acknowledgement. I thank the referee for his/her thorough review and highly appreciate the comments and suggestions, which significantly contributed to improving the quality of the paper.

References

[**1**] A. Adouani and H. Marzougui. Sur les homéomorphismes du cercle de classe PC^r par morceaux ($r \geq 1$) qui sont conjugués C^r par morceaux aux rotations irrationnelles. *Ann. Inst. Fourier (Grenoble)* **58** (2008), 755–775.

[**2**] A. Adouani and H. Marzougui. On piecewise smoothness of conjugacy of class P circle homeomorphisms to diffeomorphisms and rotations. *Dyn. Syst.* **27** (2012), 169–186.

[**3**] A. Adouani and H. Marzougui. Singular measures for class P-circle homeomorphisms with several break points. *Ergod. Th. & Dynam. Sys.* **34** (2014), 423–456.

[**4**] H. Akhadkulov, A. Dzhalilov and D. Mayer. On conjugations of circle homeomorphisms with two break points. *Ergod. Th. & Dynam. Sys.* **34** (2014), 725–741.

[**5**] Kh. A. Akhadkulov. Some circle homeomorphisms with break-type singularities. *Russian Math. Surveys* **61**(5) (2006), 981–983.

[**6**] W. de Melo and S. van Strien. *One-Dimensional Dynamics (Ergebnisse der Mathematik und ihrer Grenzgebiete, 25)*. Springer, Berlin, 1993.

[**7**] A. Denjoy. Sur les courbes définies par les équations differentielles à la surface du tore. *J. Math. Pures Appl.* (9) **11** (1932), 333–375.

[**8**] A. Dzhalilov. Piecewise smoothness of conjugate homeomorphisms of a circle with corners. *Theoret. Math. Phys.* **120** (1999), 961–972.

[**9**] A. Dzhalilov, H. Akin and S. Temir. Conjugations between circle maps with a single break point. *J. Math. Anal. Appl.* **366** (2010), 1–10.

[**10**] A. Dzhalilov and K. M. Khanin. On invariant measure for homeomorphisms of a circle with a break point. *Funct. Anal. Appl.* **32** (1998), 153–161.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press

[**11**] A. Dzhalilov, I. Liousse and D. Mayer. Singular measures of piecewise smooth circle homeomorphisms with two break points. *Discrete Contin. Dyn. Syst.* **24**(2) (2009), 381–403.
[**12**] A. Dzhalilov, D. Mayer and U. A. Safarov. Piecewise-smooth circle homeomorphisms with several break points. *Izv. Math.* **76** (2012), 94–112.
[**13**] A. Finzi. Sur le problème de la génération d'une transformation donnée d'une courbe fermée par une transformation infinitésimale. *Ann. Sci. Éc. Norm. Supér.* (3) **67** (1950), 243–305.
[**14**] B. Hasselblatt and A. Katok. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, Cambridge, 1995.
[**15**] J. Hawkins and K. Schmidt. On C^2-diffeomorphisms of the circle which are of type III1. *Invent. Math.* **66** (1982), 511–518.
[**16**] M. Herman. Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations. *Publ. Math. Inst. Hautes Études Sci.* **49** (1979), 5–234.
[**17**] E. Hewitt and K. Stromberg. *Real and Abstract Analysis (Graduate Texts in Mathematics, 25)*. Springer, Berlin, 1965.
[**18**] Y. Katznelson and D. Ornstein. The differentiability of the conjugation of certain diffeomorphisms of the circle. *Ergod. Th. & Dynam. Sys.* **9** (1989), 643–680.
[**19**] K. Khanin and A. Teplinsky. Renormalization horseshoe and rigidity theory for circle diffeomorphisms with breaks. *Comm. Math. Phys.* **320** (2013), 347–377.
[**20**] K. M. Khanin and D. Khmelev. Renormalizations and rigidity theory for circle homeomorphisms with singularities of the break type. *Comm. Math. Phys.* **235** (2003), 69–124.
[**21**] K. M. Khanin and Ya. G. Sinai. Smoothness of conjugacies of diffeomorphisms of the circle with rotations. *Russian Math. Surveys* **44** (1989), 69–99.
[**22**] K. M. Khanin and A. Teplinsky. Herman's theory revisited. *Invent. Math.* **178** (2009), 333–344.
[**23**] I. Liousse. Nombre de rotation, mesures invariantes et ratio set des homéomorphismes affines par morceaux du cercle. *Ann. Inst. Fourier (Grenoble)* **55** (2005), 1001–1052.
[**24**] A. Navas. *Groups of Circle Diffeomorphisms (Chicago Lectures in Mathematics)*. University of Chicago Press, Chicago, IL, 2011.
[**25**] H. Poincaré. Sur les courbes définies par les équations différentielles. *J. Math. Pures Appl.* **1** (1885), 167–244.
[**26**] Y. G. Sinai. *Topics in Ergodic Theory*. Princeton University Press, Princeton, NJ, 1994.
[**27**] J. Stark. Smooth conjugacy and renormalisation for diffeomorphisms of the circle. *Nonlinearity* **1** (1988), 541–575.
[**28**] J. C. Yoccoz. Il n'y a pas de contre-exemple de Denjoy analytique. *C. R. Math. Acad. Sci. Paris* I **298** (1984), 141–144.

https://doi.org/10.1017/etds.2015.32 Published online by Cambridge University Press