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FROOT AND STEIN REVISITED ONCE AGAIN

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ABSTRACT

Høgh, Linton and Nielsen (2006) showed that the famous result in the award winning paper of Froot and Stein (1998) is not correct in the sense that their result does not follow from their assumptions. In this paper we show that the economic intuition behind the paper of Froot and Stein (1998) is correct and that their result can be obtained when the market is reformulated in a continuous time setting and modern market theory is employed.

KEYWORDS

Financial Risk; CAPM Transaction Costs

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1. Introduction

Froot and Stein (1998) developed an optimisation theory for a company's exposure to market risk. Their result is remarkable: all companies owned by stock holders should hedge all the financial risk that can be hedged without transaction costs. Of course, no financial risk can be hedged without some transaction costs, however, the result is nevertheless remarkable and important as a guideline for how companies should act towards financial risk. The result is also intuitively appealing, because why should stockholders use a company for investment purposes when they can take care of their investments themselves. The underlying logic behind the result is based on the company model of Froot *et al.* (1993) that tells you that a company's business opportunities are decreasing as a function of capital with decreasing first derivative. Combining this insight with a standard CAPM model correlating the value of a financial cash flow with its correlation to the financial market, Froot and Stein (1998) produced their surprising results. When we write almost, it is because the result of Froot and Stein (1998) is incorrect in the sense that their main theorem does not follow from their assumptions. Høgh *et al.* (2006) derive the more complex and less intuitive result that actually follows from the assumption of Froot and Stein (1998).

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However, this paper shows that the economic intuition behind the result of Froot and Stein (1998) is correct and can be technically verified under a revised set of assumptions.

2. The Model Setup

We shortly resume that part of the economic model of Froot *et al.* (1993) and Froot and Stein (1998) that we will maintain for our purposes. We do not use the CAPM model used by Froot and Stein (1998) because the CAPM model only consistently describes an arbitrage free market when we either use a quadratic function instead of an utility function or when we have only normally distributed markets. However, we may question an economic utility function that has to be increasing and concave everywhere. Nor can we assume that the market is normally distributed both before and after a concave transformation as Froot and Stein (1998) implicitly assumed by using CAPM after a concave transformation. While some economic insight still can be extracted from the CAPM model, see Høgh *et al.* (2006), a much clearer message results if a modern approach to arbitrage free markets is combined with the economic model of a company used in Froot and Stein (1998). So, let us consider a company exposed to some initial risk that has to decide on its strategy of financial hedging for the next period. The company takes this decision by maximising its market value conditioning on what its future funding and investments opportunities are.

To model this we consider two time points: time 0 and time 1. At time 0 the company is exposed to the initial risk w_0 and has to choose its hedge position. Let the stochastic variable, w_1, denote the value of this initial risk at time 1. At time 1, w_1 is realised and the company can reinvest the cash flow.

We assume that the company in comparison to the market has other investment opportunities that level off with the capital invested.

Denote by $f(w_1)$ the PV at time 1 of the company's future investments opportunities at time 1. We assume that f is strictly increasing and concave, i.e. $f' > 0$ and $f'' < 0$.

We assume that the company can raise external capital in the market at time 1. However, there exists friction between the market and the company in such a way that raising external capital is a negative NPV transaction. Let e and $C(e)$ denote the level of external capital raised and the NPV of that transaction, respectively. We model C by an increasing, convex function, i.e. positive first and second derivatives. Hence, the cost of raising capital externally is marginally increasing reflecting the increasing credit risk.

The company can maximise its NPV at time 1, $f(w_1 + e) - w_1 + e - C(e)$, with respect to the level of external capital e. Let U be the NPV of the company at time 1 given the maximising level of external capital. Then U can be expressed as

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$$U(w_1) = \max_{e \geq 0} \big(f(w_1 + e) - w_1 + e - C(e) \big).$$

One can show that $U' > 0$ and $U'' < 0$. This means that the return is marginal decreasing in the level of w_1.

In order to analyse the effect of any hedging we have to determine the market value at time 0 of $U(w_1)$.

3. Arbitrage Free Market

Let $(\Omega, \mathbb{F}, P, \{\mathbb{F}_0, \mathbb{F}_1\})$ denote a discrete filtered finite probability space carrying a M-dimensional, $\mathbb{F}_0$ and $\mathbb{F}_1$-adapted, stochastic process denoted by $\mathbf{X}_0 = (X_0^1, \ldots, X_0^{M-1})^\top$ and $\mathbf{X}_1 = (X_1^1, \ldots, X_1^{M-1})^\top$. Let $\mathbf{X}_0$, $\mathbf{X}_1$, X_0^0 and X_1^0 describe the value at time 0 and 1 of the M assets traded in the market, where X_0^0 and X_1^0 are deterministic and positive.

Assuming the market is arbitrage free, ensures the existence of at least one other probability measure, equivalent to P, under which $X_0^0/X_0^0 \cdot X_0^i = X_0^i$ and $X_0^0/X_1^0 \cdot X_1^i \;\; i \in \{0, \ldots, M-1\}$, are martingale processes. Denote one of these other probability measures by Q. An arbitrage-free price at time 0, $\Pi_0(\Phi_1)$, of cash flow Φ realised at time 1 is then given by

$$\Pi_0(\Phi_1) = \frac{X_0^0}{X_1^0} \cdot E_0^Q[\Phi_1],$$

see Björk (2004).

4. Risk Neutral Valuation of U

Assume without loss of generality that $X_0^i = 1$, $i \in \{0, \ldots, M-1\}$ given $\mathbb{F}_0$. Assume that the company only carries risk, w, that can be written as an affine function of $\mathbf{X}_n$ and X_n^0. I.e. for $\mathbf{A} = (\alpha_0, \ldots, \alpha_{M-1})^\top$:

$$w_n(\mathbf{A}) = \Delta \cdot \sum_{i=0}^{M-1} \alpha_i \cdot X_n^i$$

where $\sum_{i=0}^{M-1} \alpha_i = 1$ and $\Delta > 0$ is the constant amount that is to be invested and is decided by the firm in an earlier date. $\Delta \cdot \alpha_i$ denotes the position at time 0 with respect to the ith asset. Note that $\sum_{i=0}^{M-1} \alpha_i = 1$ implies $\alpha_0 = 1 - \sum_{i=1}^{M-1} \alpha_i$.

Adapting the pricing formula to the Froot and Stein (1998) setup, we choose Φ as

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$$\Phi(X_1^0, \mathbf{X}_1) = U(w_1(\mathbf{A})).$$

Hence, the arbitrage-free price at time 0 of $U(w_1(\mathbf{A}))$ is given by

$$\Pi_0^{\Phi}(\mathbf{A}) = E_0^{Q}\left[\frac{U(w_1(\mathbf{A}))}{X_1^0}\right].$$

Under a regularity condition, it follows that for $i \in \{1, \ldots, M-1\}$

$$\frac{\partial}{\partial \alpha_i}\pi_0^{\Phi}(\mathbf{A}) = E_0^{Q}\left[U'(w_1(\mathbf{A}))\Delta\frac{(X_1^i - X_1^0)}{X_1^0}\right].$$

It also follows that for $i, j \in \{1, \ldots, M-1\}$ the second derivatives are:

$$\frac{\partial^2}{\partial \alpha_i \partial \alpha_j}\Pi_0^{\Phi}(\mathbf{A}) = E_0^{Q}\left[U''(w_1(\mathbf{A}))\Delta^2\frac{(X_1^i - X_1^0)(X_1^j - X_1^0)}{X_1^0}\right]$$

so the Hessian is negative semi-definite because $U'' < 0$ and it can be written as a quadratic form.

We have that

$$\frac{\partial}{\partial \alpha_i}\Pi_0^{\Phi}(\mathbf{A})|_{\alpha_0=1} = 0,$$

since $w_1(\mathbf{A})$ becomes independent of X_1^i for $\alpha_0 = 1$. i.e. the market value is maximised by eliminating all market risk in the internal cash flow at time 0. Indeed, since X_1^0 is deterministic then, using the martingale properties and the definitions in the previous section:

$$\frac{\partial}{\partial \alpha_i}\Pi_0^{\Phi}(\mathbf{A})|_{\alpha_0=1} = E_0^{Q}\left[U'(\Delta X_1^0)\Delta\frac{(X_1^i - X_1^0)}{X_1^0}\right] = U'(\Delta X_1^0)\Delta E_0^{Q}\left[\frac{(X_1^i - X_1^0)}{(X_1^0)^2}\right] = 0.$$

Assuming every asset is perfectly traded, the company can freely choose its initial position $\mathbf{A}$ by trading in the market at time 0, as long as $\sum_{i=0}^{M-1}\alpha_i = 1$.

We must acknowledge the referee's contribution in providing us with a simpler (and very slightly more general) proof of the result that does not rely on optimality conditions, and also makes the financial reasoning clearer. Since X_1^0 is assumed to be constant and U is strictly concave, Jensen's inequality and the martingale condition on the price processes give:

$$E_0^{Q}\left[\frac{U(w_1(\mathbf{A}))}{X_1^0}\right] \leq U\left(E_0^{Q}\left[\frac{w_1(\mathbf{A})}{X_1^0}\right]\right) = U(\Delta).$$

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The upper bound is clearly attained by investing all wealth in the risk-free asset (actually, by the condition for equality in Jensen's inequality the bound is attained if and only if all wealth is in the risk-free asset). This result is really just a restatement of classical results that are well known in utility theory, combined with basic facts about the risk-neutral measure. In particular, it is well known that risk-averse investors will prefer a deterministic payoff of the expected value of an investment to the investment itself (this is Jensen's in- equality) and that the expected return of all (tradeable) assets under the risk-neutral measure is equal to the risk-free interest rate. This makes optimisation problems of this sort under Q degenerate. All portfolios have expected return (under Q) equal to the risk-free rate, so risk-averse investors will naturally prefer (value highest, in this context) the safest one, i.e. investing all wealth in the risk-free asset.

5. Non-tradeable Risk

We can also derive the conjecture result of Froot and Stein (1998) when it comes to hedging in a company with non-tradeable risks. Assume that one of the assets, say X_0^j, is not traded in market and therefore cannot be excluded from the internal risk, w, at time 0. Also assume that (X_0^j, X_1^j) are independent from other assets in the market at each corresponding time point. Let the company's initial and now unchangeable position with respect to X_0^j be $\tilde{\alpha}_j$. (1) and (2) above and the condition $\sum_{i=0}^{M-1} \alpha_i = 1$ give that $\Pi_0^{\Phi}(\mathbf{A})$ is maximised by choosing $\alpha_0 = 1 - \tilde{\alpha}_j$.

As also suggested by the referee, the same reasoning as above also gives the result when there is an un-hedgeable risk. In this case, the assumed independence and the conditional version of Jensen's inequality give

$$E_0^Q\left[\frac{U(w_1(\mathbf{A}))}{X_1^0}\right] = E_0^Q\left[E_0^Q\left[\frac{U(w_1(\mathbf{A}))}{X_1^0}\middle| X_0^j\right]\right]$$
$$\leq E_0^Q\left[U\left(E_0^Q\left[\frac{w_1(\mathbf{A})}{X_1^0}\middle| X_0^j\right]\right)\right] = E_0^Q\left[\frac{U\left(\Delta\cdot(1-\tilde{\alpha}_j)+\tilde{\alpha}_j X_0^j\right)}{X_1^0}\right]$$

and again equality is attained if and only if all available wealth is invested in the risk-free asset. It seems relevant to note here that because the market is incomplete in this case (and (X_0^j, X_1^j) is unattainable), the value above will depend on the choice of the risk-neutral measure Q (out of infinitely many possible choices). This is important for the implications of the result to the Froot-Stein model, where the above value function is an input to a 0th stage decision model (amount of capital).

We do not discuss here how to choose the Q measure. The choice of Q in

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this case seems to be non-trivial and to be depending on the application at hand.

6. Conclusion

When experiencing decreasing marginal payoffs in the level of capital invested and when trading at fair value in market, shareholders develop aversion towards market risk in the internal cash flow of the company. Hence the economic intuition of Froot and Stein (1998) is correct.

As suggested by one of the referees, the model is subject to a criticism that could also be made of the Froot and Stein (1998) model that a firm would not in reality know today what its future investment opportunities will be. However, incorporating this added dimension of uncertainty into the model would require significant work, and the model undoubtedly still has much value, so long as the concerns raised above can be addressed. Another comment in the same category is that the model considers market frictions at time one (there are frictions involved in raising additional capital), but not at time zero (the arbitrage pricing model used is one that assumes away market frictions).

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