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Bilateral generating functions involving Jacobi polynomials

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1. *Introduction.* In paper(1) it has been proved that

$$\sum_{n=0}^{\infty}\binom{m+n}{m}P_{m+n}^{(\alpha-n,\beta-n)}(x)\,t^n=[1+\tfrac{1}{2}(x+1)\,t]^{\alpha}[1+\tfrac{1}{2}(x-1)\,t]^{\beta}P_m^{(\alpha,\beta)}[x+\tfrac{1}{2}(x^2-1)\,t] \tag{1·1}$$

where the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ is defined as ((3), p. 255)

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha+\beta)_{2n}}{n!(1+\alpha+\beta)_n}\left(\frac{x-1}{2}\right)^n{}_2F_1\left[\begin{matrix}-n,\ -\alpha-n;\\ -\alpha-\beta-2n;\end{matrix}\ \frac{2}{1-x}\right]. \tag{1·2}$$

Using Euler's transformation

$${}_2F_1(\alpha,\beta;\gamma;x)=(1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\ \gamma;\ -\frac{x}{1-x}\right),$$

we rewrite (1·1) as

$$\sum_{n=0}^{\infty}\binom{m+n}{m}P_{m+n}^{(\alpha-n,\beta-n)}(x)\,t^n$$

$$=\frac{(1+\alpha+\beta+m)_m}{m!}\left(\frac{x+1}{2}\right)^m\left(\frac{x-1}{x+1}\right)^{-\alpha}[1+\tfrac{1}{2}(x-1)\,t]^{\alpha+\beta+m}$$

$$\times\,{}_2F_1\left[\begin{matrix}-\alpha-\beta-m,\ -\alpha-m;\\ -\alpha-\beta-2m;\end{matrix}\ \frac{2}{(x+1)\,(1+\tfrac{1}{2}(x-1)\,t)}\right]. \tag{1·4}$$

Now, with the help of (1·4), we obtain bilateral generating functions, each involving a Jacobi polynomial and one of the terminating Appell's functions F_1, F_2, and F_4.

2. The generating functions that we intend to prove are

$$\sum_{n=0}^{\infty}\binom{m+n}{m}P_{m+n}^{(\lambda-n,\mu-n)}(x)\,F_1(-n;\ \alpha,\beta;\ \gamma;\ y,z)\,t^n$$

$$=\frac{(1+\lambda+\mu+m)_m}{m!}\left(\frac{x-1}{x+1}\right)^{-\lambda}\left(\frac{x+1}{2}\right)^m\rho^{\lambda+\mu+m}$$

$$\times F_G\left[-\lambda-\mu-m,\ -\lambda-\mu-m,\ -\lambda-\mu-m,\ -\lambda-m,\alpha,\beta;\right.$$

$$\left.-\lambda-\mu-2m,\gamma,\gamma;\ \frac{2}{\rho(x+1)},\ \frac{-yt(1-x)}{2\rho},\ \frac{-zt(1-x)}{2\rho}\right], \tag{2·1}$$

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$$\sum_{n=0}^{\infty}\binom{m+n}{m}P_{m+n}^{(\lambda-n,\mu-n)}(x)\,F_2(-n;\,\alpha,\beta;\,\gamma,\delta;\,y,z)\,t^n$$

$$=\frac{(1+\lambda+\mu+m)_m}{m!}\left(\frac{x+1}{2}\right)^m\left(\frac{x-1}{x+1}\right)^{-\lambda}\rho^{\lambda+\mu+m}$$

$$\times F_A\left[-\lambda-\mu-m;\,-\lambda-m,\alpha,\beta;\right.$$

$$\left.-\lambda-\mu-2m,\gamma,\delta;\,\frac{2}{\rho\,(x+1)},\,\frac{-yt\,(1-x)}{2\rho};\,\frac{-zt(1-x)}{2\rho}\right] \quad (2\cdot2)$$

and

$$\sum_{n=0}^{\infty}\binom{m+n}{m}P_{m+n}^{(\lambda-n,\mu-n)}(x)\,F_4(-n,\alpha;\,\beta,\gamma;\,y,z)\,t^n$$

$$=\frac{(1+\lambda+\mu+m)_m}{m!}\left(\frac{x+1}{2}\right)^m\left(\frac{x-1}{x+1}\right)^{-\lambda}\rho^{\lambda+\mu+m}$$

$$\times F_E\left[-\lambda-\mu-m,\,-\lambda-\mu-m,\,-\lambda-\mu-m;\,-\lambda-m,\alpha,\alpha;\right.$$

$$\left.-\lambda-\mu-2m,\beta,\gamma;\frac{2}{\rho(1+x)},\,\frac{-zt(1-x)}{2\rho},\,\frac{-yt(1-x)}{2\rho}\right], \quad (2\cdot3)$$

where $$\rho=1+\tfrac{1}{2}(x-1)\,t.$$

Saran's functions F_G and F_E in (2·1) and (2·3) are defined as (4)

$$F_G(a_1,a_1,a_1,b_1,b_2,b_3;c_1,c_2,c_2;x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}\,(b_1)_m\,(b_2)_n\,(b_3)_p}{m!\,n!\,p!\,(c_1)_m\,(c_2)_{n+p}}\,x^m y^n z^p,$$

$$F_E(a_1,a_1,a_1,b_1,b_2,b_2;\,c_1,c_2,c_3;\,x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a_1)_{m+n+p}(b_1)_m\,(b_2)_{n+p}}{m!\,n!\,p!\,(c_1)_m\,(c_2)_n\,(c_3)_p}\,x^m y^n z^p,$$

while Lauricella's function F_A in (2·2) is given by (2)

$$F_A(a;\,b,c,d;\,e,f,g;\,x,y,z)=\sum_{m,n,p=0}^{\infty}\frac{(a)_{m+n+p}(b)_m(c)_n\,(d)_p}{m!\,n!\,p!\,(e)_m\,(f)_n\,(g)_p}\,x^m y^n z^p.$$

We give below the proof of only (2·1); others can be proved likewise.

We start with the left-hand side of (2·1).

$$\sum_{n=0}^{\infty}\binom{m+n}{m}P_{m+n}^{(\lambda-n,\mu-n)}(x)\,F_1(-n;\,\alpha,\beta;\,\gamma;\,y,z)\,t^n$$

$$=\frac{1}{m!}\sum_{n=0}^{\infty}(m+n)!\,P_{m+n}^{(\lambda-n,\mu-n)}(x)\,t^n\sum_{k+r\leqslant n}\frac{(\alpha)_k\,(\beta)_r}{(n-k-r)!\,k!\,r!\,(\gamma)_{k+r}}\,(-y)^k\,(-z)^r$$

$$=\frac{1}{m!}\sum_{k,r=0}^{\infty}\frac{(m+k+r)!\,(\alpha)_k\,(\beta)_r}{k!\,r!\,(\gamma)_{k+r}}\,(-yt)^k\,(-zt)^r$$

$$\times\sum_{n=0}^{\infty}\binom{m+k+r+n}{m+k+r}P_{m+k+r+n}^{(\lambda-k-r-n,\mu-k-r-n)}(x)\,t^n$$

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$$= \frac{(1+\lambda+\mu+m)_m}{m!}\left(\frac{x+1}{2}\right)^m\left(\frac{x-1}{x+1}\right)^{-\lambda}\rho^{\lambda+\mu+m}$$

$$\times \sum_{k,r=0}^{\infty} \frac{(-\lambda-\mu-m)_{k+r}(\alpha)_k(\beta)_r}{k!\,r!\,(\gamma)_{k+r}}\left[\frac{-yt(1-x)}{2\rho}\right]^k\left[\frac{-zt(1-x)}{2\rho}\right]^r$$

$$\times {}_2F_1\left[\begin{matrix}-\lambda-\mu-m+k+r, & -\lambda-m;\\ & -\lambda-\mu-2m;\end{matrix}\ \frac{2}{\rho(x+1)}\right] \quad \text{by} \quad (1\cdot4)$$

$$= \frac{(1+\lambda+\mu+m)_m}{m!}\left(\frac{x+1}{2}\right)^m\left(\frac{x-1}{x+1}\right)^{-\lambda}\rho^{\lambda+\mu+m}$$

$$\times F_G\left[\begin{matrix}-\lambda-\mu-m, & -\lambda-\mu-m, & -\lambda-\mu-m, & -\lambda-m, & \alpha, & \beta;\\ & & & -\lambda-\mu-2m, & \gamma, & \gamma;\end{matrix}\ \frac{2}{\rho(x+1)}, \frac{-yt(1-x)}{2\rho}, \frac{-zt(1-x)}{2\rho}\right].$$

This completes the proof of (2·1). Similarly for (2·2) and (2·3).

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