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Sparse Recovery via ℓ_q-Minimization for Polynomial Chaos Expansions

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Abstract. In this paper we consider the algorithm for recovering sparse orthogonal polynomials using stochastic collocation via ℓ_q minimization. The main results include: 1) By using the norm inequality between ℓ_q and ℓ_2 and the square root lifting inequality, we present several theoretical estimates regarding the recoverability for both sparse and non-sparse signals via ℓ_q minimization; 2) We then combine this method with the stochastic collocation to identify the coefficients of sparse orthogonal polynomial expansions, stemming from the field of uncertainty quantification. We obtain recoverability results for both sparse polynomial functions and general non-sparse functions. We also present various numerical experiments to show the performance of the ℓ_q algorithm. We first present some benchmark tests to demonstrate the ability of ℓ_q minimization to recover exactly sparse signals, and then consider three classical analytical functions to show the advantage of this method over the standard ℓ_1 and reweighted ℓ_1 minimization. All the numerical results indicate that the ℓ_q method performs better than standard ℓ_1 and reweighted ℓ_1 minimization.

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1. Introduction

In the field of uncertainty quantification (UQ), one of the main task is to quantify the effect of uncertain model parameters on model output [14, 24, 26]. When the simulation model is computationally expensive to run, we want to build a cheap surrogate of the the response of the model output to variations in the model input. In this paper, we consider the approximation of a function model $f(\boldsymbol{z}) : \mathbb{R}^d \to \mathbb{R}$ ($d \geq 1$) via a generalized Polynomial Chaos Expansion (PCE), which consists of a polynomial basis whose elements are orthogonal under the probability measure of the input variable $\boldsymbol{z}$ [27–29]. In particular, we focus on identifying the PCE coefficients from small number of function samples, which

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means that the number of samples is (severely) less than the cardinality of the linear approximation space.

Recently, based on the idea of compressive sensing [7, 10, 20], stochastic collocation method via ℓ_1-minimization techniques [11, 15, 17, 19, 21, 22, 30, 31] have been shown to be an efficient method to recover the sparse PCE coefficients from the underdetermined system. This is a natural approach to relax the original ℓ_0 minimization problem and trying to seek a sparse solution of the PCE vector.

Instead of ℓ_1-minimization, noticing the fact that $\lim_{q\to 0^+} \|\mathbf{c}\|_q^q = \|\mathbf{c}\|_0$, the ℓ_q minimization may provide a better approximation to the sparse solution. The advantage of this approach can be found in [18] and it has been widely used to recover sparse solutions [9, 13, 18]. The theoretical results in [25] give an interpretation to the intuitive observation that ℓ_q minimization provides a better approximation to ℓ_0 minimization than that ℓ_1 minimization can provide when the order of restricted isometry constants(RIC) is greater than 2k. Numerical comparisons between ℓ_1, ℓ_q minimization and other non-convex methods, such as reweighted ℓ_1 minimization, can be found in [13].

Based on the above observations, we will consider the stochastic collocation method via ℓ_q minimization to approximate the sparse PCE coefficients. The main contribution of this paper is that we establish some theroetical estimates for both exact recovery of sparse signals and the recoverability for general non-sparse signals. Following the works in [3], these estimates are obtained by using the key inequality between ℓ_2 and ℓ_q norm developed in [16]. The estimates are slightly different from the existing estimates of ℓ_q minimization in [13, 16, 25] and references therein. Although these estimates are not able to verify the improvement of the ℓ_q minimization over the ℓ_1 minimization in terms of the RIC bound δ_s, it is the foundation results to get the sparse recovery analysis for our bounded orthogonal system. We apply the ℓ_q minimization to the stochastic collocation method and obtain the sparse PCE coefficients for Legendre orthogonal polynomials. We present the recoverability results for both sparse polynomial functions and general non-sparse functions based on the new RIC bound estimates of the ℓ_q minimization. Various numerical results are presented to show that the ℓ_q minimization make a significant improvement over ℓ_1 minimization to achieve sparse solutions, not only in low dimensions but also in high dimensions.

The rest of the paper is organized as follows. In Section 2, we introduce the generalized polynomial chaos (gPC) approximation, set up the ℓ_1 minimization and ℓ_q minimization problems. Section 3 introduces some auxiliary results, presents the recoverability estimates for both sparse signals and non-sparse signals via the ℓ_q minimization approach, gives the theorems for recovering high-dimensional Legendre polynomial chaos. In Section 4, some numerical experiments are provided to explore the performance and accuracy of stochastic collocation via ℓ_q minimization comparing with ℓ_1 minimization and reweighted ℓ_1 minimization. We finally give some conclusions in Section 5.

2. Problem setup

Let $\boldsymbol{z} = (z^1, \cdots, z^d)^T$ be a set of random vectors defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with d mutually independent components, where each z^i takes values in $\Gamma^i \subset \mathbb{R}$.

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Since the variables $\{z^i\}_{i=1}^d$ are mutually independent, they have marginal probability density functions ρ^i associated with random variable z^i that completely characterize the distribution of $\boldsymbol{z}$. We define $\Gamma := \otimes_{i=1}^d \Gamma^i \subset \mathbb{R}^d$, and let $\rho(\boldsymbol{z}) = \prod_{i=1}^d \rho^i(z^i) : \Gamma \to \mathbb{R}^+$ denote the joint probability density function (PDF) of $\boldsymbol{z}$.

In the stochastic collocation method, we are concerned with the approximation of the model output, a d-variate function $f(\boldsymbol{z}) : \Gamma \to \mathbb{R}$, from a finite (ideally small) set of function evaluations. Let $\Theta_M = \{\boldsymbol{z}_1, \cdots, \boldsymbol{z}_M\} \in \Gamma$ be a set of points at which the function values of f are available, and denote these values as $f_j = f(\boldsymbol{z}_j), j = 1, \cdots, M$. We want to construct the approximation of f based on $\{\boldsymbol{z}_j, f_j\}_{j=1}^M$. Let V be a linear space from which the approximation is sought and let $N = \dim(V)$. This paper is concerned with the special case of the above setup where V is a polynomial subspace, and the $N = \dim(V)$ coefficients defining the approximation are underdetermined from $M \ll N$ samples.

2.1. Polynomial chaos expansion (PCE)

We seek the approximation in the Generalized polynomial chaos (gPC) framework. The basic idea of gPC is to represent the function $f(\boldsymbol{z})$ as an expansion of a polynomial of random variables $\boldsymbol{z} = (z^1, \cdots, z^d)^T$, the basis for approximation is typically taken as a set of orthogonal polynomials [29]. For each marginal density ρ^i, we can define the univariate gPC basis elements ϕ_n^i, polynomials of degree n, via the orthogonality relation

$$\mathbb{E}\left[\phi_n^i(z^i)\phi_\ell^i(z^i)\right] = \int_{\Gamma^i} \phi_n^i(z^i)\phi_\ell^i(z^i)\rho^i(z^i)dz^i = \delta_{n,\ell}, \quad n,\ \ell \geq 0, \tag{2.1}$$

with $\delta_{n,\ell}$ the Kronecker delta function. For example, the Gaussian (normal) distribution yields the Hermite polynomials, the uniform distribution pairs with Legendre polynomials, etc. For a detailed account of the correspondence, see [29].

In the multi-dimensional case($d > 1$), the orthogonal polynomials are constructed as tensor products of the one-dimensional polynomials in each variable:

$$\Phi_{\boldsymbol{n}}(\boldsymbol{z}) = \prod_{i=1}^{d} \phi_{n_i}^i\left(z^i\right), \tag{2.2}$$

where $\boldsymbol{n} = \left(n_1, \cdots, n_d\right)$ is a multi-index set with $|\boldsymbol{n}| = \sum_{i=1}^d n_i$. The product functions $\Phi_{\boldsymbol{n}}$ are L^2 orthogonal under the joint probability density function ρ for $\boldsymbol{z}$:

$$\mathbb{E}\left[\Phi_{\boldsymbol{n}}(\boldsymbol{z})\Phi_{\boldsymbol{\ell}}(\boldsymbol{z})\right] = \int_\Gamma \Phi_{\boldsymbol{n}}(\boldsymbol{z})\Phi_{\boldsymbol{\ell}}(\boldsymbol{z})\rho(\boldsymbol{z})d\boldsymbol{z} = \delta_{\boldsymbol{n},\boldsymbol{\ell}}, \tag{2.3}$$

where $\delta_{\boldsymbol{n},\boldsymbol{\ell}} = \prod_{i=1}^d \delta_{n_i,\ell_i}$.

A gPC expansion for f is given by

$$f(\boldsymbol{z}) = \sum_{\boldsymbol{n} \in \mathbb{N}_0^d} c_{\boldsymbol{n}} \Phi_{\boldsymbol{n}}(\boldsymbol{z}), \tag{2.4}$$

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if $f(\boldsymbol{z})$ has finite variance, then this expansion is well-defined and convergent in the L^2 sense under fairly weak assumptions on the distribution of $\boldsymbol{z}$ [12]. In practice, the infinite expansion in (2.4) must be truncated. The most common approach is to approximate f in a total degree space T_P^d. An element f_P in T_P^d is of the form

$$f_P = \sum_{\boldsymbol{k}\in\Lambda_P^T} c_{\boldsymbol{k}}\Phi_{\boldsymbol{k}}(\boldsymbol{z}), \qquad \Lambda_P^T = \left\{ \boldsymbol{k}\in\mathbb{N}_0^d \,|\, \sum_{i=1}^d k_i \le P \right\}. \tag{2.5}$$

The dimension of T_P^d is

$$N = \#\Lambda_P^T \triangleq \dim T_P^d = \begin{pmatrix} d+P \\ P \end{pmatrix}. \tag{2.6}$$

By placing an order for the orthogonal polynomials, we can change (2.5) into the following single index version

$$f_P = \sum_{\boldsymbol{k}\in\Lambda_P^T} c_{\boldsymbol{k}}\Phi_{\boldsymbol{k}}(\boldsymbol{z}) = \sum_{n=1}^N \mathbf{c}_n\Phi_n(\boldsymbol{z}). \tag{2.7}$$

Our main task is to identify the expansion coefficients $\mathbf{c}_n$ from a limited available knowledge of data $f(\boldsymbol{z}_j), j = 1,\cdots,M$ $(M \ll N)$.

2.2. ℓ_1 minimization approach

In recent years, the idea of compressive sensing has been extended to stochastic collocation methods to construct the polynomial approximation of f. Under certain conditions, compressive sensing approach provides a means of identifying sparse coefficient vectors from a limited amount of data. Specifically, given a small set of M unstructured realizations $\{\boldsymbol{z}_j\}_{j=1}^M$ with corresponding outputs $\mathbf{f} = (f(\boldsymbol{z}_1),\cdots,f(\boldsymbol{z}_M))$, we aim to find a solution that satisfies

$$\mathbf{Ac} = \mathbf{f}, \tag{2.8}$$

where $\mathbf{c} = (c_1,\cdots,c_N)^T \in \mathbb{R}^N$ is the coefficient vector to be determined in the gPC expansion (2.7), and $\mathbf{A} \in \mathbb{R}^{M\times N}$ is the measurement matrix whose entries are

$$[\mathbf{A}]_{jn} = \Phi_n(\boldsymbol{z}_j), \quad j = 1,\cdots,M, \quad n = 1,\cdots,N. \tag{2.9}$$

We are concerned with the case $M \ll N$ in this paper, so the problem (2.8) becomes severely underdetermined. Thus we must propose some form of regularization to obtain a unique solution. One way to circumvent the difficulty is to employ the convex ℓ_1 minimization scheme. An ℓ_1 minimization algorithm attempts to identify the dominant gPC coefficients by solving the optimization problem

$$\min\|\mathbf{c}\|_1 \quad \text{subject to} \quad \mathbf{Ac} = \mathbf{f}, \tag{2.10}$$

where $\|\cdot\|_1$ is the ℓ_1 norm on vectors. Also, we need to mention that the interpolation condition $\mathbf{Ac} = \mathbf{f}$ can be relaxed to $\|\mathbf{Ac} - \mathbf{f}\| \le \varepsilon$, for some tolerance value ϵ, resulting in a regression type "de-noising" approach.

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2.3. ℓ_q minimization approach

Instead of the ℓ_1 minimization, the other way to find the dominant gPC coefficients in (2.8) is through the ℓ_q minimization problem $(0 < q \leq 1)$

$$\min \|\mathbf{c}\|_q \quad \text{subject to} \quad \mathbf{Ac} = \mathbf{f}, \tag{2.11}$$

where $\|\mathbf{c}\|_q^q = \sum_{n=1}^{N} |c_n|^q$. Noticing the fact that $\lim_{q\to 0^+} \|\mathbf{c}\|_q^q = \|\mathbf{c}\|_0$, the ℓ_q minimization may provide a better approximation to the sparse solution. The ability of the exact recovery of the ℓ_q minimization is also related to the restricted isometry property (RIP), which was recalled as follows.

Definition 2.1 (Restricted isometry constant [8]). *Let* $\mathbf{A}$ *be an* $M \times N$ *matrix. Define the restricted isometry constant (RIC)* $\delta_s < 1$ *to be the smallest positive number such that the inequality*

$$(1 - \delta_s)\|\mathbf{c}\|_2^2 \leq \|\mathbf{Ac}\|_2^2 \leq (1 + \delta_s)\|\mathbf{c}\|_2^2 \tag{2.12}$$

holds for all $\mathbf{c} \in R^N$ *of sparsity at most* s*. Then, the matrix* $\mathbf{A}$ *is said to satisfy the* s*-restricted isometry property (RIP) with restricted isometry constant* δ_s*.*

Definition 2.2 (Restricted orthogonality constant (ROC) [4, 8]). *If* $s + s' \leq N$*, the* s,s'*-restricted orthogonality constant* $\theta_{s,s'}$ *is the smallest number that satisfies*

$$|\langle \mathbf{Ac}, \mathbf{Ac}' \rangle| \leq \theta_{s,s'} \|\mathbf{c}\|_2 \|\mathbf{c}'\|_2, \tag{2.13}$$

for all $\mathbf{c}$ *and* $\mathbf{c}'$ *are* s*-sparse and* s'*-sparse respectively, and have disjoint supports.*

3. Main results for the ℓ_q recovery

Before deducing our main results for l_q recovery, we would like to introduce some important inequalities, which is necessary for later proves.

3.1. Auxiliary Results

Here we summarize some existing results that are useful in deriving the main results in this paper. It was shown in [8] that the constants δ_s and $\theta_{s,s'}$ are related by the following inequalities

$$\theta_{s,s'} \leq \delta_{s+s'} \leq \theta_{s,s'} + \max(\delta_s, \delta_{s'}). \tag{3.1}$$

The following inequality was developed by Cai et al. [4].

Lemma 3.1 (Square root lifting inequality [4]). *For any* $a \geq 1$ *and positive integers* s*,* s' *is chosen so that* as' *is an integer. Then*

$$\theta_{s,as'} \leq \sqrt{a}\,\theta_{s,s'}. \tag{3.2}$$

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We recall some properties of the ℓ_q-quasinorm. For any vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$ and $q \in (0,1]$, one has

$$\|\mathbf{u}\|_1 \le \|\mathbf{u}\|_q, \quad \|\mathbf{u}\|_q \le N^{1/q-1/2}\|\mathbf{u}\|_2, \quad \|\mathbf{u}+\mathbf{v}\|_q^q \le \|\mathbf{u}\|_q^q + \|\mathbf{v}\|_q^q.$$

Besides the square root lifting inequality (3.2), the other main ingredients in proving the RIP conditions is the norm inequality for ℓ_2 and ℓ_q developed in [16]. This inequality is an improvement on a version of the converse of a generalized Cauchy-Schwarz inequality extended to the setting of ℓ_q-quasinorm.

Lemma 3.2. ([16]) *For $q \in (0,1)$ and $\mathbf{x} \in \mathbb{R}^n$, there is*

$$\|\mathbf{x}\|_2 \le \frac{\|\mathbf{x}\|_q}{n^{1/q-1/2}} + \lambda_q \sqrt{n}\Big(\max_{1\le i\le n}|\mathbf{x}_i| - \min_{1\le i\le n}|\mathbf{x}_i|\Big), \tag{3.3}$$

where

$$\lambda_q := \left(\frac{q}{2}\right)^{\frac{q}{2-q}} - \left(\frac{q}{2}\right)^{\frac{2}{2-q}}.$$

Moreover, λ_q is a decreasing convex function of $q \in (0,1)$ with

$$\lim_{q\to 0}\lambda_q = 1 \quad \textit{and} \quad \lim_{q\to 1}\lambda_q = \frac{1}{4}.$$

Remark 3.1. If $q = 1$, the above lemma still holds and we have

$$\|\mathbf{x}\|_2 \le \frac{\|\mathbf{x}\|_1}{\sqrt{n}} + \frac{1}{4}\sqrt{n}\Big(\max_{1\le i\le n}|\mathbf{x}_i| - \min_{1\le i\le n}|\mathbf{x}_i|\Big).$$

For more detail, one can refer to [3].

3.2. Recovery analysis of ℓ_q minimization

We now establish the recovery results for the ℓ_q minimization problem. We also consider the de-noising version, i.e. to find the sparse solution, denoted by $\hat{\mathbf{c}}$, via the following ℓ_q minimization problem:

$$\min\|\mathbf{c}\|_q \quad \text{subject to} \quad \|\mathbf{A}\mathbf{c}-\mathbf{f}\|_2 \le \varepsilon, \tag{3.4}$$

where ε is related to the noise level in data $\mathbf{f}$. We first state our new estimates for exact recovery of sparse functions.

Theorem 3.1. *Suppose $\mathbf{c}$ is s-sparse. Let s_1 and s_2 be positive integers such that $s_1 \ge s$ and $8(s_1 - s) \le s_2$. Let*

$$K = \left(\frac{s_1}{s_2}\right)^{1/q-1/2} + \lambda_q\sqrt{\frac{s_2}{s_1}} - \frac{2(s_1-s)^{1/q}}{\sqrt{s_1}s_2^{1/q-1/2}},$$

where $\lambda_q = \left(\frac{q}{2}\right)^{\frac{q}{2-q}} - \left(\frac{q}{2}\right)^{\frac{2}{2-q}}$. Then under the condition

$$\delta_{s_1} + K\theta_{s_1,s_2} < 1,$$

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the ℓ_q minimizer $\hat{\mathbf{c}}$ of (3.4) *satisfies*

$$\|\mathbf{c}-\hat{\mathbf{c}}\|_2 \le \frac{2\sqrt{2}\sqrt{1+\delta_{s_1}}}{1-\delta_{s_1}-K\theta_{s_1,s_2}}\varepsilon.$$

In particular, in the noiseless case where $\mathbf{Ac}=\mathbf{f}$, the ℓ_q minimization (2.11) *recovers $\mathbf{c}$ exactly.*

The following result is the consequence of Theorem 3.1 and the square root lifting inequality.

Theorem 3.2. *Let $\mathbf{c}$ be any vector with sparsity s, $q\in(0,1]$ satisfying*

$$\delta_s < \frac{1}{1+\eta(q)}, \tag{3.5}$$

where

$$\eta(q)=\left(\frac{2}{3}\lambda_q+\left(\frac{9}{4}\right)^{1/q-1/2}\right)\sqrt{\frac{9}{5}},\quad \lambda_q=\left(\frac{q}{2}\right)^{\frac{q}{2-q}}-\left(\frac{q}{2}\right)^{\frac{2}{2-q}}.$$

Then, the following results holds:

- *Let $\mathbf{f}=\mathbf{Ac}$, then the solution $\hat{\mathbf{c}}$ to* (2.11) *is exact, i.e., $\hat{\mathbf{c}}=\mathbf{c}$.*
- *Let $\mathbf{f}=\mathbf{Ac}+\mathbf{e}$, where $\mathbf{e}\in\mathbb{R}^M$ is any perturbation with $\|\mathbf{e}\|_2\le\varepsilon$, then the solution $\hat{\mathbf{c}}$ to* (3.4) *satisfies*

  $$\|\hat{\mathbf{c}}-\mathbf{c}\|_2 \le \tilde{C}_q\varepsilon,$$

  where

  $$\tilde{C}_q=\frac{2\sqrt{2(1+1/(1+\eta(q)))}}{1-(1+\eta(q))\delta_s}. \tag{3.6}$$

The proof of the above Theorems 3.1 and 3.2, following the framework in [3], is given in Appendices A and B, respectively.

When $\mathbf{c}$ is not s-sparse, the ℓ_q minimization can also recover $\mathbf{c}$ with accuracy. To quantify this recovery accuracy we can measure the solution against its "best" sparse version $\mathbf{c_s}$, which is the s-sparse vector containing only the s largest entries (in absolute values) of $\mathbf{c}$.

Theorem 3.3. *Let $\mathbf{c}\in\mathbb{R}^N$ be an arbitrary vector and $\mathbf{c}_s$ be the s-sparse vector containing only the s largest entries (in absolute value) of $\mathbf{c}$. Assume that*

$$\delta_{2s} < \frac{1}{1+2^{1/q-1/2}}.$$

Then the solution $\hat{\mathbf{c}}$ to (3.4) *satisfies*

$$\|\hat{\mathbf{c}}-\mathbf{c}\|_2 \le C_{1,q}s^{1/2-1/q}\|\mathbf{c}-\mathbf{c_s}\|_q + C_{2,q}\varepsilon, \tag{3.7}$$

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where

$$C_{1,q} = \frac{2^{2/q-1}(1+(\sqrt{2}-1)\delta_{2s})}{1-(1+2^{1/q-1/2})\delta_{2s}} s^{1/2-1/q}, \tag{3.8a}$$

$$C_{2,q} = \frac{(2+2^{1/q})\sqrt{1+\delta_{2s}}}{1-(1+2^{1/q-1/2})\delta_{2s}}. \tag{3.8b}$$

The proof can be found in Appendix C.

Remark 3.2. We note that the condition δ_s and δ_{2s} depend only on q and if $q = 1$, the results are the same as that in [3] and [6], respectively. When q tends to zero, it decreases exponentially fast. This means that although using the similar approach developed in the standard CS literature on ℓ_1 minimization, our estimates for ℓ_q minimization (3.5) derived in Theorem 3.2 and (3.7) derived in Theorem 3.3 are worse than those of the ℓ_1 minimization [3] and [6], respectively. However, the numerical examples in this paper demonstrate that ℓ_q minimization consistently outperforms the ℓ_1 minimization, for a variety problems. Therefore, more theoretical studies are needed for the ℓ_q minimization. These theoretical findings is similar to the $\ell_1 - \ell_2$ minimization problem proposed in [32].

3.3. Recovery anaylsis of PCE coefficients via ℓ_q minimization

Following [23], the orthogonal polynomial system $\{\Phi_n(\boldsymbol{z})\}$ is called a bounded orthonormal system if it is uniformly bounded,

$$\sup_n \|\Phi_n\|_\infty = \sup_n \sup_{\boldsymbol{z}} |\Phi_n(\boldsymbol{z})| \le K, \quad (K \ge 1), \tag{3.9}$$

where $\Phi_n(\boldsymbol{z})$ satisfies the orthogonality relation

$$\int_{[-1,1]^d} \Phi_i(\boldsymbol{z})\Phi_j(\boldsymbol{z}) d\rho(\boldsymbol{z}) = \delta_{ij}, \quad \forall i,j \in \Lambda. \tag{3.10}$$

We now recall the following result for bounded orthonormal systems, which will play essential role in the estimate the guarantee condition of successful recoverability via ℓ_q minimization.

Theorem 3.4 (RIP for bounded orthonormal systems [23]). *Let $\mathbf{A} \in \mathbb{R}^{M\times N}$ be the interpolation matrix with entries $\{a_{j,n} = \Phi_n(\boldsymbol{z}_j)\}_{1\le n\le N, 1\le j\le M}$ from* (2.9)*, where $\{\Phi_n\}$ is a bounded orthonormal system satisfying* (3.9) *and orthogonality* (3.10)*, the points $\boldsymbol{z}_j, j = 1,\cdots,M$, are i.i.d. random samples drawn from the measure ρ. Assume that*

$$M \ge C\delta^{-2}K^2 s \log^3(s)\log(N),$$

then with probability at least $1 - N^{-\gamma\log^3(s)}$, the RIC δ_s of $1/\sqrt{M}\mathbf{A}$ satisfies $\delta_s \le \delta$. Here the $C, \gamma > 0$ are universal constants.

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Based on this property of the measurement matrix **A**, we now show the recoverability results of the ℓ_q minimization for Legendre approximation. This part generally follows the lines of [31], we are ready to discuss the high-dimensional case $d \gg 1$.

Theorem 3.5 (Exact recovery of sparse functions)**.** *Let* **A** *be the Vandermonde-like measurement matrix established using the Legendre polynomials in* T_P^d *and* M *sampling points* $\{\boldsymbol{z}_j\}_{j=1}^M$ *from the uniform measure. Suppose* $d \geq P$ *and*

$$M \geq C\delta^{-2}3^P s\log^3(s)\log(N), \tag{3.11}$$

where

$$\delta < \frac{1}{1+\eta(q)}, \quad \eta(q) = \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right)\sqrt{9/5}, \quad \lambda_q = \left(\frac{q}{2}\right)^{\frac{q}{2-q}} - \left(\frac{q}{2}\right)^{\frac{2}{2-q}}$$

and $C > 0$ *is universal. Let the target function be constructed as* (2.7)*, i.e.,*

$$\mathbf{f}(\boldsymbol{z}) = \sum_{n=1}^{N} \mathbf{c}_n \Phi_n(\boldsymbol{z})$$

with the s-sparse coefficient vector **c**. *Then, with a probability exceeding* $1 - N^{-\gamma\log^3(s)}$*, the following results hold.*

- *Let* $\mathbf{f} = \mathbf{A}\mathbf{c}$*, then the solution* $\hat{\mathbf{c}}$ *to* (2.11) *is exact, i.e.,* $\hat{\mathbf{c}} = \mathbf{c}$.
- *Let* $\mathbf{f} = \mathbf{A}\mathbf{c} + \mathbf{e}$*, where* $\mathbf{e} \in \mathbb{R}^M$ *is any perturbation with* $\|\mathbf{e}\|_2 \leq \varepsilon$*, then the solution* $\hat{\mathbf{c}}$ *to* (3.4) *satisfies*

$$\|\hat{\mathbf{c}} - \mathbf{c}\|_2 \leq \frac{\tilde{C}_q}{\sqrt{M}}\varepsilon, \tag{3.12}$$

  where $\tilde{C}_q$ *is from* (3.6).

Proof. Since $M \geq C\delta^{-2}3^P s\log^3(s)\log(N)$, we can obtain from Theorem 3.4 directly that the restricted isometry constant δ_s of $\frac{1}{\sqrt{M}}\mathbf{A}$ satisfies $\delta_s \leq \delta \leq \frac{1}{1+\eta(q)}$. The proof then follows immediately from Theorem 3.2. The noiseless case can be proved by setting $\varepsilon = 0$. The proof is complete. □

We now consider the approximation of an unknown function $\widehat{f}$. Let f be its best approximation in the polynomial space T_P^d, and in general, the best polynomial approximation f is not sparse, which means that the expansion coefficient **c** is not exactly sparse. Thus we need to use Theorem 3.3 to establish the recovery ability via the ℓ_q minimization problem.

Theorem 3.6 (Recovery of non-sparse polynomial functions)**.** *Let* **A** *be the Vandermonde-like measurement matrix established using the Legendre polynomials in* T_P^d *and* M *sampling points* $\{\xi_i\}_{i=1}^M$ *from the uniform measure. Suppose* $d \geq P$ *and*

$$M \geq C\delta^{-2}3^P s\log^3(2s)\log(N), \tag{3.13}$$

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where $\delta < \frac{1}{1+2^{1/q-1/2}}$ *and* $C > 0$ *is universal. Let the target function be a Legendre series as* (2.7)*, i.e.,*

$$f(\boldsymbol{z}) = \sum_{n=1}^{N} \mathbf{c}_n \Phi_n(\boldsymbol{z})$$

with an arbitrary coefficient vector $\mathbf{c} \in \mathbb{R}^N$*,* $N = \dim T_P^d$*, and* $\mathbf{c}_s$ *be its truncated vector corresponding to the s largest entries (in absolute value). Then, with a probability exceeding* $1 - N^{-\gamma \log^3(2s)}$*, the following results hold.*

- *Let* $\mathbf{f} = \mathbf{A}\mathbf{c}$*, then the solution* $\hat{\mathbf{c}}$ *to* (2.11) *satisfies*

$$\|\hat{\mathbf{c}} - \mathbf{c}\|_2 \le C_{1,q} s^{1/2-1/q} \|\mathbf{c} - \mathbf{c}_s\|_2.$$

- *Let* $\mathbf{f} = \mathbf{A}\mathbf{c} + \mathbf{e}$*, where* $\mathbf{e} \in \mathbb{R}^M$ *is any perturbation with* $\|\mathbf{e}\|_2 \le \varepsilon$*, then the solution* $\hat{\mathbf{c}}$ *to* (3.4) *satisfies*

$$\|\hat{\mathbf{c}} - \mathbf{c}\|_2 \le \frac{C_{2,q}}{\sqrt{M}} \varepsilon + C_{1,q} s^{1/2-1/q} \|\mathbf{c} - \mathbf{c}_s\|_2, \tag{3.14}$$

  where $C_{1,q}$ *and* $C_{2,q}$ *are defined in* (3.8).

Proof. Note that here the restricted isometry constant δ_{2s} of $\frac{1}{\sqrt{M}}\mathbf{A}$ satisfies $\delta \le \delta_{2s} < \frac{1}{1+2^{1/q-1/2}}$, then we can directly obtain the results similar as Theorem 3.5. □

4. Numerical tests

In this section, we numerically demonstrate the different performance of the ℓ_1 minimization, reweighted ℓ_1 minimization and ℓ_q minimization. Concretely, we will conduct three groups of tests, namely, the manufactured polynomial function recovery, the approximation of analytical functions, the quantity of interest (QoI) of a PDE model.

We choose the function system as

$$\Phi_{\mathbf{n}} : \mathbf{n} \in \Lambda_P^T$$

and test the recovery properties in the total degree (TD) spaces T_P^d, which are very useful when dealing with high dimensional problems. In our experimental cases, we choose the dimension d as 2 and degree P as 20 for the low-dimensional case, similarly, we regard the case with dimension $d = 10$ and degree $P = 4$ as the high-dimensional case. In the later numerical examples, we are interested in the recovery successful rate for different methods, hence we declare a reconstruction is a success if the absolute error satisfies $\|\hat{\mathbf{c}} - \mathbf{c}\|_\infty < 10^{-3}$.

We employ the the available tools SPGL1 from [2] to implement the ℓ_1 minimization. To solve the ℓ_q minimization problem (2.11), we use the iterative algorithm proposed in [13]. We start from a vector $\mathbf{c}_0$ satisfying $\mathbf{A}\mathbf{c}_0 = \mathbf{f}$, which is a reasonable guess for

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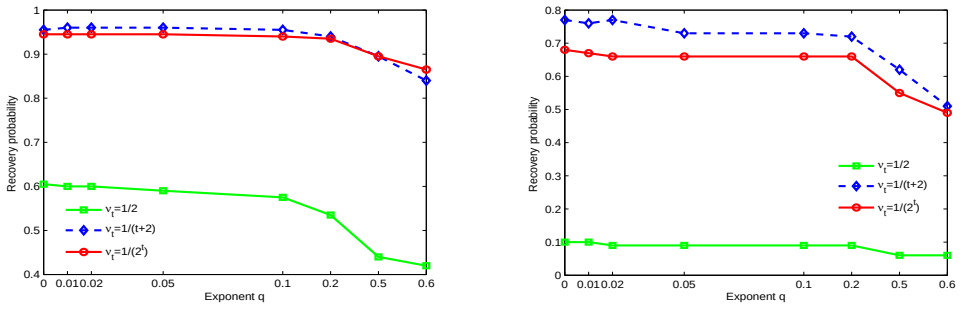


Figure 1: Frequency of success vs. parameter q for ℓ_q minimization using TD spaces, Legendre polynomials. Left: $d = 2$, $P = 20$, $N = 231$, $s = 10$, $M = 40$; Right: $d = 10$, $P = 4$, $N = 1001$, $s = 20$, $M = 80$.

$\mathbf{c}$, and then construct a sequence $(\mathbf{c}_t)$ recursively by defining $\mathbf{c}_{t+1}$ as a solution of the minimization problem [13]

$$\min_{\mathbf{c}\in\mathbb{R}^N} \sum_{i=1}^{N} \frac{\mathbf{c}_i}{(|\mathbf{c}_{t,i}| + \nu_t)^{1-q}} \quad \text{subject to } \mathbf{Ac} = \mathbf{f}.$$

According to the work [13], we will conduct two simple experiments to illustrate how to choose the parameter ν_t and q. The results are shown in Fig. 1 and we can see that $\nu_t = \left(\frac{1}{t+2}\right)$ performs best. For the exponential q, we select the best one in the set $\{0, 0.01, 0.05, 0.1\}$ for the all following experiments. Since the choice of q include $q = 0$, it is no surprising that ℓ_q minimization will perform better than reweighted ℓ_1 minimization which will be seen in the following examples.

4.1. Manufactured polynomial function recovery

In this part, we investigate the recovery performance of the ℓ_q minimization on uniform distributed points when the target function is assumed to be in sparse polynomial form. In particularly, we will recover a sparse solution vector $\mathbf{c}$ whose s non-zero entries randomly generated from standard normal distribution $\mathcal{N}(0, 1)$ while keeping the rest of coefficients zero. We repeat the experiments 100 independent realizations for each fixed sparsity s and calculate the success probability. Analogously, we fixed the number of points varying the sparsity level s from 2 to 26 with step size 3 and calculate the successful probability.

The left graph in Fig. 2 depicts the probability of recovery as a function of the sparsity when $d = 2, P = 20$, and $M = 40$ points are used. In this picture, we can see that when the sparsity $s \leq 5$, the ℓ_1-minimization can recover the coefficients $\mathbf{c}$ with high probability, however, when we use the ℓ_q minimization, even the sparsity $s = 8$, the successful probability still remains a high level, and the performance of reweighted ℓ_1 minimization is between the above two. Throughout the changing process of successful probability against the sparsity s, we obtain that the ℓ_q-minimization has overwhelming superiority, and reweighted ℓ_1 is much better than ℓ_1 minimization. It happens that there is a similar

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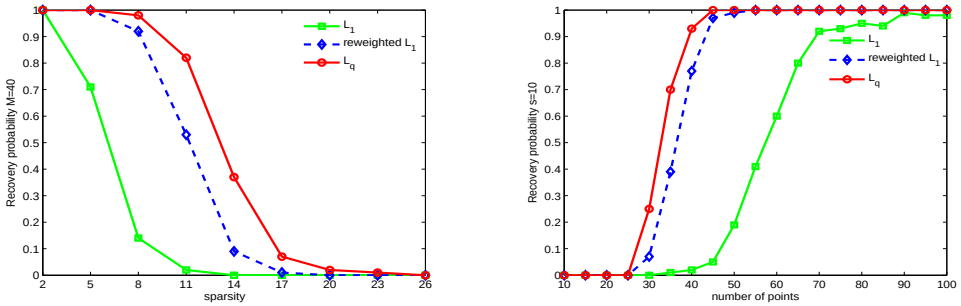


Figure 2: Numerical results of the sparse interpolation for the comparison of ℓ_1, reweighted ℓ_1 and ℓ_q minimization on the random sampling by Legendre polynomials in TD spaces. $d = 2$, $P = 20$, $N = 231$. Left: sparsity vs. approximate successful rate, $M = 40$. Right: samples vs.approximate successful rate, $s = 10$.

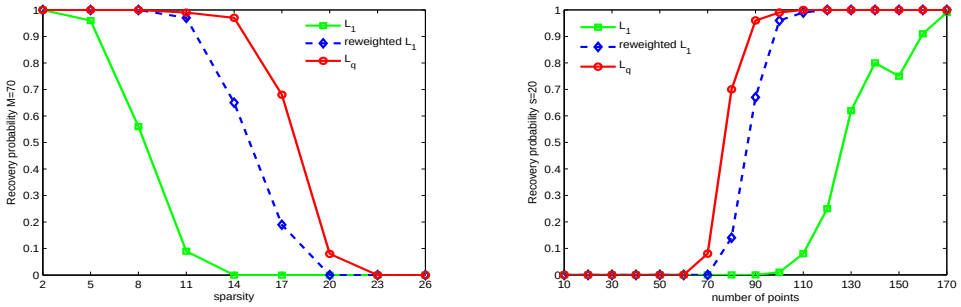


Figure 3: Numerical results of the sparse interpolation for the comparison of ℓ_1, reweighted ℓ_1 and ℓ_q minimization on the random sampling by Legendre polynomials in TD spaces. $d = 10$, $P = 4$, $N = 1001$. Left: sparsity vs. approximate successful rate, $M = 70$. Right: samples vs.approximate successful rate, $s = 20$.

case when we fixed the sparsity s while varying the number of samples. As the right graph shows that the success rate with respect to the number of samples, and fixed $s = 10$. We can obtain that, for the uniform distributed points, the performance of ℓ_q minimization outperforms than ℓ_1 minimization and reweighted ℓ_1 minimization is worse than ℓ_q but better than ℓ_1 as well. To make comparison with the low-dimensional case (Fig. 2), we consider high dimensional cases with $d = 10$. The recovery results are demonstrated in Fig. 3. Again, we see that the performance of the nonconvex solver ℓ_q minimization and reweighted ℓ_1 minimization are both superior than the convex ℓ_1 minimization, and the ℓ_q minimization is the best.

4.2. Recovery of analytical functions

Although most of the analytical functions do not have the finite representation in the orthogonal polynomials, part of them have "approximately" sparse representations. Here we consider a few functions of this form.

We report the numerical error in TD polynomial spaces for the following analytical

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functions:

$$f_1(\boldsymbol{z}) = \exp\left(-\sum_{i=1}^{d} 0.1|z_i - 1|\right),$$
$$f_2(\boldsymbol{z}) = \exp\left(-\sum_{i=1}^{d} c_i^2 (z_i - w)^2\right),$$
$$f_3(\boldsymbol{z}) = \frac{1}{\sum_{i=1}^{d} w + c_i z_i},$$

where c and w active as parameters, here, we choose $c_i = \exp(-6i/d)$ and $w = 0.5$.

To test the performance of the sparse approximation via Legendre polynomials, we employ the root-mean-square-error (RMSE). Specifically given a set of test points of $R = 10,000$ random samples $\{\boldsymbol{z}_j\}_{j=1}^{R} \in [-1,1]$ drawn from the uniform density, we evaluate the true (or regarded as reference) function $f(\boldsymbol{z}_j)$ on these points. Meanwhile, we compute the coefficients $\tilde{c}$ via the ℓ_q, reweighted ℓ_1 and ℓ_1 minimization so that obtain the approximation $\tilde{f}(\boldsymbol{z}_j)$ and compute the discrete ℓ_2 error

$$RMSE = \left(\frac{1}{R}\sum_{j=1}^{R} |\tilde{f}(\boldsymbol{z}_j) - f(\boldsymbol{z}_j)|^2\right)^{1/2}.$$

We try to recover a sparse representation of these functions in a set of Legendre polynomial basis and the numerical illustrations are given in Figs. 4, 5 and 6. In these and all subsequent plots, we show the discrete ℓ_2 error of 500 random trials. We note that in low dimension ($d = 2$) case, with the increasing of the number of sample points, the error of ℓ_q, reweighted ℓ_1 and ℓ_1 methods decrease rapidly. However, we note that the error obtained from ℓ_q minimization decays faster than reweighted ℓ_1 and ℓ_1 minimization. On the other hand, when the dimension goes high, for instance, $d = 10$, the error decays slowly comparing to the low dimension case. However, the ℓ_q minimization still has advantages than the other two methods.

4.3. Elliptic PDE with Random Inputs

Here, we will illustrate the computational performance of ℓ_q, reweighted ℓ_1 and ℓ_1 minimization for the following stochastic linear two-dimensional (in spatial) elliptic PDE problem

$$\begin{cases} -\nabla\cdot(a(\mathbf{y},\omega)\nabla u(\mathbf{y},\omega)) = f(\mathbf{y},\omega), & in\ \ \mathscr{D}\times\Omega, \\ u(\mathbf{y},\omega) = 0, & on\ \ \partial\mathscr{D}\times\Omega, \end{cases} \tag{4.1}$$

with spatial domain $\mathscr{D} = [0,1]^2$. We take a deterministic load $f(\mathbf{y},\omega) = \cos(y_1)\sin(y_2)$ for these numerical examples. And construct the random diffusion coefficient $a_N(\mathbf{y},\omega)$ with one-dimensional spatial dependence as in [1]:

$$\log(a_N(\mathbf{y},\omega) - 0.5) = 1 + z_1(\omega)\left(\frac{\sqrt{\pi}L}{2}\right)^{1/2} + \sum_{i=2}^{5} \zeta_i g_i(\mathbf{y}) z_i(\omega),$$

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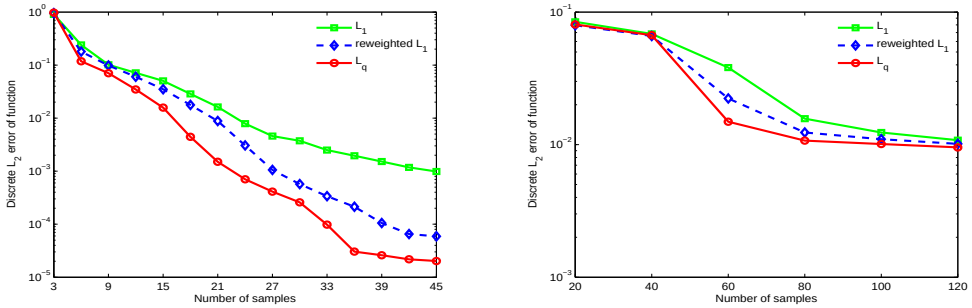


Figure 4: Numerical results of the analytical function $f_1(\boldsymbol{z})$ for the comparison of ℓ_1, reweighted ℓ_1 and ℓ_q minimization on the random sampling by Legendre polynomials in TD spaces. Left: $d=2$, $P=20$, $N=231$. Right: $d=10$, $P=4$, $N=1001$.

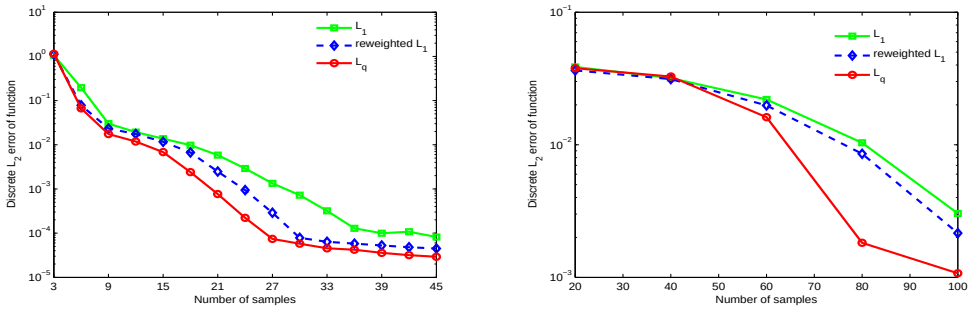


Figure 5: Numerical results of the analytical function $f_2(\boldsymbol{z})$ for the comparison of ℓ_1, reweighted ℓ_1 and ℓ_q minimization on the random sampling by Legendre polynomials in TD spaces. Left: $d=2$, $P=20$, $N=231$. Right: $d=10$, $P=4$, $N=1001$.

where

$$\zeta_i := (\sqrt{\pi}L)^{1/2}\exp\left(\frac{-(\lfloor\frac{i}{2}\rfloor\pi L)^2}{8}\right), \quad if\ i>1$$

$$g_i(\mathbf{y}) := \begin{cases} \sin\left(\frac{-(\lfloor\frac{i}{2}\rfloor\pi y_1}{L_p}\right), & if\ i\ even, \\ \cos\left(\frac{-(\lfloor\frac{i}{2}\rfloor\pi y_1}{L_p}\right), & if\ i\ odd. \end{cases}$$

Here $\{z_i\}_{i=1}^5$ are uniformly distributed on the interval $[-1,1]$. We assume that z_i are mutually independent from each other. Hence, a family of Legendre polynomials is used to approximate the quantities of interest of $\boldsymbol{z}$. For $y_1 \in [0,1]$, let $L_c = 1/32$ be a desired physical correlation length for $a(\mathbf{y},\omega)$. Then the parameter L_p and L are $L_p = \max\{1, 2L_c\}$ and $L = L_c/L_p$, respectively. The deterministic elliptic equation are solved by a standard finite element method and spatial domain $\mathscr{D}$ is partitioned into a 25×25 uniform, linear FEM mesh.

We approximate $u(\boldsymbol{z})$ in a Legendre PCE with total degree $P=6$ and seek to recover

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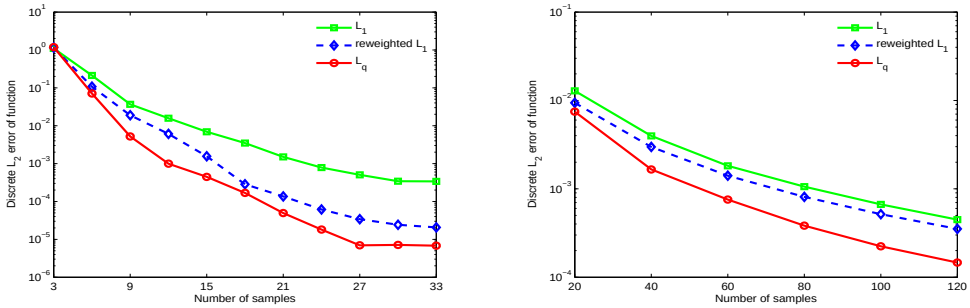


Figure 6: Numerical results of the analytical function $f_3(\boldsymbol{z})$ for the comparison of ℓ_1, reweighted ℓ_1 and ℓ_q minimization on the random sampling by Legendre polynomials in TD spaces. Left: $d=2$, $P=20$, $N=231$. Right: $d=10$, $P=4$, $N=1001$.

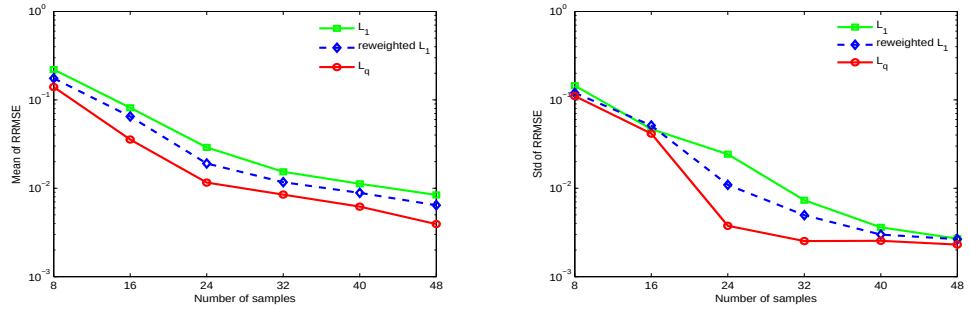


Figure 7: Comparison of the statistics of the RRMSE in reconstructing $u((0.5,0.5),\boldsymbol{z})$. Left: Mean of RRMSE; Right: Standard deviation of RRMSE. $d=5$, $P=6$, $N=462$.

the $N=462$ unknown coefficients. As the exact solution is not available, we use Monte Carlo method by 10,000 previously generated samples to obtain the reference solution of the PDE. The QoI $u(\boldsymbol{z})$ is chosen as the solution to (4.1) at location $\mathbf{y}=(0.5,0.5)$, i.e., $u(\boldsymbol{z})=u((0.5,0.5),\boldsymbol{z})$.

In Fig. 7, we compare the mean and standard deviation of RMSE for solutions computed by the ℓ_1, reweighted ℓ_1 and ℓ_q minimization. We note that the ℓ_q minimization again offers better performance over the ℓ_1 minimization and reweighted ℓ_1 minimization.

5. Conclusions

In this paper we discuss the sparse recovery of polynomial chaos expansion via the ℓ_q minimization and its applications in stochastic collocation. Some theoretical estimates are established for the recoverability of the ℓ_q minimization for both sparse and non-sparse functions. We can not reveal that ℓ_q minimization approximate the ℓ_0 minimization better than the ℓ_1 minimization from the theoretical estimates. However, the numerical experiments show that ℓ_q minimization performs better than ℓ_1 minimization and reweighted ℓ_1 minimization. Thus we believe that the ℓ_q minimization can be an efficient alternative for function approximation, especially for the high dimensional problems. It deserves more

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research to study the sharp bound of the restricted isometry constant for the ℓ_q minimization.

Appendix A: Proof of Theorem 3.1

Proof. The proof generally follows the lines of [5] and [3]. Let $\mathbf{h} = \hat{\mathbf{c}} - \mathbf{c}$. Without loss of generality, we assume $|\mathbf{h}(1)| \geq |\mathbf{h}(2)| \geq \cdots \geq |\mathbf{h}(s+1)| \geq |\mathbf{h}(s+2)| \geq \cdots$, for other cases, we can rearrange the indices if necessary.

Let $T = \{1, 2, \cdots, s\}$, and Ω be the support of $\mathbf{c}$. We get the following result, which is based on the minimality of $\hat{\mathbf{c}}$, has been used widely, see [3, 4, 6].

$$\|\mathbf{h}_\Omega\|_q \geq \|\mathbf{h}_{\Omega^c}\|_q. \tag{A.1}$$

Indeed,

$$\begin{aligned} &\|\mathbf{c}\|_q^q \geq \|\hat{\mathbf{c}}\|_q^q = \|\mathbf{h} + \mathbf{c}\|_q^q = \|\mathbf{h} + \mathbf{c}_\Omega + \mathbf{c}_{\Omega^c}\|_q^q \\ =&\|\mathbf{h}_\Omega + \mathbf{c}_\Omega + \mathbf{h}_{\Omega^c} + \mathbf{c}_{\Omega^c}\|_q^q = \|\mathbf{c}_\Omega + \mathbf{h}_\Omega\|_q^q + \|\mathbf{h}_{\Omega^c} + \mathbf{c}_{\Omega^c}\|_q^q \\ \geq&\|\mathbf{c}_\Omega\|_q^q - \|\mathbf{h}_\Omega\|_q^q + \|\mathbf{h}_{\Omega^c}\|_q^q - \|\mathbf{c}_{\Omega^c}\|_q^q. \end{aligned} \tag{A.2}$$

Meanwhile,

$$\|\mathbf{c}\|_q^q = \|\mathbf{c}_\Omega\|_q^q + \|\mathbf{c}_{\Omega^c}\|_q^q. \tag{A.3}$$

Combining (A.2) and (A.3) together, we can get

$$\|\mathbf{h}_{\Omega^c}\|_q^q \leq \|\mathbf{h}_\Omega\|_q^q + 2\|\mathbf{c}_{\Omega^c}\|_q^q.$$

In this theorem, we focus on the sparse case, i.e., $\mathbf{c}$ is s-sparse, so $\|\mathbf{c}_{\Omega^c}\|_q = 0$, then (A.1) holds:

$$\|\mathbf{h}_\Omega\|_q \geq \|\mathbf{h}_{\Omega^c}\|_q.$$

In the latter proof of Theorem 3.3, we concentrate on the non-sparse recovery, i.e., $\mathbf{c}$ is non-sparse, then the following inequality can be derived directly,

$$\|\mathbf{h}_{\Omega^c}\|_q \leq 2^{1/q-1}\big(\|\mathbf{h}_\Omega\|_q + 2^{1/q}\|\mathbf{c}_{\Omega^c}\|_q\big). \tag{A.4}$$

Note that $\Omega^c \cap T$ and $\Omega \cap T^c$ both have $s - |\Omega \cap T|$ elements, simultaneously, due to we have rearranged h_i in descending order, we obtain that

$$\|\mathbf{h}_{\Omega^c \cap T}\|_q \geq \|\mathbf{h}_{\Omega \cap T^c}\|_q.$$

Next, we shall show that this implies that

$$\|\mathbf{h}_T\|_q \geq \|\mathbf{h}_{T^c}\|_q. \tag{A.5}$$

In fact

$$\begin{aligned} &\|\mathbf{h}_T\|_q^q = \|\mathbf{h}_{\Omega\cap T}\|_q^q + \|\mathbf{h}_{\Omega^c\cap T}\|_q^q = \|\mathbf{h}_\Omega\|_q^q + \|\mathbf{h}_{\Omega^c\cap T}\|_q^q - \|\mathbf{h}_{\Omega\cap T^c}\|_q^q \\ \geq&\|\mathbf{h}_{\Omega^c}\|_q^q + \|\mathbf{h}_{\Omega^c\cap T}\|_q^q - \|\mathbf{h}_{\Omega\cap T^c}\|_q^q = \|\mathbf{h}_{\Omega^c\cap T^c}\|_q^q + 2\|\mathbf{h}_{\Omega^c\cap T}\|_q^q - \|\mathbf{h}_{\Omega\cap T^c}\|_q^q \\ =&\|\mathbf{h}_T\|_q^q + 2\|\mathbf{h}_{\Omega^c\cap T}\|_q^q - 2\|\mathbf{h}_{\Omega\cap T^c}\|_q^q \geq \|\mathbf{h}_{T^c}\|_q^q. \end{aligned}$$

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Partition $\{1,2,\cdots,N\}$ into the following sets:

$$S_0=\{1,2,\cdots,s_1\},\qquad S_1=\{s_1+1,s_1+2,\cdots,s_1+s_2\},$$
$$S_2=\{s_1+s_2+1,s_1+s_2+2,\cdots,s_1+2s_2\},\ \cdots$$

Then similar to set T, we can get $\|\mathbf{h}_{S_0}\|_q\geq\|\mathbf{h}_{S_0^c}\|_q$. Noticing that both $\Omega^c\cap S_0$ and $\Omega\cap S_0^c$ have the same number of elements, i.e $(s_1-|\Omega\cap S_0|)$, thus we have $\|\mathbf{h}_{\Omega^c\cap S_0}\|_q\geq\|\mathbf{h}_{\Omega\cap S_0^c}\|_q$. Following the same line of the proof of (A.5), we can get $\|\mathbf{h}_{S_0}\|_q\geq\|\mathbf{h}_{S_0^c}\|_q$, and then obtain

$$\|\mathbf{h}\|_q^q=\|\mathbf{h}_{S_0}\|_q^q+\|\mathbf{h}_{S_0^c}\|_q^q\leq 2\|\mathbf{h}_{S_0}\|_q^q. \tag{A.6}$$

For any fixed $0<q\leq 1$, it follows from Lemma 3.2 and Proposition 3.1 that

$$\begin{aligned}
\sum_{i\geq 1}\|\mathbf{h}_{S_i}\|_2&\leq\sum_{i\geq 1}\frac{\|\mathbf{h}_{S_i}\|_q}{s_2^{1/q-1/2}}+\lambda_q\sqrt{s_2}\big(|\mathbf{h}(s_1+1)|\\
&\qquad-|\mathbf{h}(s_1+s_2)|+|\mathbf{h}(s_1+s_2)+1|-|\mathbf{h}(s_1+2s_2)|+\cdots\big)\\
&\leq\frac{1}{s_2^{1/q-1/2}}\Big(\sum_{i\geq 1}\|\mathbf{h}_{S_i}\|_q\Big)+\lambda_q\sqrt{s_2}|\mathbf{h}(s_1+1)|\\
&\leq\frac{1}{s_2^{1/q-1/2}}\big(\|\mathbf{h}_{T^c}\|_q-(s_1-s)^{1/q}|\mathbf{h}(s_1+1)|\big)+\lambda_q\sqrt{s_2}|\mathbf{h}(s_1+1)|\\
&\leq\frac{1}{s_2^{1/q-1/2}}\big(\|\mathbf{h}_{T}\|_q-(s_1-s)^{1/q}|\mathbf{h}(s_1+1)|\big)+\lambda_q\sqrt{s_2}|\mathbf{h}(s_1+1)|\\
&\leq\frac{1}{s_2^{1/q-1/2}}\big(\|\mathbf{h}_{S_0}\|_q-2(s_1-s)^{1/q}|\mathbf{h}(s_1+1)|\big)+\lambda_q\sqrt{s_2}|\mathbf{h}(s_1+1)|\\
&\leq\frac{1}{s_2^{1/q-1/2}}\|\mathbf{h}_{S_0}\|_q+\left(\lambda_q\sqrt{s_2}-\frac{2(s_1-s)^{1/q}}{s_2^{1/q-1/2}}\right)|\mathbf{h}(s_1+1)|\\
&\leq\frac{1}{s_2^{1/q-1/2}}\|\mathbf{h}_{S_0}\|_q+\left(\lambda_q\frac{\sqrt{s_2}}{s_1^{1/q}}-\frac{2(s_1-s)^{1/q}}{s_1^{1/q}s_2^{1/q-1/2}}\right)\|\mathbf{h}_{S_0}\|_q\\
&\leq\left(\frac{1}{s_2^{1/q-1/2}}+\lambda_q\frac{\sqrt{s_2}}{s_1^{1/q}}-\frac{2(s_1-s)^{1/q}}{s_1^{1/q}s_2^{1/q-1/2}}\right)\|\mathbf{h}_{S_0}\|_q\\
&\leq\left(\Big(\frac{s_1}{s_2}\Big)^{1/q-1/2}+\lambda_q\sqrt{\frac{s_2}{s_1}}-\frac{2(s_1-s)^{1/q}}{\sqrt{s_1}s_2^{1/q-1/2}}\right)\|\mathbf{h}_{S_0}\|_2.
\end{aligned}$$

Set

$$K=\left(\frac{s_1}{s_2}\right)^{1/q-1/2}+\lambda_q\sqrt{\frac{s_2}{s_1}}-\frac{2(s_1-s)^{1/q}}{\sqrt{s_1}s_2^{1/q-1/2}},$$

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where $\lambda_q := \left(\frac{q}{2}\right)^{\frac{q}{2-q}} - \left(\frac{q}{2}\right)^{\frac{2}{2-q}}$. Now, combining (2.12) and (2.13), we can get that

$$\begin{aligned}\left|\langle \mathbf{Ah}, \mathbf{Ah}_{S_0}\rangle\right| &= \left|\langle \mathbf{Ah}_{S_0}, \mathbf{Ah}_{S_0}\rangle + \sum_{i\geq 1}\langle \mathbf{Ah}_{S_i}, \mathbf{Ah}_{S_0}\rangle\right| \\ &\geq (1-\delta_{s_1})\|\mathbf{h}_{S_0}\|_2^2 - \theta_{s_1,s_2}\|\mathbf{h}_{S_0}\|_2 \sum_{i\geq 1}\|\mathbf{h}_{S_i}\|_2 \\ &\geq \left(1-\delta_{s_1} - K\theta_{s_1,s_2}\right)\|\mathbf{h}_{S_0}\|_2^2.\end{aligned}$$

Note that

$$\|\mathbf{Ah}\|_2 = \|\mathbf{A}(\mathbf{c}-\hat{\mathbf{c}})\|_2 \leq \|\mathbf{Ac}-\mathbf{f}\|_2 + \|\mathbf{A}\hat{\mathbf{c}}-\mathbf{f}\|_2 \leq 2\varepsilon$$

and

$$\|\mathbf{h}\|_2^2 = \|\mathbf{h}_{\mathbf{S_0}}\|_2^2 + \|\mathbf{h}_{\mathbf{S_0^c}}\|_2^2 \leq 2\|\mathbf{h}_{\mathbf{S_0}}\|_2^2.$$

Putting the above two inequalities together we get

$$\begin{aligned}\|\mathbf{h}\|_2 \leq \sqrt{2}\|\mathbf{h}_{S_0}\|_2 &\leq \frac{\sqrt{2}\|\mathbf{Ah}\|_2\|\mathbf{Ah}_{S_0}\|_2}{\left(1-\delta_{s_1}-K\theta_{s_1,s_2}\right)\|\mathbf{h}_{S_0}\|_2} \\ \leq \frac{2\sqrt{2}\varepsilon\sqrt{1+\delta_{s_1}}\|\mathbf{h}_{S_0}\|_2}{\left(1-\delta_{s_1}-K\theta_{s_1,s_2}\right)\|\mathbf{h}_{S_0}\|_2} &\leq \frac{2\sqrt{2}\sqrt{1+\delta_{s_1}}}{\left(1-\delta_{s_1}-K\theta_{s_1,s_2}\right)}\varepsilon.\end{aligned}$$

The proof is completed. □

Appendix B: Proof of Theorem 3.2

Proof. Based on the Theorem 3.1, let us set $s_1 = s$ and $s_2 = \frac{4}{9}s$, where $s \equiv 0(\text{mod } 9)$. Then

$$K = \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right).$$

If $\mathbf{A}$ satisfies

$$\delta_s + \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right)\theta_{s,\frac{4}{9}s} < 1,$$

the solution $\hat{\mathbf{c}}$ of (3.4) satisfies

$$\|\hat{\mathbf{c}}-\mathbf{c}\|_2 \leq \frac{2\sqrt{2}\sqrt{1+\delta_s}}{1-\delta_s - \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right)\theta_{s,\frac{4}{9}s}}\varepsilon.$$

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Note that

$$\begin{aligned}
&\delta_s + \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right)\theta_{s,\frac{4}{9}s}\\
=&\delta_s + \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right)\theta_{\frac{9}{5}\frac{5}{9}s,\frac{4}{9}s}\\
\le&\delta_s + \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right)\sqrt{9/5}\theta_{\frac{5}{9}s,\frac{4}{9}s}\\
\le&\left(1 + \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right)\sqrt{9/5}\right)\delta_s\\
=&(1+\eta(q))\delta_s < 1.
\end{aligned}$$

Hence, under the condition

$$\delta_s < \frac{1}{1+\eta(q)},$$

we have

$$\begin{aligned}
\|\hat{\mathbf{c}} - \mathbf{c}\|_2 &\le \frac{2\sqrt{2}\sqrt{1+\delta_s}}{1-\delta_s - \left(\frac{2}{3}\lambda_q + \left(\frac{9}{4}\right)^{1/q-1/2}\right)\theta_{s,\frac{4}{9}s}}\varepsilon\\
&\le \frac{2\sqrt{2(1+1/(1+\eta(q)))}}{1-(1+\eta(q))\delta_s}\varepsilon.
\end{aligned}$$

The proof is completed. □

Appendix C: Proof of Theorem 3.3

Proof. The proof mostly follows the line in [6]. Due to the Lemma 2.1 in [6], i.e.,

$$|\langle \mathbf{Ac}, \mathbf{Ac}'\rangle| \le \delta_{s+s'}\|\mathbf{c}\|_2\|\mathbf{c}'\|_2, \tag{C.7}$$

for all $\mathbf{c}, \mathbf{c}'$ supported on the disjoint subsets $T, T' \subset \{1, \cdots, N\}$ with $|T| \le s, |T'| \le s'$.

Set $\hat{\mathbf{c}} = \mathbf{c} + \mathbf{h}$ and decompose $\mathbf{h}$ into a sum of vectors $\mathbf{h}_{T_0}, \mathbf{h}_{T_1}, \mathbf{h}_{T_2}, \cdots$, each of sparsity at most s. Let T_0 be the support of $\mathbf{c}$; T_1 to the locations of the s largest coefficients of $\mathbf{h}_{T_0^c}$; T_2 to the locations of the next s largest coefficients of $\mathbf{h}_{T_0^c}$, and so on. Also denote $T_{01} = T_0 \cup T_1$ and $\mathbf{c}_{T_0^c} = \mathbf{c} - \mathbf{c_s}$.

For each $j \le 2$, for $i_1 \in T_j$ and $i_2 \in T_{j-1}$, from the inequality $|\mathbf{h}_{i_1}| \le |\mathbf{h}_{i_2}|$, hence $\left|\mathbf{h}_{i_1}\right|^q \le s^{-1}\left\|\mathbf{h}_{T_{j-1}}\right\|_q^q$, so $\left|\mathbf{h}_{i_1}\right| \le s^{-1/q}\left\|\mathbf{h}_{T_{j-1}}\right\|_q$, finally, we have

$$\left\|\mathbf{h}_{T_j}\right\|_2 \le s^{1/2-1/q}\left\|\mathbf{h}_{T_{j-1}}\right\|_q.$$

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Thus,

$$\sum_{j\geq 2}\left\|\mathbf{h}_{T_j}\right\|_2 \leq s^{1/2-1/q}\left\|\mathbf{h}_{T_0^c}\right\|_q.$$

Now, we will show that $\left\|h_{(T_0\cup T_1)^c}\right\|_2$ is essentially bounded by $\|h_{(T_0\cup T_1)}\|_2$.
First, we can easily get that

$$\left\|\mathbf{h}_{(T_0\cup T_1)^c}\right\|_2 = \left\|\sum_{j\geq 2}\mathbf{h}_{T_j}\right\|_2 \leq \sum_{j\geq 2}\left\|\mathbf{h}_{T_j}\right\|_2 \leq s^{1/2-1/q}\left\|\mathbf{h}_{T_0^c}\right\|_q.$$

Similar to (A.4), with T_0 instead of Ω, we have

$$\left\|\mathbf{h}_{T_0^c}\right\|_q \leq 2^{1/q-1}\left(\left\|\mathbf{h}_{T_0}\right\|_q + 2^{1/q}\left\|\mathbf{c}_{T_0^c}\right\|_q\right), \tag{C.8}$$

So we obtain

$$\left\|\mathbf{h}_{(T_0\cup T_1)^c}\right\|_2 \leq 2^{1/q-1}s^{1/2-1/q}\left(\left\|\mathbf{h}_{T_0}\right\|_q + 2^{1/q}\left\|\mathbf{c}_{T_0^c}\right\|_q\right). \tag{C.9}$$

Next, we set bounds for $\|\mathbf{h}_{T_0\cup T_1}\|_2$. From the fact that $\mathbf{A}\mathbf{h}_{T_0\cup T_1} = \mathbf{A}\mathbf{h} - \sum_{j\geq 2}\mathbf{A}\mathbf{h}_{T_j}$, we obtain

$$\left\|\mathbf{A}\mathbf{h}_{T_0\cup T_1}\right\|_2^2 = \left|\langle \mathbf{A}\mathbf{h}_{T_0\cup T_1}, \mathbf{A}\mathbf{h}\rangle - \sum_{j\geq 2}\langle \mathbf{A}\mathbf{h}_{T_0\cup T_1}, \mathbf{A}\mathbf{h}_{T_j}\rangle\right|.$$

On the one hand

$$\left|\langle \mathbf{A}\mathbf{h}_{T_0\cup T_1}, \mathbf{A}\mathbf{h}\rangle\right| \leq \left\|\mathbf{A}\mathbf{h}_{T_0\cup T_1}\right\|_2\left\|\mathbf{A}\mathbf{h}\right\|_2 \leq 2\varepsilon\sqrt{1+\delta_{2s}}\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2,$$

On the other hand

$$\left|\langle \mathbf{A}\mathbf{h}_{T_i}, \mathbf{A}\mathbf{h}_{T_j}\rangle\right| \leq \delta_{2s}\left\|\mathbf{h}_{T_i}\right\|_2\left\|\mathbf{h}_{T_j}\right\|_2,\ i=0,1.$$

Therefore, due to $\left\|\mathbf{h}_{T_0}\right\|_2 + \left\|\mathbf{h}_{T_1}\right\|_2 \leq \sqrt{2}\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2$,

$$\begin{aligned}
&(1-\delta_{2s})\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2^2 \leq \left\|\mathbf{A}\mathbf{h}_{T_0\cup T_1}\right\|_2^2\\
\leq& 2\varepsilon\sqrt{1+\delta_{2s}}\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 + \sum_{j\geq 2}\left|\langle \mathbf{A}\mathbf{h}_{T_0\cup T_1}, \mathbf{A}\mathbf{h}_{T_j}\rangle\right|\\
\leq& 2\varepsilon\sqrt{1+\delta_{2s}}\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 + \delta_{2s}\left(\left\|\mathbf{h}_{T_0}\right\|_2 + \left\|\mathbf{h}_{T_1}\right\|_2\right)\sum_{j\geq 2}\left\|\mathbf{h}_{T_j}\right\|_2\\
\leq& \left\|\mathbf{h}_{T_0\cup T_1}\right\|_2\left(2\varepsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s}\sum_{j\geq 2}\left\|\mathbf{h}_{T_j}\right\|_2\right).
\end{aligned}$$

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Dividing $(1-\delta_{2s}\|\mathbf{h}_{T_0\cup T_1}\|_2)$ and combining (C.8), we obtain

$$\begin{aligned}
\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 &\le \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\varepsilon + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}\sum_{j\ge 2}\left\|\mathbf{h}_{T_j}\right\|_2\\
&\le \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\varepsilon + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}s^{1/2-1/q}\left\|\mathbf{h}_{T_0^c}\right\|_q\\
&\le \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\varepsilon + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}s^{1/2-1/q}2^{1/q-1}\left(2^{1/q}\left\|\mathbf{c}_{T_0^c}\right\|_q + \left\|\mathbf{h}_{T_0}\right\|_q\right)\\
&\le \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\varepsilon + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}s^{1/2-1/q}2^{1/q-1}\left(2^{1/q}\left\|\mathbf{c}_{T_0^c}\right\|_q + s^{1/q-1/2}\left\|\mathbf{h}_{T_0}\right\|_2\right)\\
&\le \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}\varepsilon + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}2^{1/q-1}\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 + \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}s^{1/2-1/q}2^{2/q-1}\left\|\mathbf{c}_{T_0^c}\right\|_q.
\end{aligned}$$

Set $\alpha = \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}$ and $\beta = \frac{\sqrt{2}\delta_{2s}}{1-\delta_{2s}}$, then we can rewrite the above equation:

$$(1-2^{1/q-1}\beta)\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 \le \alpha\varepsilon + 2^{2/q-1}s^{1/2-1/q}\beta\left\|\mathbf{c}_{T_0^c}\right\|_q.$$

Hence

$$\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 \le \frac{\alpha}{1-2^{1/q-1}\beta}\varepsilon + \frac{2^{2/q-1}s^{1/2-1/q}\beta}{1-2^{1/q-1}\beta}\left\|\mathbf{c}_{T_0^c}\right\|_q.$$

Finally, combining with (C.9), we have

$$\begin{aligned}
\left\|\mathbf{h}\right\|_2 &\le \left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 + \left\|\mathbf{h}_{(T_0\cup T_1)^c}\right\|_2\\
&\le \left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 + 2^{1/q-1}s^{1/2-1/q}\left\|\mathbf{h}_{T_0}\right\|_q + 2^{2/q-1}s^{1/2-1/q}\left\|\mathbf{c}_{T_0^c}\right\|_q\\
&\le \left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 + 2^{1/q-1}\left\|\mathbf{h}_{T_0}\right\|_2 + 2^{2/q-1}s^{1/2-1/q}\left\|\mathbf{c}_{T_0^c}\right\|_q\\
&\le \left(2^{1/q-1}+1\right)\left\|\mathbf{h}_{T_0\cup T_1}\right\|_2 + 2^{2/q-1}s^{1/2-1/q}\left\|\mathbf{c}_{T_0^c}\right\|_q\\
&\le \frac{1+2^{1/q-1}}{1-2^{1/q-1}\beta}\alpha\varepsilon + \frac{2^{2/q-1}(1+\beta)}{1-2^{1/q-1}\beta}s^{1/2-1/q}\left\|\mathbf{c}_{T_0^c}\right\|_q\\
&= C_{2,q}\varepsilon + C_{1,q}\left\|\mathbf{c}-\mathbf{c}_{\mathbf{s}}\right\|_q,
\end{aligned}$$

where

$$\begin{aligned}
C_{1,q} &= \frac{2^{2/q-1}(1+\beta)}{1-2^{1/q-1}\beta}s^{1/2-1/q} = \frac{2^{2/q-1}(1+(\sqrt{2}-1)\delta_{2s})}{1-(1+2^{1/q-1/2})\delta_{2s}}s^{1/2-1/q},\\
C_{2,q} &= \frac{1+2^{1/q-1}}{1-2^{1/q-1}\beta}\alpha = \frac{(2+2^{1/q})\sqrt{1+\delta_{2s}}}{1-(1+2^{1/q-1/2})\delta_{2s}}.
\end{aligned}$$

Then the proof is complete. □

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