Proc. Camb. Phil. Soc. (1974), **75**, 261
PCPS 75–28
Printed in Great Britain

Some wave solutions in general relativity

BY L. K. PATEL
Department of Mathematics, Gujarat University, Ahmedabad 9 (*India*)

Abstract. A general scheme for the derivation of wave solutions in general relativity is developed. Some solutions describing the flow of gravitational waves are discussed. Singular electromagnetic fields corresponding to one particular solution are also discussed.

1. *Introduction.* The theory of gravitational and electromagnetic waves has been discussed by many investigators. Takeno (7) has studied the mathematical theory of plane gravitational waves in detail. Peres (6) has studied a plane wave-like solution whose geometry can be described by the line element

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 - 2f(dt-dz)^2, \tag{1·1}$$

where f is a function of x, y and u, u being $t-z$. The metric (1·1) is also discussed by Vaidya and Pandya (8) in connection with gravitational and electromagnetic radiation. A particular case of the metric (1·1) is discussed by Zund (9) in connection with total pure radiation in general relativity. In fact, Peres's solution is a particular case of a more general solution obtained by Pandya and Vaidya (2). The metric studied by them can be written in the form

$$ds^2 = -e^{\phi}(dx^2+dy^2) + 2du\,dz + (1-2f)\,du^2, \tag{1·2}$$

where ϕ and f are functions of x, y and $u = t-z$.

The object of the present investigation is to formulate a general scheme for the derivation of wave solutions which include the results obtained by Peres, Pandya and Vaidya and Zund as particular cases. For the discussion of such wave solutions we require some geometrical preliminaries. We shall first discuss these.

2. *Geometrical preliminaries.* In this section, we follow closely Patel and Vaidya (3). All operations take place in flat space-time, and indices are raised and lowered by the Minkowskian metric tensor.

In Minkowskian space-time, consider an arbitrary smooth world line L that is everywhere time-like. Let x^i be a general coordinate frame in this Minkowskian space-time and let λ_i be the unit tangent vector at any point P of L. Let A_i, B_i, C_i be three mutually orthogonal space-like unit vectors lying in the 3 flat orthogonal to λ_i at the point under consideration. Thus we have the following relations:

$$\lambda^i\lambda_i = 1, \quad A^iA_i = B^iB_i = C^iC_i = -1 \tag{2·1}$$

and

$$\lambda^iA_i = \lambda^iB_i = \lambda^iC_i = A^iB_i = B^iC_i = C^iA_i = 0. \tag{2·2}$$

We define new coordinates x, y, z and t in terms of x^i by the relations

$$x = x^iA_i, \quad y = x^iB_i, \quad z = x^iC_i, \quad t = x^i\lambda_i. \tag{2·3}$$

https://doi.org/10.1017/S0305004100048465 Published online by Cambridge University Press

In Minkowskian space-time, the concept of uniform vector fields is valid. As a matter of fact, we can find four uniform vector fields such that any two of them are orthogonal, one of them is time-like and other three space-like. We take (λ_i, A_i, B_i, C_i) to be the vectors through the point P of such uniform vector fields. Thus, $A_{i,k} = 0$, $B_{i,k} = 0$, $C_{i,k} = 0$, $\lambda_{i,k} = 0$ where a comma denotes partial differentiation (e.g. $\lambda_{i,k} = \partial\lambda_i/\partial x^k$, etc.). Therefore, in what follows, these four vectors are assumed to be constant.

We use the following conventions:

A semicolon indicates covariant differentiation; round brackets indicate symmetrization over the indices enclosed and square brackets indicate anti-symmetrization.

Consider a Riemannian space described by the line element $ds^2 = g_{ik}dx^i dx^k$ with g_{ik} given by

$$g_{ik} = \eta_{ik} + SZ_iZ_k + HA_iA_k + NB_iB_k, \tag{2·4}$$

where S, H and N are functions of x, y and u and

$$Z_i = \lambda_i - C_i. \tag{2·5}$$

From (2·5) it follows that

$$Z^iZ_i = 0. \tag{2·6}$$

Thus Z_i is a null vector in Minkowskian space-time.

3. *Gravitational waves.* The determinant g and the contravariant components g^{ik} of the metric tensor g_{ik} given by (2·4) can be easily computed. They are given by

$$g = |g_{ik}| = -(1-H)(1-N) \tag{3·1}$$

and

$$g^{ik} = \eta^{ik} - SZ^iZ^k - \frac{H}{1-H}A^iA^k - \frac{N}{1-N}B^iB^k. \tag{3·2}$$

The following facts can be easily established:

(i) The raising and lowering of vector indices of Z^i can be carried out with respect to Minkowskian or Riemannian metric (2·4).

(ii) The null character of Z_i with respect to Minkowskian metric implies its null character with respect to the Riemannian metric (2·4).

(iii) $g^{ik}A_i = \dfrac{A^k}{1-H}$, $\quad g^{ik}B_i = \dfrac{B^k}{1-N}$.

Here also, we shall continue to use Minkowskian metric for raising and lowering of vector indices and any dependence on g_{ik} will be explicitly written out as in (iii). From this result we have

$$g^{ik}A_iA_k = -\frac{1}{1-H}, \quad g^{ik}B_iB_k = -\frac{1}{1-N}. \tag{3·3}$$

As g is negative, $1-H$ and $1-N$ should be of the same sign. A_i and B_i are space-like unit vectors with respect to the Minkowskian metric. In order to have the space-like

https://doi.org/10.1017/S0305004100048465 Published online by Cambridge University Press

character of A_i and B_i with respect to the metric (2·4), it is evident from (3·3) that $1-H>0$, $1-N>0$. The 3-index symbols for the metric (2·4) are given by

$$\begin{aligned}2\begin{Bmatrix}n\\ik\end{Bmatrix} &= 2Z^nS_{,(i}Z_{k)}+\frac{2A^n}{1-H}H_{,(i}A_{k)}+\frac{2B^n}{1-N}N_{,(i}B_{k)}\\ &\quad-\left(\eta^{na}S_{,a}+\frac{HS_x}{1-H}A^n+\frac{NS_y}{1-N}B^n\right)Z_iZ_k\\ &\quad-\left(\eta^{na}H_{,a}+\frac{HH_x}{1-H}A^n+\frac{NH_y}{1-N}B^n\right)A_iA_k\\ &\quad-\left(\eta^{na}N_{,a}+\frac{HN_x}{1-H}A^n+\frac{NN_y}{1-N}B^n\right)B_iB_k. \qquad (3\cdot4)\end{aligned}$$

Throughout the paper the lower suffixes attached to functional symbols denote partial derivative of the function with respect to the corresponding variable, e.g.

$$S_y=\partial S/\partial y, \quad H_{xu}=\partial^2H/\partial x\,\partial u, \quad \text{etc.}$$

It is evident from (3·4) that

$$\begin{Bmatrix}n\\ik\end{Bmatrix}A_n=\frac{1}{2(1-H)}\{S_xZ_iZ_k-H_xA_iA_k+N_xB_iB_k-2H_uA_{(i}Z_{k)}-2H_yA_{(i}B_{k)}\}, \quad (3\cdot5)$$

$$\begin{Bmatrix}n\\ik\end{Bmatrix}B_n=\frac{1}{2(1-N)}\{S_yZ_iZ_k-N_yB_iB_k+H_yA_iA_k-2N_uB_{(i}Z_{k)}-2N_xA_{(i}B_{k)}\} \quad (3\cdot6)$$

and

$$\begin{Bmatrix}n\\ik\end{Bmatrix}Z_n=0. \qquad (3\cdot7)$$

From (3·7) we conclude that the null congruence Z_i is geodetic. It can be easily verified that the congruence Z_i is irrotational, shear-free and expansion-free.

The Ricci tensor for the metric (2·4) can be expressed as

$$\begin{aligned}4R_{ik} &= \{2\psi_{xx}+\psi_x^2-\phi_x\psi_x+e^{\phi-\psi}(2\phi_{yy}+\phi_y^2-\psi_y\phi_y)\}A_iA_k\\ &\quad+\{2\phi_{yy}+\phi_y^2-\psi_y\phi_y+e^{\psi-\phi}(2\psi_{xx}+\psi_x^2-\psi_x\phi_x)\}B_iB_k\\ &\quad+\{2\phi_{uu}+\phi_u^2+2\psi_{uu}+\psi_u^2+e^{-\psi}(S_y\psi_y-S_y\phi_y+2S_{yy})\\ &\quad+e^{-\phi}(S_x\phi_x-S_x\psi_x+2S_{xx})\}Z_iZ_k+2(2\psi_{xu}+\psi_x\psi_u-\phi_u\psi_x)A_{(i}Z_{k)}\\ &\quad+2(2\phi_{yu}+\phi_y\phi_u-\psi_u\phi_y)B_{(i}Z_{k)}, \qquad (3\cdot8)\end{aligned}$$

where $e^{\phi}=1-H$ and $e^{\psi}=1-N$.

Therefore for gravitational waves ($R_{ik}=0$) ϕ, ψ, S have to satisfy the following four equations:

$$e^{\psi}(2\psi_{xx}+\psi_x^2-\phi_x\psi_x)+e^{\phi}(2\phi_{yy}+\phi_y^2-\phi_y\psi_y)=0, \qquad (3\cdot9)$$

$$2\psi_{xu}+\psi_x\psi_u-\phi_u\psi_x=0, \qquad (3\cdot10)$$

$$2\phi_{yu}+\phi_y\phi_u-\phi_y\psi_u=0 \qquad (3\cdot11)$$

and

$$2\phi_{uu}+\phi_u^2+2\psi_{uu}+\psi_u^2+e^{-\psi}(S_y\psi_y-S_y\phi_y+2S_{yy})+e^{-\phi}(S_x\phi_x-S_x\psi_x+2S_{xx})=0. \quad (3\cdot12)$$

https://doi.org/10.1017/S0305004100048465 Published online by Cambridge University Press

As ϕ and ψ have to satisfy three differential equations (3·9), (3·10) and (3·11), we must have to establish the condition of consistency of these equations. The equations (3·10) and (3·11) are readily integrable giving the first integrals

$$e^{\psi}\psi_x^2 = f(x,y)\,e^{\phi}, \quad e^{\phi}\phi_y^2 = F(x,y)\,e^{\psi}, \tag{3·13}$$

where f and F are arbitrary functions of x and y. The equations (3·9) and (3·13) lead to the following simple relation between f and F,

$$\sqrt{F}f_x + \sqrt{f}F_y = 0 \tag{3·14}$$

provided $f \neq 0$, $F \neq 0$.

The equation (3·14) is the required condition of consistency of the equations (3·9), (3·10) and (3·11). Here it should be noted that the explicit form of the line-element for the solution is

$$ds^2 = (1+S)\,du^2 + 2du\,dz - e^{\phi}\,dx^2 - e^{\psi}\,dy^2. \tag{3·15}$$

Due to the non-linearity of the equations (3·9), (3·10), (3·11) and (3·12), it seems difficult to obtain their general solution. Therefore we shall now consider some particular solutions of these equations.

Here Kerr and Schild's(4) work is worth mentioning. They have shown that the solution studied by Kerr(5) is of the form discussed by us. Kerr's designation of these solutions as 'wave solutions' must therefore be considered as problematic and unsubstantiated for the Kerr solution is not a wave solution.

4. *Particular cases*

Case (i). Let $\phi = \psi$. In this case the equations (3·9), (3·10), (3·11) and (3·12) reduce to

$$\phi_{xx} + \phi_{yy} = 0, \tag{4·1}$$

$$\phi_{xu} = \phi_{yu} = 0 \tag{4·2}$$

and

$$e^{\phi}(2\phi_{uu} + \phi_u^2) + S_{xx} + S_{yy} = 0. \tag{4·3}$$

Here the condition of consistency (3·14) is satisfied identically. The equations (4·1), (4·2) and (4·3) describe the solution discussed by Pandya and Vaidya(2).

However, if ϕ is independent of u, then for gravitational waves, we have

$$\phi_{xx} + \phi_{yy} = 0, \quad S_{xx} + S_{yy} = 0. \tag{4·4}$$

The simplest solution of $S_{xx} + S_{yy} = 0$ is $S = ax + by + c$ where a, b, c are arbitrary constants.

If we choose $a = 1$, $c = -1$, $b = 0$, we get a special case of the class of plane-fronted stationary waves discussed by Hoffman(1). The metric in this case can be expressed in the form

$$ds^2 = x\,du^2 + 2du\,dz - e^{\phi}(dx^2 + dy^2), \tag{4·5}$$

where ϕ satisfies two-dimensional Laplace's equation.

It is clear that if we get $\phi = 0$ in the solution of Pandya and Vaidya we get Peres's solution.

https://doi.org/10.1017/S0305004100048465 Published online by Cambridge University Press

Case (ii). $\phi_y = \psi_x = 0$. In this case equations (3·9), (3·10) and (3·11) are identically satisfied. The expression for Ricci tensor reduces to

$$4R_{ik} = \{2\phi_{uu} + \phi_u^2 + 2\psi_{uu} + \psi_u^2 + e^{-\psi}(S_y\psi_y + 2S_{yy}) + e^{-\phi}(S_x\phi_x + 2S_{xx})\} Z_i Z_k. \quad (4{\cdot}6)$$

For gravitational waves, we obtain

$$2\phi_{uu} + \phi_u^2 + 2\psi_{uu} + \psi_u^2 + e^{-\psi}(S_y\psi_y + 2S_{yy}) + e^{-\phi}(S_x\phi_x + 2S_{xx}) = 0. \quad (4{\cdot}7)$$

Therefore, for gravitational waves, the three unknown functions ϕ, ψ and S have to satisfy only one equation (4·7). Hence the choice of any two of those functions is at our disposal. The choice $\phi = \psi = 0$ leads to the solution discussed by Peres.

The expression for Riemann curvature tensor in this case is

$$\begin{aligned} R_{hijk} = {} & \{2S_{xx} - S_x\phi_x - e^{\phi}(2\phi_{uu} + \phi_u^2)\} A_{[k}Z_{j]} A_{[h}Z_{i]} \\ & + \{2S_{yy} - S_y\psi_y - e^{\psi}(2\psi_{uu} + \psi_u^2)\} B_{[k}Z_{j]} B_{[h}Z_{i]} \\ & + 2S_{xy}\{A_{[k}Z_{j]} B_{[h}Z_{i]} + B_{[k}Z_{j]} A_{[h}Z_{i]}\}. \end{aligned} \quad (4{\cdot}8)$$

If $S = 0$ and $\phi = 0$ (or $\psi = 0$), then the results (4·6) and (4·8) give that $R_{ik} = 0$ implies $R_{hijk} = 0$ and the geometry becomes Minkowskian.

If we choose $S \neq 0$, $\phi \neq 0$ (or $\psi \neq 0$), we get the gravitational field which is different from those discussed by Peres and Pandya and Vaidya.

5. *States of total pure radiation.* Linchnerowicz (see Zund (9)) has defined a state of total pure radiation by the equations

$$l^h R_{hijk} = 0, \quad l_p R_{hijk} + l_i R_{phjk} + l_h R_{ipjk} = 0, \quad (5{\cdot}1)$$

$$R_{ik} = \Omega l_i l_k, \quad (5{\cdot}2)$$

where l_i is a real non-zero null vector and Ω is a real non-zero scalar function of the coordinates x^i.

As Z_i is a null vector with regard to the Riemannian metric (2·4), without loss of generality we may choose

$$l_i = lZ_i, \quad (5{\cdot}3)$$

where $l \neq 0$ and it is clear that $g^{ik}l_i l_k = 0$. With the help of (4·6), (4·8) and (5·3), it is easy to verify that Lichnerowicz equations (5·1) are identically satisfied. Equation (5·2) gives us

$$4\Omega l^2 = 2\phi_{uu} + \phi_u^2 + 2\psi_{uu} + \psi_u^2 + e^{-\psi}(S_y\psi_y + 2S_{yy}) + e^{-\phi}(S_x\phi_x + 2S_{xx}). \quad (5{\cdot}4)$$

From (4·8) it can be easily seen that the Riemannian space described by the metric (2·4) with $\psi_x = \phi_y = 0$ is not recurrent in general, i.e. there exists no non-zero vector K_p such that $R_{hijl;\,p} = K_p R_{hijl}$. If we set

$$\phi = \psi = 0 \quad \text{and} \quad S = \beta(a_{11}x^2 + 2a_{12}xy + a_{22}y^2), \quad (5{\cdot}5)$$

where all a's are constants and β is a function of u only, then the expression for R_{hijk} becomes

$$\beta^{-1}R_{hijk} = 4a_{11}A_{[k}Z_{j]}A_{[h}Z_{i]} + 4a_{22}B_{[k}Z_{j]}B_{[h}Z_{i]} + 4a_{12}\{A_{[k}Z_{j]}B_{[h}Z_{i]} + B_{[k}Z_{j]}A_{[h}Z_{i]}\}.$$

https://doi.org/10.1017/S0305004100048465 Published online by Cambridge University Press

With the help of the above result, it can be easily seen that the Riemannian space-time described by (2·4) is recurrent. The recurrence vector K_p of R_{hijk} is given by

$$K_p = \frac{\partial}{\partial x^p}(\log \beta). \tag{5·6}$$

The result (5·4), in this case, becomes

$$\Omega l^2 = \beta(a_{11} + a_{22}). \tag{5·7}$$

Hence, it follows that the Riemannian space-time described by (2·4) with $\psi_x = \phi_y = 0$ admits two types of total pure radiation fields

$$\text{(i)} \quad \Omega = a_{11} + a_{22},\ l = \sqrt{|\beta|}, \qquad \text{(ii)} \quad \Omega = \beta,\ l = \sqrt{\{|a_{11} + a_{22}|\}}$$

discussed by Zund (9).

6. *Singular electromagnetic fields.* If the electromagnetic field tensor F_{ik} satisfies the source-free Maxwell equations

$$\left.\begin{aligned} (g^{pi}g^{qk}F_{ik})_{;p} &= 0 \\ \text{and} \qquad F_{ik;p} + F_{kp;i} + F_{pi;k} &= 0, \end{aligned}\right\} \tag{6·1}$$

then the electromagnetic field is singular if and only if

$$F_{ik} = l_i m_k - l_k m_i, \tag{6·2}$$

where

$$l^i l_i = 0, \quad l^i m_i = 0, \quad g^{ik} m_i m_k = -1. \tag{6·3}$$

Now $l_i = lZ_i$. Therefore we can take m_i to be given by

$$m_i = e^{\frac{1}{2}\phi} \cos\theta A_i + e^{\frac{1}{2}\psi} \sin\theta B_i, \tag{6·4}$$

where θ is a function of x, y and u. If we set

$$A = l e^{\frac{1}{2}\phi} \cos\theta, \quad B = l e^{\frac{1}{2}\psi} \sin\theta \tag{6·5}$$

then

$$l^2 = e^{-\phi} A^2 + e^{-\psi} B^2 \tag{6·6}$$

and

$$F_{ik} = 2AZ_{[i}A_{k]} + 2BZ_{[i}B_{k]}. \tag{6·7}$$

The Maxwell equations (6·1) imply

$$e^{\frac{1}{2}\psi}(A\, e^{-\frac{1}{2}\phi})_x = e^{\frac{1}{2}\phi}(e^{-\frac{1}{2}\psi} B)_y$$

and

$$A_y = B_x. \tag{6·8}$$

Here it should be noted that l and Ω must be chosen in accordance with (5·4). Let us make a simple choice $l = 1$ and

$$4\Omega = 2\phi_{uu} + \phi_u^2 + 2\psi_{uu} + \psi_u^2 + e^{-\phi}(S_x \phi_x + 2S_{xx}) + e^{-\psi}(S_y \psi_y + 2S_{xx}). \tag{6·9}$$

For this choice, A, B, ϕ, ψ have to satisfy (6·8) and (6·6) with $l = 1$. In this case

$$F_{ik} = 2e^{\frac{1}{2}\phi} \cos\theta Z_{[i}A_{k]} + 2e^{\frac{1}{2}\psi} \sin\theta Z_{[i}B_{k]}. \tag{6·10}$$

As shown by Zund (9) the choice (5·5) gives rise to two types.

Type P: $\Omega = a_{11} + a_{22}, \quad l = \sqrt{|\beta|}$.

Type Q: $\Omega = \beta, \quad l = \sqrt{(a_{11} + a_{22})}$.

For this choice Maxwell's equations (6·8) reduce to

$$A_x = -B_y, \quad A_y = B_x. \tag{6·11}$$

https://doi.org/10.1017/S0305004100048465 Published online by Cambridge University Press

The electromagnetic field tensor F_{ik} in these two types is given by

Type P: $F_{ik} = 2\sqrt{|\beta|}\{\cos\theta Z_{[i}A_{k]} + \sin\theta Z_{[i}B_{k]}\}$.

Type Q: $F_{ik} = 2\sqrt{(|a_{11}+a_{22}|)}\{\cos\theta Z_{[i}A_{k]} + \sin\theta Z_{[i}B_{k]}\}$.

Thus the singular electromagnetic fields discussed in this section are more general than those discussed by Zund (9).

The author is indebted to Professor P. C. Vaidya for his help and guidance in the preparation of this paper. Author's thanks are also due to Dr M. Misra for many helpful discussions.

The author wishes to thank the referee for his valuable comments.

REFERENCES

(1) Hoffman, R. B. Stationary 'noncanonical' solutions of the Einstein vacuum field equations. *J. Mathematical Phys.* **10** (1969), 953–956.

(2) Pandya, I. M. and Vaidya, P. C. Wave solutions in general relativity. *Proc. Nat. Inst. Sci. India, Part A* **27** (1960), 620–624.

(3) Patel, L. K. and Vaidya, P. C. A generalized plane wave metric. *Ann. Inst. H. Poincaré* **15** (1971), 169–175.

(4) Kerr, R. P. and Schild, A. Some algebraically degenerate solutions of Einstein's field equations. *Proc. Symposia Appl. Math.* Vol. xvii AMS, Providence, R.I. (1965), 199.

(5) Kerr, R. P. Gravitational field of a spinning mass as an example of algebraically special metrics. *Phys. Rev. Lett.* **11** (1963), 237.

(6) Peres, A. Some gravitational waves. *Phys. Rev. Lett.* **3** (1959), 571–572.

(7) Takeno, H. *The mathematical theory of plane gravitational waves in general relativity*. Scientific Reports of research institute of theoretical physics, Hiroshima University No. 1 (1961).

(8) Vaidya, P. C. and Pandya, I. M. Radiation: gravitational and electromagnetic. *Curr. Sci.* **29** (1960), 268.

(9) Zund, J. D. A special class of states of total pure radiation in general relativity. *Tensor* **23** (1972), 144–146.

https://doi.org/10.1017/S0305004100048465 Published online by Cambridge University Press