ON AVERAGE VALUES OF ARITHMETIC FUNCTIONS

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Introduction

The problem considered in this paper is that of finding the least possible $h = h(x)$ such that a given arithmetic function $a(n)$ should keep its average order in the interval $x, x+h$, i.e. that we have

$$\sum_{n=1}^{x} a(n) = A\psi(x) + o(\psi(x)) \tag{1}$$

and

$$\sum_{n=x}^{x+h} a(n) \sim A\psi(x+h) - A\psi(x), \tag{2}$$

as $x \to \infty$*.

There are some arithmetic functions for which the error term in (1) is of comparatively small order. Thus for the Euler's function $\phi(n)$ it is known that†

$$\sum_{n=1}^{x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x),$$

whence we get (2) with $h = \log^{1+\epsilon} x$.

The method developed in this paper is mainly of interest in the case of $a(n)$ with a large error term in (1). Examples of this kind are given by the arithmetic functions $\lambda(n)$, $\mu(n)$, $\Lambda(n)$, ... defined as follows:

$\lambda(n) = (-1)^{\rho}$, ρ being the total number of prime factors of n, when multiple factors are counted multiply; $\lambda(1) = 1$.

$$\mu(n) = \begin{cases} \lambda(n) \text{ if } n \text{ contains no square factor} > 1, \\ 0 \text{ otherwise.} \end{cases}$$

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^m \ (p \text{ prime}, m \geqslant 1), \\ 0 \text{ otherwise.} \end{cases}$$

For these functions the known error terms in (1) give (2) directly with $h = x \log^{-\Delta} x$ and arbitrary $\Delta > 0$, but not with

$$h = x^{\theta}, \tag{3}$$

where $\theta < 1$.

* When $A = 0$ the relation (2) is to be interpreted as meaning

$$\sum_{n=x}^{x+h} a(n) = o\{\psi(x+h) - \psi(x)\}.$$

† See, for example, Hardy and Wright(16), Chap. XVIII.

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A result of the type (3) was first proved by Hoheisel* for $\Lambda(n)$ with $\theta > \frac{32999}{33000}$. Using another method Heilbronn† got for $\mu(n)$ and $\Lambda(n)$ the value $\theta = \frac{249}{250}$, while Čudakov's results about the zeros of the Riemann zeta function‡ reduced θ to $\frac{3}{4}+\epsilon$. Ingham§ improved Hoheisel's method by making use of a convexity theorem for integrals|| and got for $\Lambda(n)$ the value

$$\theta = \frac{1+4c}{2+4c}+\epsilon, \tag{4}$$

where c is a constant for which

$$\zeta(\tfrac{1}{2}+it) = O(t^c), \tag{5}$$

as $t \to \infty$.

The best known value¶ of c equal to $\frac{19}{116}$ gives $\theta = \frac{48}{77}+\epsilon$.

In the present paper we use Heilbronn's method together with a convexity theorem of Ingham** and obtain the solution given by (3), (4) for the functions $\mu(n)$, $\lambda(n)$, $\Lambda(n)$ (Theorem A). The result stated for $\Lambda(n)$ may be expressed in the form

$$\pi(x+h)-\pi(x) \sim \frac{h}{\log x} \quad (x \to \infty),$$

$\pi(x)$ being the number of primes not exceeding x. Putting

$$h = x^\theta \ (0<\theta<1), \quad \vartheta = \frac{1}{1-\theta}, \quad y^\vartheta = x,$$

we get

$$\pi\{(y+1)^\vartheta\}-\pi(y^\vartheta) \geqslant \pi(y^\vartheta+y^{\vartheta-1})-\pi(y^\vartheta) = \pi(x+h)-\pi(x) \sim \frac{h}{\log x} > 0$$

[since $(y+1)^\vartheta > y^\vartheta + y^{\vartheta-1}$ for all $\vartheta > 1$, $y > y_0(\vartheta)$]. This proves that *between consecutive ϑth powers*

$$y^\vartheta, \quad (y+1)^\vartheta \quad (y > y_0),$$

there is at least one prime. The best known value of θ gives $\vartheta = \frac{77}{29}+\epsilon$.

The other results of theorem A expressed in a similar form state that *in the interval x, $x+x^\theta$ $(x > x_0)$ or y^ϑ, $(y+1)^\vartheta$ $(y > y_0)$ the functions $\lambda(n)$ and $\mu(n)$ change their signs at least once.*

Theorems B and C give a solution of the problem stated for the arithmetic functions

$$q(n) = \begin{cases} 1 \text{ if } n \text{ is not divisible by a } k\text{th power of a prime } (k \geqslant 2), \\ 0 \text{ otherwise.} \end{cases}$$

$2^{\nu(n)}$, $\nu(n)$ being the number of different prime factors of n.

$d(n^2)$ and $d^2(n)$, where $d(n)$ is the number of divisors of n.

* Hoheisel(4). † Heilbronn(3). ‡ Čudakov(11).
§ Ingham(1). || Hardy, Ingham and Pólya(13), Theorem 7.
¶ See Ingham(1), note on p. 256. ** Ingham(1), Theorem 2.

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The result about $q(n)$ with $k = 2$ proves that *in the interval* $x, x + x^{\frac{1}{2}+\epsilon}$ $(x > x_0(\epsilon))$ *or* $y^{\vartheta_0}, (y+1)^{\vartheta_0}$ $(\vartheta_0 = \frac{5}{3} + \epsilon, y > y_0(\epsilon))$ *there is at least one quadratfrei number**.

Theorems B and C may be proved without making use of Čudakov's results.

To prove the results stated we use a formula of Landau (Lemma 2) for the sum $\sum_{n \leqslant x} a(n)$ of the coefficients of the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

called *the generating function of* $a(n)$. Generating functions of the $a(n)$ considered in this paper are given by the formulae†

$$\frac{1}{\zeta(s)} = \Sigma \frac{\mu(n)}{n^s}, \quad \frac{\zeta(2s)}{\zeta(s)} = \Sigma \frac{\lambda(n)}{n^s}, \quad -\frac{\zeta'(s)}{\zeta(s)} = \Sigma \frac{\Lambda(n)}{n^s},$$

$$\frac{\zeta(s)}{\zeta(ks)} = \Sigma \frac{q(n)}{n^s}, \quad \frac{\zeta^2(s)}{\zeta(2s)} = \Sigma \frac{2^{\nu(n)}}{n^s}, \quad \frac{\zeta^3(s)}{\zeta(2s)} = \Sigma \frac{d(n^2)}{n^s},$$

$$\frac{\zeta^4(s)}{\zeta(2s)} = \Sigma \frac{d^2(n)}{n^s} \quad (s = \sigma + it, \sigma > 1).$$

Other arithmetic functions I shall consider elsewhere. There are no special difficulties in extending the results to arithmetic progressions as Heilbronn has done.

I wish to express my thanks to Mr Ingham for drawing my attention to this problem and for his valuable suggestions and criticism.

PRELIMINARY LEMMAS

We make use of the following properties of the Riemann zeta-function and an auxiliary function $M_X(s)$ defined by (16):

1) *There are positive numbers* $\alpha < 1$, $\alpha_1 < 2$ *and* $A = A(\alpha)$ *such that for all* $s = \sigma + it$ *of the region*

$$1 - A \log^{-\alpha} t < \sigma < 2, \quad t > t_0 > 1, \tag{6}$$

$$\zeta(s) \neq 0, \tag{7}$$

and we have uniformly

$$\zeta(s) = O(\log^{\alpha_1} t), \tag{8}$$

$$\frac{\zeta'(s)}{\zeta(s)} = O(\log^{\alpha} t), \tag{9}$$

$$\frac{1}{\zeta(s)} = O(\log^{\alpha} t). \tag{10}$$

* From a result $\sum_{n \leqslant x} |\mu(n)| = \frac{6}{\pi^2} x + o(\sqrt{x})$ of Landau (*Handbuch*, p. 606) we cannot deduce anything better than $\vartheta_0 = 2$.

† See, for example, Hardy and Wright(16), Chap. XVII.

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2) *For c and α_1 satisfying* (5), (8) *we have uniformly in σ*

$$\zeta(\sigma+iT) = \begin{cases} O(T^{2c(1-\sigma)}\log^{\alpha_1} T) & \text{for } \frac{1}{2} \leqslant \sigma \leqslant 1 \\ O(T^{(2c-1)\sigma+\frac{1}{2}}\log^{\alpha_1} T) & \text{for } 0 \leqslant \sigma \leqslant \frac{1}{2}. \end{cases} \tag{11}$$

$$\int_0^T |\zeta(\sigma+it)|^2 dt = \begin{cases} O(T^{2-2\sigma}) & \text{for fixed } \sigma < \frac{1}{2} \\ O(T\log T) & \text{for } \sigma = \frac{1}{2}. \end{cases} \tag{12}$$

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt = O(T\log^4 T). \tag{13}$$

3) *Let $\omega = \omega(\sigma)$ $(\sigma_0 \leqslant \sigma \leqslant \sigma_1)$ be a function satisfying a Lipschitz condition $|\omega(\sigma)-\omega(\sigma')| \leqslant k|\sigma-\sigma'|$ and $\zeta(\sigma+it) = O(t^{\omega}\log^{\beta} t)$ uniformly in σ as $t\to\infty$. Then we have uniformly in σ $(\sigma_0+\delta \leqslant \sigma \leqslant \sigma_1-\delta,\ \delta>0)$*

$$\zeta'(\sigma+it) = O(t^{\omega(\sigma)}\log^{\beta+1} t) \quad (t\to\infty). \tag{14}$$

4) *For $\sigma \geqslant \sigma_0 > 0$, $|t| < bT$, $b < 2\pi$ we have uniformly*

$$\zeta(s) = \sum_{n\leqslant T} n^{-s} - \frac{T^{1-s}}{1-s} + O(T^{-\sigma}). \tag{15}$$

5) *Let*

$$M_X(s) = \sum_{n\leqslant X} \mu(n) n^{-s} \quad (X>2), \tag{16}$$

and $1 \leqslant k \leqslant k_0$. Then there exists a number C depending on k_0 only such that for $\sigma \geqslant \frac{1}{2}$ we have

$$\int_0^T |M_X(\sigma+kit)|^2 dt < C(T+X)\log X. \tag{17}$$

(7) and (8) are proved by Titchmarsh* with $\alpha = \frac{4}{5}+\epsilon$†, $A = A_1$, $\alpha_1 = \frac{6}{5}$, and hence by a known method‡ we can show that (9), (10) are true for $A = \frac{1}{8}A_1$.

(11) is a consequence of the Phragmén-Lindelöf theorem§, (5), (8) and the functional equation for $\zeta(s)$.

For (12) see Ingham(2), formulae (4), (5) and the references given there.

(13) is a result of Hardy and Littlewood(15), Theorem D (3.12); see also Titchmarsh(8).

(14) can be proved applying Cauchy's theorem

$$f'(s) = \frac{1}{2\pi i}\int_C \frac{f(w)\,dw}{(w-s)^2}$$

to $f(s) = \zeta(s)$, where C is a circle with centre s and radius $\log^{-1} t$.

For a proof of (15) see Hardy and Littlewood(14), Lemma 2, or Titchmarsh(9), Theorem 19.

* Titchmarsh(7). † For a better value ($\alpha = \frac{3}{4}+\epsilon$) see Čudakov(12).
‡ See Landau(6) or Titchmarsh(9), Theorem 14.
§ See, for example, Titchmarsh(10), p. 180.

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(17) is proved by Ingham* for $k = 1$, $\sigma = \frac{1}{2}$, and the proof still holds for $\sigma > \frac{1}{2}$, $1 < k \leqslant k_0$.

LEMMA 1. *Let*

$$f_X(s) = \zeta(s) M_X(s) - 1, \tag{18}$$

where $M_X(s)$ is defined by (16). *Then for $X = O(T)$, $T = O(X)$, $1 \leqslant k \leqslant k_0$ and all σ of the interval $\frac{1}{2} \leqslant \sigma \leqslant 1$ we have uniformly*

$$\int_1^T |f_X(\sigma + kit)|^2 dt = O(T^{(2+4c)(1-\sigma)} \log^4 T), \tag{19}$$

c being any constant satisfying (5).

This is a consequence of Theorem 2 of Ingham(1).

LEMMA 2. *Let $\alpha(x)$ be a positive non-decreasing function of x such that*

$$\alpha(2x) = O(\alpha(x))$$

and for all $n \geqslant n_0$

$$a_n = O(\alpha(n)).$$

Further suppose the series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is convergent for $\sigma > 1$, and for $1 < \eta < 2$ and for some fixed l

$$(\eta - 1)^l \Sigma |a_n| n^{-\eta}$$

is bounded. Then we have uniformly for $1 < \eta < 2$, $T > 0$, $x > 2$

$$\sum_{n=1}^{x} a_n - \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} \frac{x^s}{s} f(s)\, ds = O\left(\frac{x^\eta}{T(\eta-1)^l} + \frac{x\alpha(x)\log x}{T} + \alpha(x)\right). \tag{20}$$

For $l = 1$ and $\alpha(x) = \log x$ the lemma is proved by Landau†. But there are no difficulties in repeating the same argument and proving (20) under the generalized conditions.

LEMMA 3. *Let $f(s)$ satisfy the conditions of Lemma 2 and $\alpha(x) = o(x^\epsilon)$ for all $\epsilon > 0$. Further, suppose*

$$T > t_0, \quad x = T^a, \quad 1 < a \leqslant a_0, \quad \eta = 1 + \log^{-1} T, \quad h < x,$$

and

$$h \geqslant x^{1-\frac{1}{a}} \log^{1+\delta} x \,.\, \mathrm{Max}\{\log^{l-1} x,\ \alpha(x)\}, \quad \delta \geqslant \delta_0 > 0. \tag{21}$$

Then

$$\sum_{n=x}^{x+h} a_n - \frac{1}{2\pi i} \int_{\eta - iT}^{\eta + iT} \{(x+h)^s - x^s\} \frac{f(s)}{s} ds = o(h), \tag{22}$$

as $x \to \infty$.

* Ingham(1), p. 260.

† Landau(5), Hilfssatz 3.

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Putting in turn $x+h$ and x in (20) we get for the left-hand side of (22) the estimate

$$O\left(\frac{(x+h)^\eta}{T(\eta-1)^l}+\frac{(x+h)\alpha(x+h)\log(x+h)}{T}+\alpha(x+h)\right)$$
$$=O(xT^{-1}\log^l x+xT^{-1}\alpha(x)\log x+\alpha(x))=O(x^{1-\frac{1}{a}}\log^l x+x^{1-\frac{1}{a}}\alpha(x)\log x)=o(h).$$

THEOREM A

Notation. In the following paragraph $g(s)$ denotes any of the functions

$$1, \quad -\zeta'(s), \quad \zeta(2s),$$

and $f(s)=g(s)/\zeta(s)$. Further let A, α and t_0 be the constants of (6) and T, η, a, x, h be as explained in Lemma 3. We define

$$\sigma_0 = 1-A\log^{-\alpha}T,$$
$$\sigma_1 = \begin{cases} \frac{1}{2}+\log^{-1}T & \text{for} \quad g(s)=\zeta(2s), \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

L, L_1, ... are straight lines joining the pairs of points

$$(L)\,\eta-iT,\ \eta+iT;\ (L_1)\,\sigma_0+iT,\ \eta+iT;\ (L_2)\,\sigma_0-iT,\ \sigma_0+iT;\ (L_3)\,\eta-iT,\ \sigma_0-iT;$$
$$(L_{21})\,\sigma_1+iT,\ \sigma_0+iT;\ (L_{22})\,\sigma_1-iT,\ \sigma_1+iT;\ (L_{23})\,\sigma_0-iT,\ \sigma_1-iT,$$

respectively.

y is any number satisfying the inequality

$$x\leqslant y\leqslant x+h,$$

and relations involving y hold uniformly in this range.

LEMMA A 1. *We have for* $x\to\infty$

$$I_1=\int_{L_1}\{(x+h)^s-x^s\}\frac{f(s)}{s}ds=o(h), \quad I_3=\int_{L_3}\{(x+h)^s-x^s\}\frac{f(s)}{s}ds=o(h).$$

Let us consider for example the integral I_1. On L_1 we have $(x+h)^s-x^s=O(x)$ and $f(s)=O(\log^\alpha T)$ by (9), (10). Since the length of L_1 is $O(\log^{-\alpha}T)$, we have

$$I_1=O(xT^{-1})=O(x^{1-\frac{1}{a}})=o(h).$$

The same argument proves $I_3=o(h)$.

LEMMA A 2. *If* $f_T(s)$ *is the function of Lemma* 1, *we have for* $a>2+4c$, $x\to\infty$

$$J_1=\int_{L_2}y^{s-1}f(s)\,f_T^2(s)\,ds=o(1).$$

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Let us divide the path of integration into two parts

$$L_2' \ (|t| \leqslant t_0), \quad L_2'' \ (|t| > t_0)$$

and denote the corresponding parts of J_1 by J_1', J_1''. If T is large enough, $\zeta(s)$ does not vanish in the neighbourhood of L_2' and hence for any of the $f(s)$ defined above we have along L_2'

$$f(s) = O\left(\frac{1}{1-\sigma_0}\right) = O(\log^\alpha T).$$

Similarly $f_T^2(s) = \zeta^2(s) M_T^2(s) - 2\zeta(s) M_T(s) + 1 = O\left(\frac{1}{(1-\sigma_0)^2} \frac{T^{2(1-\sigma_0)}}{(1-\sigma_0)^2}\right)$

$$= O(T^{2(1-\sigma_0)} \log^{4\alpha} T).$$

Therefore we have

$$J_1' = O(y^{\sigma_0-1} T^{2(1-\sigma_0)} \log^{5\alpha} T) = O(T^{-(a-2)(1-\sigma_0)} \log^{5\alpha} T)$$

$$= O \exp\{-(a-2) A \log^{1-\alpha} T + 5\alpha \log\log T\} = o(1).$$

Dealing with J_1'' we make use of Lemma 1, (9), (10) and obtain the estimate

$$J_1'' = O(y^{\sigma_0-1} \log^\alpha T) \int_{t_0}^{T} |f_T(\sigma_0 + it)|^2 dt = O(T^{(2+4c-a)(1-\sigma_0)} \log^{\alpha+4} T).$$

Hence $J_1'' = o(1)$ for all $a > 2 + 4c$. This proves the lemma.

LEMMA A 3. *Let $M_T(s)$ be defined by* (16). *Then we have for $a > 2$, $x \to \infty$*

$$J_2 = \int_{L_2} y^{s-1} g(s) M_T(s) ds = o(1).$$

We have to consider the different cases of $g(s)$ separately.

1) For $g(s) = 1$ we have

$$J_2 = iy^{\sigma_0-1} \sum_{n \leqslant T} \mu(n) n^{-\sigma_0} \int_{-T}^{T} \left(\frac{y}{n}\right)^{it} dt = O\left(y^{\sigma_0-1} \sum_{n \leqslant T} n^{-\sigma_0} \log^{-1} \frac{y}{n}\right)$$

$$= O(y^{\sigma_0-1} \log^{-1} T) \sum_{n \leqslant T} n^{-\sigma_0} = O(T^{(1-a)(1-\sigma_0)} \log^{\alpha-1} T) = o(1).$$

2) In the case of $g(s) = \zeta(2s)$ we make use of Cauchy's theorem and replace L_2 by L_{21}, L_{22}, L_{23}. If the corresponding parts of J_2 are J_{21}, J_{22}, J_{23}, we have

$$J_{21} = O(y^{-1} \log^{\alpha_1} T) \sum_{n \leqslant T} \int_{\sigma_1}^{\sigma_0} \left(\frac{y}{n}\right)^{\sigma} d\sigma = O(y^{-(1-\sigma_0)} \log^{\alpha_1-1} T) \sum_{n \leqslant T} n^{-\sigma_0}$$

$$= O(T^{(1-a)(1-\sigma_0)} \log^{\alpha+\alpha_1-1} T) = o(1),$$

and similarly $J_{23} = o(1)$.

Obviously $\int_{L_{22}} |\zeta(2s)|^2 dt \leqslant \log^2 T \int_{-T}^{T} dt = O(T \log^2 T).$

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Hence we get, by (17),

$$J_{22} = O(y^{-\frac{1}{2}}) \int_{L_{23}} |\zeta(2s) M_T(s)|\, dt = O(y^{-\frac{1}{2}}) \left(\int_{L_{23}} |\zeta(2s)|^2 dt \int_{L_{23}} |M_T(s)|^2 dt \right)^{\frac{1}{2}}$$

$$= O(T^{1-\frac{a}{2}} \log^{\frac{1}{2}} T) = o(1).$$

3) We could deal similarly with the case of $g(s) = -\zeta'(s)$ by making use of (11), (14) and the known* mean value of $\zeta'(s)$

$$\int_0^T |\zeta'(\tfrac{1}{2}+it)|^2 dt = O(T \log^3 T).$$

Or, if we integrate by parts,

$$\int y^{s-1} \zeta'(s) M_T(s)\, ds = y^{s-1} \zeta(s) M_T(s) - y^{-1} \sum_{n \leqslant T} \mu(n) \log \frac{y}{n} \int \left(\frac{y}{n}\right)^s \zeta(s)\, ds,$$

and use the approximate functional equation for $\zeta(s)$, (15), we get J_2 as a sum of four terms $J_{21}, \ldots, J_{24}$, say. They are

$$J_{21} = \left[y^{s-1} \zeta(s) M_T(s) \right]_{\sigma_0 - iT}^{\sigma_0 + iT}, \qquad J_{22} = -y^{-1} \sum_{m,\, n \leqslant T} \mu(n) \log \frac{y}{n} \int_{L_2} \left(\frac{y}{mn}\right)^s ds,$$

$$J_{23} = y^{-1} T \sum_{n \leqslant T} \mu(n) \log \frac{y}{n} \int_{L_3} \left(\frac{y}{nT}\right)^s \frac{ds}{1-s}, \quad J_{24} = O(y^{\sigma_0 - 1} T^{1-\sigma_0}) \sum_{n \leqslant T} n^{-\sigma_0} \log \frac{y}{n}.$$

Obviously

$$J_{21} = O(T^{(1-a)(1-\sigma_0)} \log^{\alpha+\alpha_1} T) = o(1),$$

$$J_{24} = O(T^{(1-a)(1-\sigma_0)} \log T) \sum_{n \leqslant T} n^{-\sigma_0} = O(T^{(2-a)(1-\sigma_0)} \log^{1+\alpha} T) = o(1).$$

Assuming $yT^{-2} > e$ we get

$$J_{22} = -iy^{\sigma_0 - 1} \sum_{m,\, n \leqslant T} \mu(n) (mn)^{-\sigma_0} \log \frac{y}{n} \int_{-T}^{T} \left(\frac{y}{mn}\right)^{it} dt$$

$$= O(y^{\sigma_0 - 1} \log T) \Big(\sum_{n \leqslant T} n^{-\sigma_0} \Big)^2 = O(T^{(2-a)(1-\sigma_0)} \log^{1+2\alpha} T) = o(1),$$

and similarly

$$J_{23} = O(y^{\sigma_0 - 1} T^{1-\sigma_0} \log T) \sum_{n \leqslant T} n^{-\sigma_0} \int_0^T \frac{dt}{|1 - \sigma_0 + it|} = O(T^{(2-a)(1-\sigma_0)} \log^{2+\alpha} T) = o(1).$$

This proves the lemma.

LEMMA A 4. *For $a > 2 + 4c$, $x \to \infty$ we have*

$$J_3 = \int_{L_3} y^{s-1} g(s) \zeta(s) M_T^2(s)\, ds = o(1).$$

* Ingham (2), Theorem A''.

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Let us replace the path of integration by L_{21}, L_{22}, L_{23} and denote by J_{31}, J_{32}, J_{33} the corresponding parts of J_3. On L_{21} we use (11), (14) and get for all defined $g(s)$

$$J_{31} = O(y^{-1}\log^{2\alpha_1+1} T) \sum_{m,n \leqslant T} \int_{\sigma_1}^{\sigma_0} y^{\sigma} T^{4c(1-\sigma)} (mn)^{-\sigma}\, d\sigma$$

$$= O(y^{-1} T^{4c} \log^{2\alpha_1+1} T) \sum_{m,n} \int_{\sigma_1}^{\sigma_0} \left(\frac{y}{mnT^{4c}}\right)^{\sigma} d\sigma.$$

For $a = 2+4c+\epsilon$ and $T > T_0(\epsilon)$ we have $y/mnT^{4c} > e$. Therefore

$$J_{31} = O(y^{-(1-\sigma_0)} T^{4c(1-\sigma_0)} \log^{2\alpha_1+1} T) \Big(\sum_{n \leqslant T} n^{-\sigma_0}\Big)^2$$

$$= O(T^{(2+4c-a)(1-\sigma_0)} \log^{2\alpha+2\alpha_1+1} T) = o(1).$$

The same argument proves $J_{33} = o(1)$. By (5) and (14) we have for $t \to \infty$

$$g(\tfrac{1}{2}+it) = O(t^c \log t).$$

Hence using (17) we get

$$J_{32} = O(y^{-\frac{1}{2}} T^{2c} \log T) \int_0^T |M_T(\sigma_1+it)|^2\, dt = O(T^{1+2c-\frac{a}{2}} \log^2 T) = o(1).$$

This proves the lemma.

LEMMA A 5. *For $a > 2+4c$, $x \to \infty$ we have*

$$I_2 = \int_{L_2} \{(x+h)^s - x^s\} \frac{f(s)}{s}\, ds = o(h).$$

Let us consider first the integral

$$J = \int_{L_2} y^{s-1} f(s)\, ds.$$

Multiplying the identity*

$$f_T^2(s) = \zeta^2(s)\, M_T^2(s) - 2\zeta(s)\, M_T(s) + 1$$

by $f(s) = g(s)/\zeta(s)$, we get an expression for $f(s)$

$$f(s) = f(s)\, f_T^2(s) + 2g(s)\, M_T(s) - g(s)\, \zeta(s)\, M_T^2(s),$$

which enables us to write J as a sum of the integrals J_1, J_2, J_3 considered in Lemmas A 2–A 4. Hence $J = o(1)$ for the stated values of a.

Going back to I_2 we get at once

$$I_2 = \int_x^{x+h} J\, dy = o(h).$$

THEOREM A. *For h defined by* (3), (4) *we have*

$$\sum_{n=x}^{x+h} \mu(n) = o(h), \quad \sum_{n=x}^{x+h} \lambda(n) = o(h), \quad \sum_{n=x}^{x+h} \Lambda(n) = h + o(h),$$

as $x \to \infty$.

Let us consider the integral of Lemma 3

$$I = \int_L \{(x+h)^s - x^s\} \frac{f(s)}{s}\, ds.$$

* See Lemma 1.

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The integrand is regular in the rectangle with the sides L, L_1, L_2, L_3, except (possibly) for a simple pole at $s = 1$ with residue γh. Therefore

$$I = 2\pi i\gamma h + I_1 + I_2 + I_3.$$

Since I_1, I_2, I_3 are each $o(h)$ by Lemmas A 1, A 5, and $\gamma = 1$ for $g(s) = -\zeta'(s)$, $\gamma = 0$ for $g(s) = 1$, $\zeta(2s)$, the theorem follows by Lemma 3.

Theorem B

Let $f(s)$ denote now the function

$$f(s) = \zeta(s)/\zeta(ks), \quad k \text{ an integer}, \quad 1 < k \leqslant k_0.$$

We denote by L, L_1, ... straight lines joining the pairs of points

$$(L)\,\eta - iT,\ \eta + iT; \quad (L_1)\,1/k + iT,\ \eta + iT; \quad (L_2)\,1/k - iT,\ 1/k + iT; \quad (L_3)\,\eta - iT,\ 1/k - iT;$$

$$(L_{21})\,1/2k + iT,\ 1/k + iT; \quad (L_{22})\,1/2k - iT,\ 1/2k + iT; \quad (L_{23})\,1/k - iT,\ 1/2k - iT.$$

Other symbols keep their previous meaning*.

Lemma B 1. *For $a > 1$, $x \to \infty$ we have*

$$I_1 = \int_{L_1} \{(x+h)^s - x^s\} \frac{f(s)}{s}\, ds = o(h),$$

$$I_3 = \int_{L_3} \{(x+h)^s - x^s\} \frac{f(s)}{s}\, ds = o(h).$$

Let us divide L_1 into three parts as follows

$$L_{11}\ (1/k \leqslant \sigma \leqslant \tfrac{1}{2}), \quad L_{12}\ (\tfrac{1}{2} \leqslant \sigma \leqslant \sigma_0), \quad L_{13}\ (\sigma_0 \leqslant \sigma \leqslant \eta)$$

(for $k = 2$ the first part does not exist, its length being zero). On L_{11}, L_{12} we shall consider instead of I_1 the integral

$$F = \int y^{s-1} f(s)\, ds$$

and denote by F_1, F_2 the parts of F corresponding to L_{11}, L_{12}. Making use of (10), (11) we get the results

$$F_1 = O(y^{-1} T^{\frac{1}{2}} \log^{\alpha+\alpha_1} T) \int_{1/k}^{\frac{1}{2}} (yT^{2c-1})^\sigma\, d\sigma = O(T^{c-\frac{a}{2}} \log^{\alpha+\alpha_1-1} T),$$

$$F_2 = O(y^{-1} T^{2c} \log^{\alpha+\alpha_1} T) \int_{\frac{1}{2}}^{\sigma_0} (yT^{-2c})^\sigma\, d\sigma.$$

* As it stands the following proof of Theorem B uses Čudakov's results (8) [in the region (6)] and (10) [only for $\sigma \geqslant 1$]. But these can be avoided by taking an arbitrary α ($0 < \alpha < 1$) in the definition of σ_0, putting $h = x^{1-\frac{1}{a}+\varepsilon}$, replacing α_1 and α by 1 in (8) and (10) for $\sigma \geqslant 1$, and using (11) (with $\alpha_1 = 1$) in place of (8) when $\sigma < 1$. A further simplification would be to dispense with σ_0 and use the estimate $\{(x+h)^s - x^s\}/s = O(x^\sigma/T)$ throughout the proof of Lemma B 1. [A.E.I., 7 August 1941.]

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Since c can be taken to be less than $\frac{1}{2}$ we have

$$F_1 = o(1), \quad F_2 = O(T^{(2c-a)(1-\sigma_0)} \log^{\alpha+\alpha_1-1} T) = o(1).$$

Hence the part of I_1 corresponding to L_{11}, L_{12} is

$$I_1' = \int_x^{x+h} (F_1 + F_2)\, dy = o(h).$$

The remaining part of I_1 is obviously, by (8),

$$I_1'' = (\eta - \sigma_0)\, O(xT^{-1} \log^{\alpha_1} T) = O(x^{1-\frac{1}{a}} \log^{\alpha_1-\alpha} x) = o(h)$$

(since $\alpha_1 - \alpha$ can be taken < 1). Hence $I_1 = I_1' + I_1'' = o(h)$.

The same argument holds for I_3.

LEMMA B 2. *If $f_T(s)$ is the function of Lemma 1, we have for $a > 1$, $x \to \infty$*

$$J_1 = \int_{L_2} y^{s-1} f(s) f_T(ks)\, ds = o(1).$$

Dividing L_2 into the parts

$$L_2' \ (|t| < t_0), \quad L_2'' \ (|t| \geqslant t_0)$$

we consider the corresponding parts J_1', J_1'' of J_1. On L_2' we have

$$f(s) f_T(ks) = \zeta(s) M_T(ks) - \frac{\zeta(s)}{\zeta(ks)} = O(\log T);$$

therefore

$$J_1' = O(y^{\frac{1}{k}-1} \log T) = o(1).$$

On L_2'' we make use of (12) and Lemma 1 and get the result

$$J_1'' = O(T^{-a(k-1)/k} \log^{\alpha} T) \left(\int_1^T \left| \zeta\left(\frac{1}{k} + it\right) \right|^2 dt \int_{t_0}^T |f_T(1 + kit)|^2 dt \right)^{\frac{1}{2}}$$

$$= O(T^{(1-a)(k-1)/k} \log^{\alpha+\frac{1}{2}} T) = o(1).$$

This proves the lemma.

LEMMA B 3. *For $\dfrac{3k-1}{2k-1} < a < 2$, $x \to \infty$ we have*

$$J_2 = \int_{L_2} y^{s-1} \zeta(s) M_T(ks)\, ds = o(1).$$

Let us replace L_2 by L_{21}, L_{22}, L_{23}. On L_{21} we use (11) and the property

$$M_T(k\sigma + kiT) = O(T^{1-k\sigma} \log T) \quad \left(0 \leqslant \sigma \leqslant \frac{1}{k}\right).$$

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The corresponding part of J_2 is

$$J_{21} = O(y^{-1}T^{\frac{1}{2}}\log^{\alpha_1+1}T)\int_{1/2k}^{1/k}(yT^{2c-1-k})^{\sigma}\,d\sigma.$$

Since $a<2$ and c can be taken to be less than $\frac{1}{2}$, $a+2c-1-k$ is less than some negative number $-\delta<0$. Therefore we have

$$J_{21} = O(y^{-1+1/2k}T^{\frac{1}{2}-(k+1-2c)/2k}\log^{\alpha_1}T) = O\left(T^{\frac{2k-1}{2k}\left(1+\frac{2c}{2k-1}-a\right)}\log^{\alpha_1}T\right)$$
$$= o(1).$$

The same result holds for J_{23}, the part of J_2 corresponding to L_{23}.

By (12) and (17) we have for the remaining part of J_2

$$J_{22} = O(y^{1/2k-1})\left(\int_0^T |M(\tfrac{1}{2}+kit)|^2\,dt\int_0^T\left|\zeta\left(\frac{1}{2k}+it\right)\right|^2 dt\right)^{\frac{1}{2}}$$
$$= O(T^{-a(2k-1)/2k+\frac{1}{2}-1/2k}\log^{\frac{1}{2}}T) = O(T^{(3k-1-a(2k-1))/2k}\log^{\frac{1}{2}}T).$$

Hence $J_{22} = o(1)$ for the stated values of a.

LEMMA B 4. *For $x\to\infty$ and the values of a of Lemma* B 3 *we have*

$$I_2 = \int_{L_2}\{(x+h)^s - x^s\}\frac{f(s)}{s}\,ds = o(h).$$

From the identity $f_T(ks) = \zeta(ks)M_T(ks) - 1$ (see Lemma 1) we deduce

$$f(s) = \zeta(s)M_T(ks) - f(s)f_T(ks).$$

Hence by Lemmas B 2, B 3 we have

$$J = \int_{L_2} y^{s-1}f(s)\,ds = J_2 - J_1 = o(1)$$

and consequently

$$I_2 = \int_x^{x+h} J\,dy = o(h).$$

THEOREM B. *We have*

$$\sum_{n=x}^{x+h} q(n) = \frac{h}{\zeta(k)} + o(h) \quad \textit{for} \quad h = x^{\frac{k}{3k-1}+\epsilon},\ x\to\infty.$$

In particular $k=2$ gives

$$\sum_{n=x}^{x+h}|\mu(n)| = \frac{6}{\pi^2}h + o(h) \quad \textit{for} \quad h = x^{\frac{1}{2}+\epsilon},\ x\to\infty.$$

Since the integrand of

$$I = \int_L\{(x+h)^s - x^s\}\frac{f(s)}{s}\,ds, \quad f(s) = \frac{\zeta(s)}{\zeta(ks)}$$

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is regular in the rectangle whose sides are L, L_1, L_2, L_3, except for a simple pole at $s = 1$ with residue $h/\zeta(k)$, we have

$$I = 2\pi i h/\zeta(k) + I_1 + I_2 + I_3.$$

I_1, I_2, I_3 being the integrals of Lemmas B 1, B 4 are $o(h)$ and we get the theorem, applying Lemma 3.

THEOREM C

In this section $f(s)$ denotes any of the functions

$$f(s) = \zeta^k(s)/\zeta(2s), \quad k = 2, 3, 4,$$

and L, L_1, ... are straight lines joining the pairs of points

$$(L)\ \eta - iT,\ \eta + iT; \quad (L_1)\ \tfrac{1}{2} + iT,\ \eta + iT; \quad (L_2)\ \tfrac{1}{2} - iT,\ \tfrac{1}{2} + iT; \quad (L_3)\ \eta - iT;\ \tfrac{1}{2} - iT.$$

Let $\psi_k(x)$ for $k = 2, 3, 4$ denote the residue of $x^s\zeta^k(s)/s\zeta(2s)$ at $s = 1$, i.e.

$$\psi_2(x) = \frac{x}{\zeta(2)}\left\{\log x + 2\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}\right\},$$

$$\psi_3(x) = \frac{x}{\zeta(2)}\left\{\frac{\log^2 x}{2} + \left(3\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}\right)\log x + B\right\},$$

$$\psi_4(x) = \frac{x}{\zeta(2)}\left\{\frac{\log^3 x}{6} + \left(4\gamma - 1 - 2\frac{\zeta'(2)}{\zeta(2)}\right)\frac{\log^2 x}{2} + C\log x + D\right\},$$

where γ is Euler's constant, B, C and D are some other constants whose calculation does not appear simple.

The notation remains in other respects as in Theorem A*.

LEMMA C 1. *For $a > \frac{4}{3}$, $h \geqslant x^{1-\frac{1}{a}}\log^5 x$, $x \to \infty$ we have*

$$I_1 = \int_{L_1}\{(x+h)^s - x^s\}\frac{f(s)}{s}\,ds = o(h), \quad I_3 = \int_{L_3}\{(x+h)^s - x^s\}\frac{f(s)}{s}\,ds = o(h).$$

Let us divide L_1 into the parts

$$L_{11}\ (\sigma_0 \leqslant \sigma \leqslant \eta), \quad L_{12}\ (\tfrac{1}{2} \leqslant \sigma \leqslant \sigma_0).$$

The part of I_1 corresponding to L_{11} is

$$I_{11} = O(x^{1-\frac{1}{a}}\log^{k\alpha_1 - \alpha} T) = o(h),$$

since α_1 can be taken to be less than $\frac{5}{4}$. On L_{12} let us estimate first the integral

$$F = \int_{L_{12}} y^{s-1} f(s)\,ds.$$

* See the footnote on page 367.

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By (11) we have $F = O(y^{-1} T^{2kc} \log^{\alpha+k\alpha_1} T) \int_{\frac{1}{2}}^{\sigma_0} (yT^{-2kc})^{\sigma} d\sigma.$

Since $a > \frac{4}{3}$ and c can be taken less than $\frac{1}{6}$, we have $yT^{-2kc} > e$. Therefore

$$F = O(T^{(2kc-a)(1-\sigma_0)} \log^{\alpha+k\alpha_1} T) = o(1),$$

$$I_{12} = \int_x^{x+h} F dy = o(h), \quad I_1 = I_{11} + I_{12} = o(h).$$

In the same manner we prove $I_3 = o(h)$.

LEMMA C 2. *For $a > 1 + 4c$, $x \to \infty$ we have*

$$J_1 = \int_{L_2} y^{s-1} f(s) f_T(2s)\, ds = o(1).$$

The proof is similar to that of Lemma B 2. The part J_1' of J_1 with $|t| \leqslant t_0$ ($t_0 \geqslant 2$) is obviously $o(1)$. For the remaining part J_1'' we use (5), (13) and Lemma 1, obtaining the result

$$J_1'' = O(y^{-\frac{1}{2}} T^{2c} \log^{\alpha} T) \left(\int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt \int_{t_0}^T |f_T(1+2it)|^2 dt \right)^{\frac{1}{4}}$$

$$= O(T^{-\frac{a}{2}+\frac{1}{2}+2c} \log^{\alpha+4} T) = o(1).$$

LEMMA C 3. *For $k \leqslant 4$, $a > 2$, $x \to \infty$ we have*

$$J_2 = \int_{L_2} y^{s-1} \zeta^k(s)\, M_T(2s)\, ds = o(1).$$

Since* $\int_0^T |\zeta(\tfrac{1}{2}+it)|^k dt = O(T \log^4 T), \quad M_T(1+2it) = O(\log T)$

we have $$J_2 = O(T^{1-\frac{a}{2}} \log^5 T) = o(1).$$

LEMMA C 4. *For $a > 2$, $x \to \infty$ we have*

$$I_2 = \int_{L_2} \{(x+h)^s - x^s\} \frac{f(s)}{s}\, ds = o(h).$$

From the identity $f(s) = \zeta^k(s)\, M_T(2s) - f(s) f_T(2s)$

and Lemmas C 2, C 3 we deduce

$$J = \int_{L_2} y^{s-1} f(s)\, ds = o(1), \quad I_2 = \int_x^{x+h} J dy = o(h).$$

* In the case of $k = 3$ we have by (12), (13)

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^3 dt \leqslant \left(\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 dt \int_0^T |\zeta(\tfrac{1}{2}+it)|^4 dt \right)^{\frac{1}{2}} = O(T \log^{\frac{1}{2}} T).$$

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THEOREM C. *For* $h = x^{\frac{1}{2}+\epsilon}$, $x \to \infty$ *we have*

$$\sum_{n=x}^{x+h} 2^{\nu(n)} = \psi_2(x+h) - \psi_2(x) + o(h),$$

$$\sum_{n=x}^{x+h} d(n^2) = \psi_3(x+h) - \psi_3(x) + o(h),$$

$$\sum_{n=x}^{x+h} d^2(n) = \psi_4(x+h) - \psi_4(x) + o(h).$$

The residue of $\{(x+h)^s - x^s\}\frac{\zeta^k(s)}{s\zeta(2s)}$ at $s = 1$ is $\psi_k(x+h) - \psi_k(x)$ $(k = 2, 3, 4)$

In other respects the proof remains the same as that of Theorem A or B.

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