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ON THE DETERMINATION OF STAR BODIES FROM THEIR HALF-SECTIONS

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Abstract. We obtain explicit inversion formulas for the Radon-like transform that assigns to a function on the unit sphere the integrals of that function over hemispheres lying in lower-dimensional central cross-sections. The results are applied to the determination of star bodies from the volumes of their central half-sections.

§1. *Introduction.* Let K be a compact subset of $\mathbb{R}^n$ which contains the origin o as an interior point and is star-shaped with respect to o. Such a set K is called a *star body* if the radial function

$$\rho_K(\theta) = \max\{r \geqslant 0 : r\theta \in K\}, \quad \theta \in S^{n-1}, \tag{1.1}$$

that determines the shape of K is continuous (see Gardner [2]). We denote by $G_{n,k}$, $2 \leqslant k \leqslant n-1$, the Grassmann manifold of all k-dimensional linear subspaces ξ of $\mathbb{R}^n$. Passing to spherical coordinates, one can evaluate the k-dimensional volume of $K \cap \xi$ as

$$\mathrm{vol}_k(K \cap \xi) = \frac{1}{k} \int_{S^{n-1}\cap\xi} \rho_K^k(\theta)\, d_\xi\theta, \tag{1.2}$$

where $d_\xi\theta$ stands for the corresponding $(k-1)$-dimensional surface element. The right-hand side of (1.2) is a constant multiple of the Funk-type transform

$$(F_k f)(\xi) = \int_{S^{n-1}\cap\xi} f(\theta)\, d_\xi\theta, \quad \xi \in G_{n,k}, \tag{1.3}$$

that can be explicitly inverted in different ways provided that f is an even integrable function on S^{n-1} (see, e.g., [7, 8, 11]). This fact makes it possible to determine the shape of K from the knowledge of the volumes of $K \cap \xi$ for all $\xi \in G_{n,k}$ provided that K is origin-symmetric. The star bodies which are not origin-symmetric cannot be uniquely reconstructed from the integrals (1.2) (see, e.g., Gardner [2, §7.2]). This statement agrees with a known fact that the kernel of F_k on $L^1(S^{n-1})$ consists of odd functions.

In the case $k = n-1$, Groemer [6], who followed some ideas from Backus [1], suggested replacing the hyperplane central sections by the "half-sections" $K \cap H(u, v)$, where

$$H(u, v) = \{x \in \mathbb{R}^n : x \cdot u = 0, x \cdot v \geqslant 0\}, \quad u \perp v, \tag{1.4}$$

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is a half-plane determined by mutually orthogonal unit vectors u and v, so that the origin o lies on the boundary of $H(u, v)$. Thus (1.2) is substituted by the lower-dimensional hemispherical integral

$$\mathrm{vol}_{n-1}(K \cap H(u, v)) = \frac{1}{n-1} \int_{S^{n-1} \cap H(u,v)} \rho_K^{n-1}(\theta)\, d_\xi \theta. \tag{1.5}$$

One of the main results of [**6**] states that if the star bodies K and L satisfy

$$\mathrm{vol}_{n-1}(K \cap H(u, v)) = \mathrm{vol}_{n-1}(L \cap H(u, v))$$

for all $u, v \in S^{n-1}$, $u \perp v$, then $K = L$. The case of half-sections of lower dimensions is a direct consequence of Groemer's result, as was pointed out by Goodey and Weil [**5**].

To the best of our knowledge, the following important questions remain open.

Question 1. How can we explicitly reconstruct the shape of K (or the radial function ρ_K) from volumes (1.5) or, more generally, from the corresponding k-dimensional volumes?

Question 2. How can we eliminate overdeterminedness of the inversion problem in Question 1?

Regarding Question 2, we observe that (1.4) parameterizes the corresponding half-sections by the elements of the Stiefel manifold $V_{n,2} = \{(u, v) : u, v \in S^{n-1}, u \perp v\}$, so that $\dim V_{n,2} = 2n - 3 > n - 1 = \dim S^{n-1}$ if $n > 2$. In other words, the dimension of the target space is greater than the dimension of the source space. The latter means that the inversion problem in Question 1 can be overdetermined. In the case $k < n - 1$, the difference between the dimensions of the target space and source space is bigger because the corresponding lower-dimensional Funk transform (1.3) is overdetermined itself. This situation is pursuant to Gel'fand's celebrated question [**3**] about how to reduce overdeterminedness of transformations in integral geometry. In our case, it means the following problem.

Problem. Find an $(n - 1)$-dimensional submanifold $\mathfrak{M}$ of the manifold of all k-dimensional central half-sections of K so that K could be recovered from the volumes of the half-sections belonging to $\mathfrak{M}$ only.

In the present article, we solve this problem and give an answer to Questions 1 and 2. The basic idea is to consider separately the half-sections lying in the open half-spaces

$$H_\pm = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \pm x_n > 0\}. \tag{1.6}$$

The case of all $2 \leqslant k \leqslant n - 1$ is considered in §2. Here our inversion formulas remain overdetermined if $k < n-1$. This overdeterminedness is eliminated in §3.

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§2. *Inversion formulas.* We consider the following hemispherical modifications of the Funk-type transform (1.3) when a function $f \in L^1(S^{n-1})$ is integrated over $(k-1)$-dimensional hemispheres $S^{n-1}_{\pm} \cap \xi$, $\xi \in G_{n,k}$, lying in the $(n-1)$-dimensional hemispheres

$$S^{n-1}_{\pm} = \{\theta = (\theta_1, \dots, \theta_n) \in S^{n-1} : \pm\theta_n > 0\},$$

respectively. Specifically, for any subspace $\xi \in G_{n,k}$ not orthogonal to $e_n = (0, \dots, 0, 1)$, we set

$$(F_k^+ f)(\xi) = \int_{S^{n-1}_+ \cap \xi} f(\theta)\, d_\xi\theta \qquad \left(= \int_{S^{n-1}_- \cap \xi} f(-\theta)\, d_\xi\theta\right), \tag{2.1}$$

$$(F_k^- f)(\xi) = \int_{S^{n-1}_- \cap \xi} f(\theta)\, d_\xi\theta \qquad \left(= \int_{S^{n-1}_+ \cap \xi} f(-\theta)\, d_\xi\theta\right). \tag{2.2}$$

Clearly,

$$(F_k^+ f)(\xi) = \tfrac{1}{2}(F_k f_1)(\xi), \qquad f_1(\theta) = \begin{cases} f(\theta) & \text{if } \theta \in S^{n-1}_+, \\ f(-\theta) & \text{if } \theta \in S^{n-1}_-, \end{cases} \tag{2.3}$$

$$(F_k^- f)(\xi) = \tfrac{1}{2}(F_k f_2)(\xi), \qquad f_2(\theta) = \begin{cases} f(-\theta) & \text{if } \theta \in S^{n-1}_+, \\ f(\theta) & \text{if } \theta \in S^{n-1}_-, \end{cases} \tag{2.4}$$

where f_i $(i = 1, 2)$ are integrable even functions on S^{n-1} that can be reconstructed from $\varphi^{\pm} = F_k^{\pm} f$ by the formulas $f_1 = 2F_k^{-1}\varphi^+$ and $f_2 = 2F_k^{-1}\varphi^-$. Combining these formulas, we obtain

$$f(\theta) = \begin{cases} 2(F_k^{-1}\varphi^+)(\theta) & \text{if } \theta \in S^{n-1}_+, \\ 2(F_k^{-1}\varphi^-)(\theta) & \text{if } \theta \in S^{n-1}_-. \end{cases} \tag{2.5}$$

Thus we have proved the following theorem.

THEOREM 2.1. *For $2 \leqslant k \leqslant n-1$, a function $f \in L^1(S^{n-1})$ can be recovered from the integrals $\varphi^{\pm} = F_k^{\pm} f$ by the formula (2.5).*

Some comments are in order.

(i) Let $\xi \in G_{n,k}$ be a subspace which is not orthogonal to $e_n = (0, \dots, 0, 1)$. We define $\xi_{\pm} = \xi \cap \{x \in \mathbb{R}^n : \pm x_n > 0\}$. Then Theorem 2.1 implies the following corollary.

COROLLARY 2.2. *The radial function ρ_K of the star body K can be recovered from the volumes $v^{\pm}(\xi) = \mathrm{vol}_k(K \cap \xi_{\pm})$ by the formula*

$$\rho_K^k(\theta) = \begin{cases} 2k(F_k^{-1}v^+)(\theta) & \text{if } \theta \in S^{n-1}_+, \\ 2k(F_k^{-1}v^-)(\theta) & \text{if } \theta \in S^{n-1}_-. \end{cases} \tag{2.6}$$

For $\theta_n = 0$, it can be determined from (2.6) by continuity.

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(ii) In the case $k = n-1$, we can set $\xi = u^{\perp}$, $u \in S^{n-1}$ and write (1.3) as the usual Funk transform

$$(Ff)(u) = \int_{\{\theta \in S^{n-1} : u\cdot\theta = 0\}} f(\theta)\, d_u\theta. \tag{2.7}$$

Similar notation $(F^{\pm}f)(u)$ can be used for the hemispherical transforms (2.1) and (2.2).

(iii) The above reasoning shows that to reconstruct f on S^{n-1}_{+} (or S^{n-1}_{-}) the knowledge of $\varphi^{+} = F^{+}_{k} f$ (or $\varphi^{-} = F^{-}_{k} f$, respectively) is sufficient.

(iv) A theory of the Funk transform provides a variety of inversion formulas for F_k (see, e.g., [**4**, **7**–**9**, **11**]). The functions f_1 and f_2 in (2.3) and (2.4) can have discontinuity on the equator $\theta_n = 0$. It means that we cannot apply inversion formulas for F_k (at least, not straightforwardly) in which the smoothness is crucial. However, if $f \in L^p(S^{n-1})$, $1 \leqslant p < \infty$, then f_1 and f_2 also belong to $L^p(S^{n-1})$ and can be reconstructed, for example, by the method of mean value operators as follows.

For $r = \cos\psi$, $\psi \in [0, \pi/2]$ and $\theta \in S^{n-1}$, consider the shifted dual transform

$$(F^{*}_{k,\theta}\varphi)(r) = \int_{d(\theta,\xi)=\psi} \varphi(\xi)\, d\mu(\xi), \tag{2.8}$$

where φ is a function on $G_{n,k}$, $d(\cdot,\cdot)$ denotes the geodesic distance and $d\mu(\xi)$ stands for the corresponding canonical measure (see [**9**, §5] for details).

THEOREM 2.3 [**9**, Theorem 5.3]. *An even function $f \in L^p(S^{n-1})$, $1 \leqslant p < \infty$, can be recovered from $\varphi = F_k f$ by the formula*

$$f(\theta) = \lim_{s\to 1}\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^{k}\left[\frac{\pi^{-k/2}}{\Gamma(k/2)}\int_0^s (s^2 - r^2)^{k/2-1}(F^{*}_{k,\theta}\varphi)(r) r^k\, dr\right]. \tag{2.9}$$

In particular, for k even,

$$f(\theta) = \lim_{s\to 1}\frac{1}{2\pi^{k/2}}\left(\frac{1}{2s}\frac{\partial}{\partial s}\right)^{k/2}[s^{k-1}(F^{*}_{k,\theta}\varphi)(s)]. \tag{2.10}$$

Alternatively,

$$f(\theta) = \lim_{s\to 1}\left(\frac{\partial}{\partial s}\right)^{k}\left[\frac{2^{-k}\pi^{-k/2}}{\Gamma(k/2)}\int_0^s (s^2 - r^2)^{k/2-1}(F^{*}_{k,\theta}\varphi)(r)\, dr\right]. \tag{2.11}$$

The limit in these formulas is understood in the L^p-norm.

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§3. *The case $k < n-1$.* If $k < n-1$, the inversion problem for $F_k^{\pm}$ and F_k is overdetermined because the dimension of the target space is greater than the dimension of the source space: that is,

$$\dim G_{n,k} = k(n-k) > n-1 = \dim S^{n-1}.$$

Below we eliminate this overdeterminedness by choosing an $(n-1)$-dimensional submanifold $\mathfrak{M}$ of $G_{n,k}$ which is sufficient to reconstruct f from $(F_k^{\pm} f)(\xi)$ with $\xi \in \mathfrak{M}$. We proceed as in [**10**].

Suppose that $\{e_1, \dots, e_n\}$ is a standard orthonormal basis in $\mathbb{R}^n$ and denote

$$\mathbb{R}^{n-k} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{n-k}, \qquad \mathbb{R}^k = \mathbb{R}e_{n-k+1} \oplus \cdots \oplus \mathbb{R}e_n, \tag{3.1}$$

$$\mathbb{R}^{k+1} = \mathbb{R}e_{n-k} \oplus \mathbb{R}^k, \qquad S^k = S^{n-1} \cap \mathbb{R}^{k+1}, \qquad S_{\pm}^k = S_{\pm}^{n-1} \cap \mathbb{R}^{k+1}. \tag{3.2}$$

Given a point $v \in S^{n-k-1} = S^{n-1} \cap \mathbb{R}^{n-k}$, we fix an orthogonal transformation γ_v in $\mathbb{R}^{n-k}$ so that $\gamma_v e_{n-k} = v$. Let

$$\tilde{\gamma}_v = \begin{bmatrix} \gamma_v & 0 \\ 0 & I_k \end{bmatrix}, \tag{3.3}$$

where I_k is the identity $k \times k$ matrix. We denote by $G_k(\tilde{\gamma}_v \mathbb{R}^{k+1})$ the Grassmannian of all k-dimensional linear subspaces of $\tilde{\gamma}_v \mathbb{R}^{k+1}$ and set

$$\mathfrak{M} = \bigcup_{v \in S^{n-k-1}} G_k(\tilde{\gamma}_v \mathbb{R}^{k+1}). \tag{3.4}$$

The restrictions $\tilde{F}_k^{\pm} f$ of $F_k^{\pm} f$ onto $\mathfrak{M}$ can be identified with functions on the set $\tilde{S}_{n,k} = \{(v, w) : v \in S^{n-k-1}, w \in S^k\}$. These functions are represented by the hemispherical integrals

$$(\tilde{F}_k^{\pm} f)(v, w) = \int_{\{\eta \in S_{\pm}^k : \eta \cdot w = 0\}} f_v(\eta)\, d_w \eta, \qquad f_v(\eta) = f(\tilde{\gamma}_v \eta), \tag{3.5}$$

over $(k-1)$-dimensional hemispheres on S^k. Thus $\tilde{F}_k^{\pm}$ can be inverted as in the previous section.

Let us explain the details. We equip $\tilde{S}_{n,k}$ with the product measure $dv\, dw$, where dv and dw stand for the corresponding surface elements on S^{n-k-1} and S^k. Using [**10**, Theorem 3.2] and taking into account that the restrictions of f_v onto $S_{\pm}^k$ and their even extensions (cf. (2.3) and (2.4)) belong to the same L^p spaces, we obtain the following existence result.

THEOREM 3.1. *Let $1 \leqslant k < n-1$, $f \in L^p(S^n)$ and $n-k < p \leqslant \infty$. Then $(\tilde{F}_k^{\pm} f)(v, w)$ are finite for almost all $(v, w) \in \tilde{S}_{n,k}$. If $p \leqslant n-k$, then there are functions $\tilde{f}_{\pm} \in L^p(S^{n-1})$ for which $(\tilde{F}_k^{\pm} f_{\pm})(v, w) = \infty$.*

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Due to (2.5), to reconstruct f from $\varphi_v^{\pm} = \tilde{\mathcal{F}}_k^{\pm} f$, it suffices to invert the usual codimension one Funk transform $\tilde{F}$ on S^k. This gives

$$f_v(\eta) = \begin{cases} 2(\tilde{F}^{-1}\varphi_v^+)(\eta) & \text{if } \eta \in S_+^k, \\ 2(\tilde{F}^{-1}\varphi_v^-)(\eta) & \text{if } \eta \in S_-^k. \end{cases} \tag{3.6}$$

If f is a continuous function, its value at a point $\theta \in S^{n-1}$ can be found as follows. We interpret θ as a column vector $\theta = (\theta_1, \dots, \theta_{n+1})^T$ and set

$$\theta' = (\theta_1, \dots, \theta_{n-k})^T \in \mathbb{R}^{n-k}, \qquad \theta'' = (\theta_{n-k+1}, \dots, \theta_n)^T \in \mathbb{R}^k.$$

Suppose that $\theta' \neq 0$ and set

$$v = \theta'/|\theta'| \in S^{n-k-1}, \qquad \eta = (0, \dots, 0, |\theta'|, \theta'')^T \in S^k. \tag{3.7}$$

Then $\tilde{\gamma}_v \eta = \theta$ and we get $f(\theta) = (\mathcal{F}^{-1}\varphi_v)(\eta)$ for v and η, as in (3.7). If $\theta' = 0$, then $f(\theta)$ can be reconstructed by continuity from its values at the neighboring points.

If f is an arbitrary function in $L^p(S^n)$, $n - k < p \leqslant \infty$, then f can be explicitly reconstructed at almost all points on almost all spheres $S_v^k = \tilde{\gamma}_v S^k$ by making use of known inversion formulas for the Funk transform on this class of functions (see, e.g., [**10**, p. 297]).

The above reasoning is obviously applicable to reconstruction of the radial function ρ_K of the star body K from the volumes $v^{\pm}(\xi) = \mathrm{vol}_k(K \cap \xi_{\pm})$ with $\xi \in \mathfrak{M}$, as in Corollary 2.2.

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